—	*Minus or leſs.*	The Sign of *Subſtraction*; as 9—6 is 9 leſs 6, and ſignifies that 6 is to be taken from 9, that ſo their Difference may be found.
×	*Into or with.*	The Sign of *Multiplication*; as 9×6, is 9 into 6, and ſignifies that 9 is to be Multiplied into or with 6.
÷	*By.*	The Sign of *Diviſion*; as 8÷2, is 8 by 2, and ſignifies that 8 is to be Divided by 2, alſo thus 2) 8 (4 or thus $\frac{8}{2}$ each ſignifying the ſame thing, to wit, 8 Divided by 2.
=	*Equal.*	The Sign of *Equality* or *Equation*, viz. whenever this Sign = is placed betwixt *Numbers* (or *Quantities*) it denotes them to be Equal, as 9=9, or 9+6=15, or 9—6=3, *&c.* That is, 9 is Equal to 9, or 9 more 6 is Equal to 15, and 9 leſs 6 is Equal to 3, *&c.*
::	*So is.*	The Sign of *Proportion*, or that commonly called the *Golden Rule*, or *Rule of Three*, and :: is always placed betwixt the Two middle *Terms* or *Numbers* in Proportion. Thus 2 : 8 :: 6 : 24. To be read thus; as 2, is to 8; ſo is 6, to 24.

Theſe *Signs* and their *Significations*, being perfectly learnt, will help to ſhorten the Work.

CHAP. II.

Concerning the Principal Rules *in* Arithmetick, *and how they are performed in whole Numbers.*

THE *Rules* by which *Numerical Operations* are perform'd in all the Parts of *Arithmetick*, are many and various, ſeveral of them being form'd and raiſed as Occaſion requires, when applied to *Practice*; yet they are all comprehended within the due Conſideration of theſe Six, *viz.* Numeration (or Notation),

tion), Addition, Substraction, Multiplication, Division, and Evolution, or *Extraction of Roots*.

Sect. 1. *Of* Numeration *or* Notation.

Numeration or *Notation*, teacheth to Read or Expreſs the true Value of any *Number* when writ down; and conſequently to write down any propoſed *Number* according to it's true Value when it is named: And this conſiſteth of Two Parts.

1. The due Order of placing down Figures.
2. The true valuing of each Figure in it's Place.

Both which are plainly exhibited in the following Table.

&c.	*Hundreds of Thouſands of Millions,*	*Tens of Thouſands of Millions*	*Thouſands of Millions*	*Hundreds of Millions*	*Tens of Millions*	*Millions*	*Hundreds of Thouſands*	*Tens of Thouſands*	*Thouſands*	*Hundreds*	*Tens*	*Units* { *And is the firſt Place in Counting.*
6	7	8	9	8	7	6	5	4	3	2	1	
Period of Thouſands of Millions.			*Period of Millions.*			*Period of Thou-ſands.*			*Period of Units.*			

By this Numeration *Table* it is apparent, that the Order of Places is reckoned from the Right-hand towards the Left; the firſt Place of any *Number* being always that which is the outmoſt *Figure* to the Right-hand: and whatever *Figure ſtands* in that Place, doth only ſignify it's own ſimple Value, *viz.* ſo many *Units* as that *Figure* repreſents.

The ſecond Place is that of *Tens*, and any *Figure* ſtanding in that Place ſignifieth ſo many *Tens* as that *Figure* repreſents *Units*.

The

The third Place is *Hundreds*, the fourth Place *Thouſands*, &c. That is, each Place towards the Left-hand is *Ten* Times the Value of that next it, towards the Right.

For Inſtance, ſuppoſe 759 were propoſed to be read or pronounced according to the Value of each *Figure* as they now ſtand. The firſt *Figure* in this *Sum* is 9, becauſe it ſtands in the Place of *Units*, and therefore ſignifies but it's own ſimple Value, to wit, 9 *Units*, or 9. The ſecond *Figure* 5 ſtands in the Place of Tens, and therefore ſignifies Five *Tens* or *Fifty*. The *Figure* 7 ſtands in the third Place, or Place of *Hundreds*, and therefore it ſignifies *Seven Hundred*; and the whole Sum is to be read or pronounced thus, *Seven Hundred Fifty Nine*.

Note, Although the *Figure* 7 ſtands in the third Place (according to the Order of *Numbering*) yet when the whole *Sum* comes to be read, it is firſt pronounced; the reading of *Numbers* being performed like that of Letters or Words, always beginning with the outmoſt *Figure* towards the Left-hand, and ſo many *Figures* as are placed together without any Point, Comma, Line, or other Note of Diſtinction between them, are all but one *Sum*, and muſt be read as ſuch.

For Example, 763596 is but one entire *Sum* or *Number*, notwithſtanding it conſiſts of ſix Places of *Figures*, and is thus read; *Seven Hundred Sixty Three Thouſand, Five Hundred Ninety Six*.

The like is to be obſerved in reading or expreſſing the true Value of any *Sum* or *Rank* of *Numbers* conſiſting of *Seven*, *Eight*, *Nine*, or more Places of *Figures*, each *Figure* being to be valued according to it's Diſtance from the Place of *Unity*: As in the foregoing Table.

Now ſuch Values may as well ariſe by *Cyphers*, as by other *Figures*; for Inſtance, 6 ſtanding by itſelf, repreſents but Six *Units*: But if a *Cypher* be annext to it thus, 60, then it becomes *Sixty*; for the *Cypher* poſſeſſing the Place of *Units*, hath thereby removed the 6 into the Place of *Tens*; and another *Cypher* more would make it 600, *Six Hundred*, &c.

Whence it may be noted, that although a *Cypher* of itſelf ſignify nothing (as hath been ſaid before) yet being placed on the Right-hand of any *Figure*, it augments the Value of that *Figure* by advancing it into a higher Place than otherwiſe it would have been, had not the *Cypher* been there.

Take one Example more in *Numeration* (if you pleaſe, that in the Table) *viz.* 6789876543 21, which is, according as is there ſignified,

Six

Six Hundred Seventy Eight Thouſand Millions,
Nine Hundred Eighty Seven Millions,
Six Hundred Fifty Four Thouſand,
Three Hundred Twenty One Units. Of any propoſed *Species* or *Quantities* whatſoever.

And here it may be obſerved, that every third *Figure* from the Place of *Units*, bears the Name of *Hundreds*; which ſhews that if any great *Sum* be parted, or rather diſtinguiſhed into Periods, of Three *Figures* in each *Period* (as in the foregoing Table), it will be of good Uſe to help the young learner in the eaſier valuing and expreſſing that *Sum.*

Sect. 2. *Of* Addition.

Poſtulate or Petition.

That any given Number *may be increaſed or made more, by putting another* Number *to it.*

Addition is that *Rule* by which ſeveral *Numbers* are collected and put together, that ſo their *Sum* or *Total* Amount may be known.

In this *Rule* Two things being carefully obſerved, the Work will be eaſily performed.

1. The firſt is the true placing of the *Numbers*, ſo as that each *Figure* may ſtand directly underneath thoſe *Figures* of the ſame Value, *viz.* place *Units* under *Units*, *Tens* under *Tens*, and *Hundreds* under *Hundreds*, &c.

Then underneath the loweſt Rank (always) draw a Line to ſeparate the given *Numbers* from their *Sum* when it is found.

Example. If theſe *Numbers* 54327, and 2651, were given to be added together, they muſt be placed

Thus, $\left\{\begin{array}{r} 54327 \\ 2651 \\ \hline \end{array}\right.$

2. The ſecond thing to be obſerved is the due Collecting or Adding together each Row of *Figures* that ſtand over one another of the ſame Value: And that is thus performed.

Rule.

Always begin your Addition *at the Place of* Units, *and Add together all the* Figures *that ſtand in that Place, and if their* Sum *be under* Ten, *ſet it down below the Line underneath it's own Place; but if their* Sum *be more than* Ten, *you muſt ſet down only the overplus, or odd* Figure *above the* Ten (*or* Tens) *and ſo many* Tens *as the* Sum *of thoſe* Units *amounts to, you muſt carry*

to

to the place of Tens; *Adding them and all the* Figures *that ſtand in the place of* Tens *together, in the ſame manner as thoſe of the* Units *were added; then proceed in the ſame order to the place of* Hundreds, *and ſo on to each place until all is done.*

The *Sum* ariſing from thoſe *Additions* will be the *Total* Amount required.

EXAMPLE 1.

Let it be required to find the Sum of the aforeſaid *Numbers*, *viz.* { 54327 / 2651

56978 the *Sum* required.

Beginning at the place of *Units*, I ſay 1 and 7 is 8, which being leſs than 10, I ſet it down (according to the *Rule*) underneath it's own place of *Units*; and then proceed to the place of *Tens*, ſaying 5 and 2 is 7, which being leſs than 10, I ſet it down underneath it's own place of *Tens*, and proceed to do the like at the place of *Hundreds*, and then at *Thouſands*, ſetting each of their *Sums* underneath their own reſpective places: Laſtly, becauſe there is not any *Figure* in the lower Rank to be added to the Figure 5, which ſtands in the place of *Ten Thouſands*, in the upper Rank, I therefore bring down the ſaid 5 to the reſt, placing it underneath it's own place, and then I find that 54327+2651=56978, the true *Sum* required.

EXAMPLE 2.

Suppoſe it were required to find the *Sum* of theſe *Numbers*, 3578+496+742+184+95. Theſe being placed, as before directed, will ſtand as in the Margin. Then beginning (as before) at the place of *Units*, ſay 5 and 4 is 9, and 2 is 11, and 6 is 17, and 8 is 25; ſet down the 5 *Units* underneath its own place of *Units*, and carry the 20, or two *Tens*, to the place of *Tens* (at which place they are only 2) ſaying, 2 and 9 is 11, and 8 is 19, and 4 is 23, and 9 is 32, and 7 is 39; ſet down the 9 underneath it's own place of *Tens*, and carry the 30, or three *Tens* (which indeed is 300) to the place of *Hundreds*, at which place they are but 3, ſaying, 3 I carry and 1 is 4, and 7 is 11, and 4 is 15, and 5 is 20; here becauſe there is no *Figure* overplus (as before) I ſet down a *Cypher* underneath the place of *Hundreds*, and carry the 2 *Tens* (or rather the 2000) to the place of *Thouſands*, ſaying

3578
496
742
184
95
5095

(as before) 2 I carry and 3 is 5, which being the laſt, I ſet it down underneath it's own place, and all is finiſhed. And find the *Sum* or *Total* amount to be 5095=3578+496+742+184+95.

If this Example be well conſidered, it will be ſufficient to ſhew the uſual Method of *Addition* in whole *Numbers*; but to make all plain and clear, I ſhall ſhew the young Learner the Reaſon of carrying the *Tens* from one Degree or Row of *Figures*, to the next Superior Degree, which is done purely to ſave Trouble, and prevent the uſing of more *Figures* than are really neceſſary, as will appear by the following Method of adding together the ſame *Numbers* of the laſt Example.

Thus, add together each ſingle Row of Figures by itſelf; as if there were no more but that one Row, ſetting down the *Sum* underneath it's own place.

3	5	7	8
	4	9	6
	7	4	2
	1	8	4
		9	5

The *Sum* of the Row of *Units*, is			2	5
The *Sum* of the Row of *Tens*, is		3	7	0
The *Sum* of the Row of *Hund.* is	1	7	0	0
The three *Thouſand* brought down	3	0	0	0
The *Sum* or *Total* Amount as before, is	5	0	9	5

Add

From hence I preſume it will be eaſy to conceive the true Reaſon of carrying the aforeſaid *Tens*; and alſo that *Cyphers* do not augment or increaſe the *Sum* in *Addition*. (See Page 4.)

I might have here inſerted a Lineal Demonſtration of this *Rule* of *Addition*; but I thought it would rather puzzle than improve a young Learner, eſpecially in this place; beſides the Reaſon of it is ſufficiently evident from that Natural Truth of *the Whole being Equal to all it's Parts taken together*. Euclid 1. Axiom 19.

That is, the *Numbers* which are propoſed to be added together, are by that *Axiom* underſtood to be the ſeveral Parts, and their *Sum* or *Total* Amount found by *Addition* is underſtood to be the Whole.

And from thence is deduced the Method of proving the Truth of any *Operation* in *Addition*, *viz.* By parting or ſeparating the given *Numbers* into Two Parcels (or more, according to the Largeneſs of it) and then adding up each Parcel by itſelf: For if thoſe particular *Sums* ſo found, be *added* into one *Sum*, and that *Sum* prove Equal or the ſame with the *Total Sum* firſt found,

found, then all is right; if not, care muſt be taken to diſcover and correct the Error.

EXAMPLE.

Add	5647 3289 4016	The Sum of theſe Parts is,	12952
	2900 5007 1606	The Sum of theſe is,	9513
The *Total Sum* of all theſe Parts	22465	The *Sum* of each Parcel put together	22465

Sect. 3. *Of* Subſtraction.

Poſtulate or Petition.

That any Number *may be diminiſhed, or made leſs, by taking another* Number *from it.*

Subſtraction is that *Rule* by which one *Number* is *deducted* or taken out of another, that ſo the *Remainder*, *Difference*, or *Exceſs* may be known.

As 6 taken out of 9, there *remains* 3. This 3 is alſo the *Difference* between 6 and 9, or it is the *Exceſs* of 9 above 6.

Therefore the *Number* (or *Sum*) out of which *Subſtraction* is required to be made, muſt be greater than (or at leaſt equal to) the *Subtrahend* or *Number to be ſubſtracted*.

Note, *This Rule is the Converſe or Direct contrary to* Addition.

And here the ſame Caution that was given in *Addition*, of placing *Figures* directly under thoſe of the ſame Value, *viz. Units* under *Units*, *Tens* under *Tens*, and *Hundreds* under *Hundreds*, &c. muſt be carefully obſerved; alſo underneath the loweſt Rank there muſt be drawn a Line (as before in *Addition*) to ſeparate the given *Numbers* from their *Difference* when it is found.

Then having placed the leſſer *Number* under the greater, the *Operation* may be thus performed.

RULE.

Begin at the Right Hand Figure *or place of* Units (*as in* Addition) *and take or ſubſtract the lower* Figure *in that place from*

from the Figure *that ſtands over it, ſetting down the* Remainder *or* Difference *underneath it's own place. If the Two* Figures *chance to be Equal, ſet down a* Cypher: *But if the upper* Figure *be leſs than the lower* Figure, *then you muſt* add 10 *to the upper* Figure, *or mentally call it* 10 *more than it is, and from that* Sum ſubſtract *the lower* Figure, *ſetting down the* Remainder (*as* before directed). *Now becauſe the* 10 *thus added, was ſuppoſed to be borrowed from the next ſuperior place* (viz. *of* Tens) *in the upper* Figures, *therefore you muſt either call the upper* Figure *in that place from whence the* 10 *was borrowed, one leſs than really it is, or elſe (which is all one, and moſt uſual) you muſt call the lower* Figure *in that place one more than it really is, and then proceed to* Subſtraction *in that place, as in the former; and ſo gradually on from one Row of* Figures *to another until all be done.*

EXAMPLE 1.

Let it be required to find the *Difference* between 6785, and 4572. That is, let 4572 be *ſubſtracted* from 6785.

Theſe *Numbers* being placed down, as before directed, will ſtand

$$\text{Thus}\left\{\begin{array}{r} 6785 \\ 4572 \\ \hline 2213 \end{array}\right.$$

Beginning at the place of *Units*, take 2 from 5 and there will *remain* 3 which muſt be ſet down underneath it's own place, and then proceed to the place of *Tens*, taking 7 from 8, and there will *remain* 1, to be ſet down underneath it's own place; again, at the place of *Hundreds*, take 5 from 7, and there *remains* 2, which ſet down, as before; laſtly, take 4 from 6 and there will *remain* 2, which being ſet down underneath it's own place, the Work is finiſhed, and the *Difference* ſo found will be 2213=6785—4572, as was required.

EXAMPLE 2.

The *Difference* between 5849, and 7496 is required.

Having placed the *Numbers* as in the Margin, begin at the place of *Units*, (as before) and ſay 9 from 6 cannot be, but 9 from 16 and there *remains* 7, to be ſet down under it's own place; next proceed to the place of *Tens*, where you muſt now pay the 10 that was borrowed to make the 6 16, by counting the upper *Figure* 9 in that place one leſs than it is, ſaying 4 from 8 and there *remains* 4, or elſe (which is the moſt practiſed) ſay 1 I borrowed and 4 is 5, from

$$\begin{array}{r} 7496 \\ 5849 \\ \hline 1647 \end{array}$$

from 9 and there *remains* 4, to be ſet down under it's own place (as before); again, at the place of *Hundreds*, ſay 8 from 4 that cannot be, but 8 from 14 there will *remain* 6 to be ſet down; and here I have borrowed 10 (as before) which muſt be paid in the ſame manner as the other 10 was, *viz.* either by calling the 7 in the upper Rank but 6, ſaying 5 from 6 there *remains* 1, or elſe by ſaying 1 borrowed and 5 is 6 from 7 and there *remains* 1, which being ſet down under it's own place all is done, and the *Difference* required will be 1647=7496—5849.

EXAMPLE 3.

From 830476
Take 741068
———
Remains 89408

By this Example you may perceive that *Cyphers* in the *Subtrahend*, viz. in the *Numbers* to be *ſubſtracted*, do not diminiſh the *Number* from whence *Subſtraction* is made. See *Page* 4.

Theſe Three Examples, I preſume, may be ſufficient to ſhew the young Learner the Method of *Subſtracting* whole *Numbers*; as for the Reaſon thereof it is the ſame with that of *Addition*, Page 10. *viz.* of *the Whole being Equal to all it's Parts taken together*.

That is, in this *Rule* the *Number* from which *Subſtraction* is required to be made, is underſtood to be the Whole, and the *Subtrahend* or *Number* to be *ſubſtracted*, is ſuppoſed to be a part of that Whole; conſequently if that Part be taken from the Whole, the *Remainder* will be the other part.

From hence is deduced the common Method of proving *Subſtraction*, by adding together the *Subtrahend* and the *Remainder*. For if the *Sum* of thoſe Two (which are here called Parts) be equal to the *Number* from whence *Subſtraction* was made (which is here called the Whole) then the Work is right; if not, care muſt be taken to diſcover and correct the Error.

EXAMPLE.

From 59435
Take 47608 } Add
———
Proof { 11827 } The *Sum* which is equal to the *Number* from
——— } whence *Subſtraction* was made.
59435 }

Or

Or from the aboveſaid Reaſon, it will be eaſy to conceive how to prove the Truth of *Subſtraction* by *Subſtraction*.

For if from	59435	being here the whole,
there be taken	47608	as part of that whole;
there will *remain*	11827	the other part (as before)
And if from	59435	the whole, there be *ſubſtracted* the
laſt part, *viz.*	11827	
there will *remain*	47608	the firſt part, or *Number* which was required to be firſt *Subſtracted*.

From 75643	From 7000000
Take 9000	Take 986432
Remains 66643	Remains 6013568

Sect. 4. *Of* Multiplication.

Multiplication is a *Rule* by which any given *Number* may be ſpeedily increaſed, according to any propoſed *Number* of *Times*.

That is, *One* Number *is ſaid to* Multiply *another*, *when the* Number multiplied *is ſo often* added *to itſelf*, *as there are* Units *in the* Number multiplying; *and another* Number *is produced*. (Euclid 7. Def. 15.)

To perform *Multiplication*, there is required two given *Numbers*, called *Factors*.

The Firſt is the *Number* to be *multiplied*, which is generally put the greater of the Two *Numbers*, and is commonly called the *Multiplicand*.

The other is that *Number* by which the Firſt is to be *multiplied*, and is uſually called the *Multiplicator* or *Multiplier*; and this denotes the *Number* of *Times* that the *Multiplicand* is required to be *added* to itſelf. For ſo many *Units* as are contained in the *Multiplier*, ſo many times will the *Multiplicand* be really *added* to itſelf (*as per* Euclid *above*). And from thence will ariſe a Third *Number*, called the *Product*. *But in* Geometrical Operations *it is called the* Rectangle *or* Plain.

For inſtance; ſuppoſe it were required to increaſe 6 four times, that is, to *multiply* 6 into or with 4. Theſe two *Numbers* are to be ſet (or placed) down as in *Addition* or *Subſtraction*.

Thus

Thus { 6 *Multiplicand*, / 4 *Multiplier*, } or *Factors*.

Product 24 *viz.* 4 times 6 is 24, as plainly appears by *Addition*, *viz.* By ſetting down 6 four times, and then adding them together into one *Sum*, Thus

1	6
2	6
3	6
4	6

} Add

24

From hence it is evident, that Multiplication *is only a* Conciſe *or* Compendious *Way* of adding *any given* Number *to itſelf, ſo often as any* Number *of Times may be propoſed.*

Before any Operation can be readily performed in *Multiplication*, the ſeveral *Products* of the ſingle *Figures* one into another muſt be perfectly learned by Heart, *viz.* That 2 times 2 is 4, that 3 times 3 is 9, and 3 times 6 is 18, *&c.* According as they are expreſſed in the following *Table*; wherein I have omitted *multiplying* with 2, it being ſo very eaſy that any one may do it.

Multiplication Table.

3×3= 9	4×4=16	5×5=25	6×6=30	7×7=49	8×8=64
3×4=12	4×5=20	5×6=30	6×7=42	7×8=56	8×9=72
3×5=15	4×6=24	5×7=35	6×8=48	7×9=63	9×9=81
3×6=18	4×7=28	5×8=40	6×9=54		
3×7=21	4×8=32	5×9=45			
3×8=24	4×9=36				
3×9=27					

I think it needleſs to give any *Explanation* of this *Table*; for if the *Signs* and their *Significations* be well underſtood, *(vide page* 5) it muſt needs be eaſy. Only this may be noted, that $4 \times 3 = 3 \times 4$, or $7 \times 5 = 5 \times 7$, *&c.*

That is, 3 times 4 is the ſame with 4 times 3, or 5 times 7 is the ſame with 7 times 5, *&c.* The like muſt be underſtood of all the reſt in the *Table.*

And when all theſe ſingle *Products* are ſo perfectly learned by Heart, as to be ſaid without pauſing; you may then proceed (but not 'till then) to the Buſineſs of *Multiplication*; which will be found very eaſy, if the following *Rule* (and *Examples)* be carefully obſerved.

RULE.

Always begin with that Figure *which ſtands in the* Units *place with the* Multiplier, *and with it* multiply *the* Figure *which ſtands*

in

in the Units *place of the* Multiplicand; *if their* Product *be leſs than* Ten, *ſet it down underneath it's own place of* Units, *and proceed to the next* Figure *of the* Multiplicand. *But if their* Product *be above* Ten *(or* Tens) *then ſet down the Overplus only (or odd* Figure, *as in* Addition) *and bear (or carry) the ſaid* Ten *or* Tens *in mind until you have* multiplied *the next* Figure *of the* Multiplicand, *with the ſame* Figure *of the* Multiplier; *then to their* Product *add the* Ten *or* Tens *carried in mind, ſetting down the Overplus of their* Sum *above the* Tens, *as before: and ſo proceed on in the very ſame manner, until all the* Figures *of the* Multiplicand *are* multiplied *with that* Figure *of the* Multiplier.

EXAMPLE 1.

Suppoſe it were required to *multiply* 3213 into or with 3.

3213 *Multiplicand*
3 *Multiplier* } or *Factors.*

Product 9639

Beginning at the *Units* place, ſay 3 times 3 is 9, which, becauſe it is leſs than *Ten*, ſet down underneath it's own place; and proceed to the next place of *Tens*, ſaying 3 times 1 is 3, which ſet down underneath it's own place; then to the next place, *viz.* of *Hundreds*, ſaying 3 times 2 is 6, which ſet down, as before; laſtly, at the place of *Thouſands*, ſay 3 times 3 is 9, which being ſet down underneath it's own place, the Operation is finiſhed; and the true *Product* is 9639=3213×3, as was required.

EXAMPLE 2.

Let it be required to *multiply* 8569 into 8. Set down theſe *Numbers* as before,

Thus { 8569
8

68552

Beginning at the *Units* place, ſay 8 times 9 is 72, ſet down the 2 underneath it's own place of *Units*, and bear the 70, or 7 *Tens* in mind, and proceed to the next *Figure* of the *Multiplicand* (at which place the 7 *Tens* are only 7) ſaying 8 times 6 is 48, and the 7 carried in mind is 55; ſet down the odd 5 underneath it's own place of *Tens*, and carry the 50 (which is really 500) to the next place *(viz.* of *Hundreds)* at which place it is only 5, where ſay, 8 times 5 is 40, and the 5 carried in mind is 45; ſet down the 5 underneath it's own place, and carry the 40 or 4 *Tens* (which is really 4000) to the next

next place, *viz.* of *Thouſands*, ſaying, 8 times 8 is 64, and 4 carried in mind is 68. (Now this being the laſt Place or *Figure* to be *multiplied)* Set down the whole *Product* 68, and the Work is done.

So that, 8569×8=68552, the *Product* required.

Now the reaſon of this, and all other the like Operations, may be eaſily conceived from this which follows.

8 5 6 9 } The ſame *Factors* as before.
8 }

			7	2
		4	8	0
	4	0	0	0
6	4	0	0	0

Here 8 times 9 is 72, as before, becauſe the 9 ſtands in the *Units* place.

Now here it is not really 8 times 6=48, but it is 8 times 60=480, becauſe the 6 ſtands in the place of *Tens*.

And here it is not 8 times 5=40, but is is really 8 times 500=4000, becauſe the 5 ſtands in the place of *Hundreds*.

Laſtly, becauſe the 8 in the *Multiplicand* ſtands in the place of the *Thouſands*, it is therefore 8 times 8000=64000, and not 8 times 8=64.

6 8 5 5 2 } The *Sum* of the particular *Products*, which gives the true *Product*, as before.

By what hath been already ſaid, with a little Conſideration had to the *Examples*, I preſume the Learner may eaſily underſtand how to *multiply* whole *Numbers* with any ſingle *Figure*. And when it is required to *multiply* with more than one; then ſo many Figures as there are in the *Multiplier*, ſo many particular *Products* there muſt be.

That is, all the *Figures* of the *Multiplicand* muſt be *multiplied* with every ſingle *Figure* of the *Multiplier*, as if there were but one ſingle *Figure:* and the *Sum* of all thoſe particular *Products*, will be the true *Product* required. But in thoſe Operations, great Care muſt be taken in ſetting down the particular *Products* (which ariſe by each *multiplying Figure)* in their proper places. Which will be eaſily done, if the following Directions be carefully obſerved.

Viz. Always place the firſt *Figure* (or *Cypher)* of every particular *Product*, directly underneath the *multiplying Figure*. Or thus:

The Firſt Figure *(or* Cypher) *of the ſecond particular* Product *muſt ſtand directly under the ſecond* Figure *(or place) of the Firſt* Product; *and the Firſt* Figure *(or* Cypher) *of the Third*

particular Product, *muſt ſtand directly underneath the Third* Figure *of the Firſt* Product: *And ſo on until all is done.*

Now the Reaſon of placing the firſt *Figure* of every particular *Product* in their Order, will be very obvious to any one that conſiders the laſt *Example*; wherein the *Cyphers* are only ſet down to ſhew the true Diſtance of the firſt *Figure* in each particular *Product* from the *Units* place. And altho' it is not uſual to ſet down *Cyphers* in this manner; yet they are always ſuppoſed to be there: That is, their Places are always left void, as in the two following *Examples*; wherein I have placed *Points* inſtead of *Cyphers*.

EXAMPLE 3.

Let it be required to *multiply* 78094, into or with 7563.

78094 } *Factors.*
7563 }

234282	The Firſt particular *Product* with	3
468564.	The Second particular *Product* with	60
390470..	The Third particular *Product* with	500
546658...	The Fourth particular *Product* with	7000
590624922	The *Total*, or true *Product* required.	

EXAMPLE 4.

Suppoſe it be required to *multiply* 57498 into 60008.

57498
60008

459984	The *Product* with	8
3449 8....	The *Product* with	60000

345033984=57498×60008, as was required.

Here you may obſerve, that I paſs over the *Cyphers*, and only take care of placing the firſt *Product* of the laſt *Figure*, *viz.* of 60000 according to the foregoing Directions.

When there is a *Cypher* or *Cyphers*, to the Right-hand either of the *Multiplicand* or *Multiplicator*, or to both; in that caſe *multiply* the *Figures* as before; neglecting the *Cyphers* until the particular *Products* are added together; Then to their *Sum* annex ſo many *Cyphers* as are in either or both the *Factors*. As in theſe:

EXAMPLE

EXAMPLE 5.	*EXAMPLE* 6.	*EXAMPLE* 7.
9538	87600	785000
4600	79	56900
57228	7884	7065
38152	6132	4710
43874800	6920400	3925
		44666500000

Take a few Examples without their Work, at large.

$$75649\times579=43800771$$
$$687000\times356=244572000$$
$$530674\times45007=23884044718$$
$$7901375\times30000=237041250000$$
$$537084000\times590700=317255518800000$$
$$102030405\times504030201=5142640554026140 5$$
$$98765432\text{1}\times123456789=12193264111263526 9$$

Note, If it be required to *multiply* any *Number* with 10, 100, 1000, 10000, *&c.* it is only annexing the *Cyphers* of the *Multiplier* to the *Figures* of the *Multiplicand*, and the Work is done.

Thus $\begin{cases} 578\times10=5780. & 578\times1000=578000 \\ 578\times100=57800. & 578\times10000=5780000, \&c. \end{cases}$

These *Examples* (being well understood) are sufficient to instruct the Learner all the Varieties that can happen in *multiplying* of whole *Numbers*, according to the Method generally practised: However it may not be amiss to shew here how *Multiplication* may be performed (with many *Figures*) by *Addition* only.

EXAMPLE.

Let it be required to *multiply* 879654 into 79863.

In order to perform this (or any other Operation of this kind) by *Addition* only; you must make a Tariffa or small Table of the given *Multiplicand*, in this manner:

First, Make a small Column, and in it place gradually downward the Nine single *Figures*; viz. 1, 2, 3, 4, 5, *&c.*

Then againſt the *Figure* 1, ſet down the *Multiplicand* (which in this *Example* is 879654) and againſt the *Figure* 2, ſet down the double of the *Multiplicand*, found by *adding* it to itſelf; To this double *add* the *Multiplicand*, ſetting down their *Sum* againſt the *Figure* 3. And ſo proceed on by a continued *Addition* until there be Ten times the *Multiplicand* in the *Table*; which if the Work is true, will be the *Multiplicand* itſelf with a *Cypher* to the Right-hand of it, as in the annexed Table. This being done, it will be eaſy to conceive, that the *Figures* in the ſmall Column of the *Table*, do reſpectively repreſent thoſe of the *Multiplier*: And that the *Numbers* againſt any of thoſe *Figures* in the ſmall Column, will be the true *Product* of the *Multiplicand* agreeing to any *Figure* of the *Multiplier*; as plainly appears by the Work of this *Example*.

1	879654
2	1759308
3	2638962
4	3518616
5	4398270
6	5277924
7	6157578
8	7037232
9	7916886
10	8796540

Then 879654 } The *Factors* as before.
79863 }

Againſt	3, in the *Table* is	2638962	=879654×3
	6, is	5277924	=879654×60
	8, is	7037232	=879654×800
	9, is	7916886	=879654×9000
	7, is	6157578	=879654×70000
The Product required		70251807402	=879654×79863

Note, This Method of Tabulating the *Multiplicand*, is both eaſy and certain; being neither ſubject to Errors, nor burdenſome to the Memory, and therefore in large Calculations it may be found very uſeful. But for common Practice the uſual Method (as in *Page* 18, &c.) is beſt, and to be preferred before this.

Moſt *Maſters* that teach (and ſeveral *Authors* that write of) *Arithmetick*, do teach to prove the Truth of *Multiplication*, by caſting away all the *Nines* that are contained in both the *Factors*, and their *Product*; but becauſe that Method is very erroneous, as might be eaſily ſhewed; I ſhall therefore omit inſerting it, and leave the Proof of *Multiplication* to the next *Section*, wherein (I preſume) the Reaſon and Proof, both of it and *Diviſion*, will plainly appear.

Sect.

Sect. 5. *Of* Diviſion.

Diviſion is a *Rule* by which one *Number* may be ſpeedily *ſubſtracted* from another, ſo many times as it is contained therein.

That is, It ſpeedily diſcovers how often one *Number* is contained (or may be found) in another: And to perform that there are required Two Numbers to be given.

1. The one of them is that *Number* which is propoſed to be *divided*, and is called the *Dividend*.

2. The other is that *Number* by which the ſaid *Dividend* is to be *divided*, and is called the *Diviſor*.

And by comparing theſe Two, *viz.* the *Dividend* and the *Diviſor* together, there will ariſe a Third *Number*, called the *Quotient*; which ſhews how often the *Diviſor* is contained in the *Dividend*, or into what *Number* of Equal Parts the *Dividend* is then divided. Therefore,

Diviſion *is by* Euclid *fitly termed the meaſuring of one* Number *by another, viz.* one *Number* is ſaid to meaſure another by that *Number*, which when it *multiplies*, or is *multiplied* by it, it produceth. *Euclid* 7. *Def.* 23.

And, if a Number *meaſuring another*, multiply *that* Number *by which it meaſureth, or be* multiplied *by it, it produceth the* Number *which it meaſureth.* Euclid 7. Axiom 9.

That is to ſay, If that *Number* which *divides* another (called the *Diviſor*) be *multiplied* with the *Number* which is produced by *Diviſion* (called the *Quotient*) their *Product* will be the *Number divided* or *Dividend*. Whence it follows, that *Diviſion* and *Multiplication* are the Converſe and Direct Contrary one to another (as *Subſtraction* is to *Addition*) and do mutually prove the Truth of each other's *Operations*.

I ſhall therefore make choice of the foregoing *Examples* in *Multiplication*, in order (as I preſume) to render the Buſineſs of *Diviſion* more plain and eaſy.

Firſt, Let it be required to find how often 6 is contained in 24. That is, to *divide* 24 by 6.

N.B. Always place down the given *Numbers* in this Order; Firſt ſet down the *Diviſor*, and to the Right-hand of it draw a crooked Line; then ſet down the *Dividend*, and to the Right of it draw another crooked Line, in which muſt be placed the *Quotient Figure*, or *Figures*, as they become found.

Thus

Dividend.

Thus *Diviſor* 6) 24 (4 the *Quotient.*

Here I conſider how many times 6 there is in 24, and find it 4, *viz.* 4 times 6 is 24, therefore 4 is the true *Quotient* or *Anſwer* required.

This is apparent by *Subſtraction*, as in the Margin; where 24 the *Dividend* is ſet down, and from it 6 the *Diviſor* continually *ſubſtracted* ſo often as it can be, which is juſt 4 times. Therefore 4 is the true *Quotient* or *Anſwer* required.

Compare this with the *Example*, *Page* 15.

	24
1	6
	18
2	6
	12
3	6
	6
4	6
	0

Corollary.

From hence it is evident; that *Diviſion* is but a *conciſe* or *compendious Method* of *ſubſtracting* one *Number* from another, ſo often as it can be found therein; for if the *Diviſor* be continually *ſubſtracted* from the *Dividend*, accounting an *Unit* (or 1) for each time it is *ſubſtracted* (as above) the *Sum* of thoſe *Units* will be the *Quotient.*

All Operations in *Diviſion* do begin contrary to thoſe of *Multiplication*, *viz.* at the Firſt *Figure* to the Left-hand, or that of the higheſt Value, and decreaſe the *Dividend* by a repeated *Subſtraction* of each *Product* ariſing from the *Diviſor* when *multiplied* into the *Quotient Figure.* And the only Difficulty in *Diviſion* of whole *Numbers* (or indeed of any *Numbers*) lies in making choice of ſuch a *Quotient Figure*, as is neither too big nor too little; and that may be eaſily obtained by obſerving the following *Rule*, which hath two *Caſes.*

RULE.

Caſe 1. *As often as the Firſt* Figure *of the* Diviſor *is taken from the Firſt* Figure *of the* Dividend: *So often muſt the Second* Figure *of the* Diviſor *be taken from the Second* Figure *of the* Dividend, *when it is joined with what* Remains *of the Firſt. And as often muſt the Third* Figure *of the* Diviſor *be taken from the Third* Figure *of the* Dividend, *&c.*

But if the Firſt *Figure* of the *Diviſor* cannot be taken from the Firſt *Figure* of the *Dividend.* Then;

Caſe

Caſe 2. *So often as the Firſt* Figure *of the* Diviſor *is taken from the Two Firſt* Figures *of the* Dividend, *ſo often muſt the Second* Figure *of the* Diviſor *be taken from the Third* Figure *of the* Dividend, *when it is joined with what* remained *of the Second: And ſo often muſt the Third* Figure *of the* Diviſor *be taken from the Fourth* Figure *of the* Dividend, *&c.*

That is, the *Quotient Figure* muſt be ſuch, as being *multiplied* into the *Diviſor*, will *produce a Product* equal to ſuch a part of the *Dividend* as is then taken for that *Operation:* But if ſuch a *Product* cannot be exactly found, then the next leſs muſt be taken, and ordered, as in the following *Examples:* of which let that in *Page* 16 be the firſt, wherein there was given 8569 the *Multiplicand*, and 8 the *Multiplier*. To find the *Product* 68552. Let us here ſuppoſe the ſaid *Product* 68552, and 8 the *Multiplier*, both given; thence to find the *Multiplicand*. That is, Let it be required to *divide* 68552 by 8.

Dividend
Diviſor 8) 68552 (*Quotient* when found.

According to the *Rule*, *Caſe* 1. I compare 8 the *Diviſor* with 6 the Firſt *Figure* of the *Dividend*, and finding I cannot take it from that, I then conſider (by *Caſe* 2.) how often 8 can be taken from 68, the two firſt *Figures* of the *Dividend*, and find it may be taken 8 times; for 8 times 8 is 64, being the greateſt *Product* of 8 (into any *Figure*) that can be taken from 68. I therefore place 8 in the *Quotient*, and with it *multiply* 8 the *Diviſor*, ſetting down their *Product* underneath the ſaid Two Firſt *Figures* of the *Dividend*, *ſubſtracting* it from them, and then the Work will ſtand

Thus 8) 68552 (8
64
———
4

In order to a ſecond *Operation*, I make a *Point* under the next *Figure* of the *Dividend*, *viz.* under the 5, and bring it down underneath in its own place to the *Remainder* 4, which will by that means become 45. Then I conſider how many times 8 can be taken from 45, and find it may be 5 times, for 5 times 8 is 40, I therefore place 5 in the *Quotient*, and with it *multiply* 8 the *Diviſor*, ſetting down and *ſubſtracting* their *Product*, as before. Then the Work will ſtand

Thus

Thus 8) 68552 (85
64.
45
40
5

For a Third *Operation*, I make a *Point* under the next *Figure* of the *Dividend*, *viz.* under the 5, and bring it down, as before, proceeding in all reſpects, as before; and then the Work will ſtand

Thus 8) 68552 (856
64..
45
40
55
48
7

Laſtly, I point and bring down the 2, *viz*, the laſt *Figure* of the *Dividend* to the *Remainder* 7, which will then become 72, and proceeding as in the other *Operations*, I find that 8 the *Diviſor* can be taken juſt 9 times from 72, and the Work is finiſhed, and will ſtand

Thus 8) 68552 (8569
64...
45
40
55
48
72
72
(0)

The true *Quotient* is found to be 8569, being exactly the *Eighth* part of 68552, or the *Multiplicand* of the propoſed *Example* of *Multiplication*. As was required.

The Reaſon of the *Operations* will be very plain to any one that will a little conſider of it, as follows:

Diviſor

Divisor 8) 6 8 5 5 2 (8000. The First *Quotient Figure.*

Substract 6 4 0 0 0 — This *Product* of the *Divisor* into the *Quotient* is 64000, *viz.* 8 times 8000; the *Quotient Figure* being always of the same *Value* or *Degree* with that *Figure* under which the *Unit*'s place of its *Product* stands.

Divisor 8) 4 5 5 2 (500. The Second *Quotient Figure.*

Substract 4 0 0 0 — And here the *Product* is 4000, *viz.* 8 times 500, not 8 times 5.

Divisor 8) 5 5 2 (60. The Third *Quotient Figure.*

Substract 4 8 0 — Also here the *Product* is 480, *viz.* 8 times 60, for the *Reasons* abovesaid.

Divisor 8) 7 2 (9. The Fourth *Quotient Figure.*

Substract 2 — Now here the *Product* is but 72, *viz.* 9 times 8, because the 9 stands in the place of *Units.*

Remains (0 0) Now the *Sum* of all the several *Quotients*, *viz.* 8000+500+60+9=8569, as before.

If the *Process* of this *Example* be well considered and compared with that of *Multiplication*, *Page* 17. it will evidently appear to be only the *Converse* of that; for the particular *Products* are alike in both, only that which is *last* there, is *first* here; there they are *added*, here they are *substracted.* So that whoever understands the true *Reason* of the one, must needs understand the *Reason* of the other, and then *Division* will become very *easy*, although the *Divisor* consists of several places of *Figures.*

E X A M P L E.

Let it be required to *divide* 590624922 by 7563.

Dividend

Divisor 7563) 590624922 (

'Tis plain at the first sight, that 7563 the *Divisor*, cannot be taken from 5906, the like *Number* of *Figures* in the *Dividend.*

Therefore, by the *Second Case* of the *Rule* (*Page* 23.) there must be allowed Five *Figures* of the *Dividend*, *viz.* 59062 for the *First Operation* or *Quotient*; that so the *First Figure* 7 of the *Divisor* may be taken out of the two *First Figures*, *viz.* 59 of the *Dividend*, &c.

Then I proceed (*per Caſe* 2.) and conſider how often 7 may be taken from 59, and find it may be taken 8 times, for 8 times 7 is but 56, which I mentally *ſubſtract* from 59, and there *remains* 3; to this 3 I mentally adjoin the *Third Figure* of the *Dividend*, *viz.* 0, which makes it 30, out of which I muſt take the *Second Figure* of the *Diviſor*, *viz.* 5. ſo often as I took the 7 from 59, which was 8 times: But that cannot be, for 8 times 5 is 40, which is more than 30, therefore 8 is too big a *Figure* to be placed in the *Quotient*; yet, hence I conclude, that the next leſs, *viz.* 7 may be taken without any further *Trial*. I therefore place 7 in the *Quotient*, and with it *multiply* the *Diviſor*, ſetting down their *Product* under the *Dividend*, and *ſubſtract* it from thence, as in the other *Example*, and then the *Work* will ſtand

Thus 7563) 590624922 (7
52941
———
6121

In order to a *Second Operation*, I make a *Point* under the next *Figure* of the *Dividend*, *viz.* under the 4, and bring it down to the *Remainder* 6121, which will then become 61214, with which I proceed in all reſpects as I did before with the 59062, and find the next *Quotient Figure* will be 8, with which I *multiply* the *Diviſor*, &c. and *ſubſtract* their *Product* from the ſaid 61214. Then the *Work* will ſtand

Thus 7563) 590624922 (78
52941.
———
61214
60504
———
710

To this *Remainder* 710, I point and bring down the next *Figure* of the *Dividend*, *viz.* 9, which makes it 7109; now becauſe the *Diviſor* 7563 cannot be taken from 7109, I therefore place a *Cypher* in the *Quotient*.

And this muſt always be carefully obſerved, viz. *That for every* Figure *or* Cypher, *which is brought down from the* Dividend, *in order to a new* Operation, *there muſt always be either a* Figure *or* Cypher, *ſet down in the* Quotient. Then the Work will ſtand

Thus

Thus 7563) 590624922 (780
52941..
61214
60504
7109

To this 7109, I bring down another *Figure* of the *Dividend*, *viz.* 2, and then it will become 71092; then I consider how often 7 can be taken from 71, *&c.* (just as at the first Operation) and find it may be taken 9 times, therefore I set down 9 in the *Quotient*, and with it *multiply* the *Divisor*, setting down and *subtracting* their *Product*, as before; Then the Work will stand

Thus 7563) 590624922 (7809
52941...
61214
60504
71092
68067
3025

To this *Remainder* 3025, I point and bring down the last *Figure* 2 of the *Dividend*, which makes it 30252; then proceeding in all respects as before, I find the *Quotient Figure* to be 4, with it I *multiply* the *Divisor*, setting down and *subtracting* their *Product* as before, and then the Work will stand

Thus 7563) 590624922 (78094
52941....
61214
60504
71092
68067
30252
30252
(00000)

Here the Work is ended, and I find the *Quotient* to be 78094, being the true *Multiplicand* of the propos'd *Example* of *Multiplication*, Page 18.

That is, 7563 is contained in 590624922 just 78094 times, *&c.*

If the Work of this *Example* be conſidered and compared with the *Rule* (*Page* 22.) the whole Buſineſs of *Diviſion* will be eaſy; for indeed the only Difficulty (as I ſaid before) lies in making choice of a true *Quotient Figure*, which cannot well be done according to the Common Method of *Diviſion*, without Trials, yet thoſe Trials need not be made with the whole *Diviſor* (as appears by this laſt *Example*) for by the two Firſt *Figures* of the *Diviſor* all the reſt are generally regulated; except the Second *Figure* chance to be 2, 3, or 4, and at the ſame time the Third *Figure* be 7, 8, or 9, then indeed reſpect muſt be had to the Third *Figure*, according as the *Rule* directs.

However, if thoſe Trials are thought too troubleſome, they may be avoided, and the ſame *Quotient Figure* may both eaſily and certainly be found by help of ſuch a ſmall *Table* made of the *Diviſor*, as was of the *Multiplicand* in *Page* 20.

EXAMPLE 4.

Let it be required to *divide* 70251807402 by 79863. *See the Example of Multiplication, Page* 20, *and as there directed make a Table of the Diviſor* 79863,

Thus,

	Diviſor.	*Dividend.*	*Quotient.*
1	79863)	70251807402	(879654
2	159726	638904.....	
3	239589	636140	
4	319452	559041	
5	399315	770997	
6	479178	718767	
7	559041	522304	
8	638904	479178	
9	718767	431260	
10	798630	399315	
		319452	
		319452	
		(000000)	

The Work of this Operation I preſume may be eaſily underſtood. For thoſe *Figures* in the *Table* are the *Product* of the *Diviſor* into all the 9 *Figures*; conſequently thoſe *Figuros* in the ſmall Column do ſhew what *Figure* is to be placed in the *Quotient*; without any doubtful Trials of the *Diviſor* with the *Dividend*, as before.

This Method of Tabulating the *Diviſor* may be of good Uſe to a Learner; eſpecially until he is well practiſed in *Diviſion*; yea, and even then if the *Diviſor* be large, and a *Quotient* of many *Figures* be required; as in reſolving of high *Æquations*, and calculating of *Aſtronomical Tables*, or thoſe of Intereſt, *&c.*

Hitherto

Hitherto I have made choice of *Examples* wherein the *Dividend* is truly measured or *divided* off by the *Divisor*, without leaving any *Remainder*, being exactly composed of the *Divisor* and *Quotient*. But it most usually falls out, that the *Divisor* will not exactly measure the *Dividend*; in which case the *Remainder* (after *Division* is ended) must be set over the *Divisor* with a small Line betwixt them adjoining te the *Quotient*.

EXAMPLE 5.

Suppose it were required to *divide* 379 by 5.

5) 379 (75$\frac{4}{5}$ the Remainder / the Divisor
35.

29
25
---
(4)

EXAMPLE 6.

Again, Let it be required to *divide* 43789 by 67.

67) 43789 (653$\frac{38}{67}$ the true *Quotient* required.
402..

358
335

239
201
---
(38)

How such *Remainders* thus placed over their *Divisors* (which are indeed *Vulgar Fractions*) may be otherwise managed, shall be shewed farther on.

N. B. When the *Divisor* happens to be an *Unit*, viz. 1. with a *Cypher* or *Cyphers* annexed to it, as 10, 100, 1000, *&c.* *Division* is truly performed by cutting off with a Point or Comma, so many *Figures* of the *Dividend* as there are *Cyphers* in the *Divisor*; then are those *Figures* so cut off to be accounted a *Remainder*, and the rest of the *Figures* in the *Dividend* will be the true Quotient required, because an *Unit* or 1 doth neither *multiply* nor *divide*.

EXAMPLE 7.

Let it be required to *divide* 57842 by 100. The Work may stand thus, 100) 578,42 the *Quotient* required; or thus 100) 57842 (578$\frac{42}{100}$ the same as before.

Hence it follows, that if any *Divisor* have *Cyphers* to the Right-hand of it, you may cut off so many of the last *Figures* in

in the *Dividend*, and *divide* the other *Figures* of the *Dividend*, by thoſe *Figures* of the *Diviſor* that are left when the *Cyphers* are omitted. But when *Diviſion* is ended, thoſe *Cyphers* ſo omitted in the *Diviſor*, and the *Figures* cut off in the *Dividend*, are both to be reſtored to their own places.

EXAMPLE 8.

Suppoſe it were required to *divide* 675469 by 5400.

5400) 675469 (125

54..

135
108

274
270

Remains (4) But the true *Remainder* is 469. Conſequently the true *Quotient* is $125\frac{469}{5400}$.

As to the manner of proving the Truth of any Operation, either in *Multiplication* or *Diviſion*, I preſume it may be eaſily underſtood, by what is delivered in *Page* 21, compared with the three firſt *Examples* of *Diviſion*; for from thence it will be eaſy to conceive, that if the *Diviſor* and *Quotient* be *multiplied* together, their *Product* (with what *Remains* after *Diviſion* being added to that *Product*) will be equal to the *Dividend*. As in the Fifth *Example*, where the *Dividend* is 379, the *Diviſor* is 5, the *Quotient* is 75, and the *Remainder* is 4.

I ſay, $75\times5=375$, to which *add* the *Remainder* 4, it will be 379.

Again, in the Sixth *Example*, the *Diviſor* is 67, the *Quotient* is 653, and the *Remainder* is 38.

Then $653\times67=43751$, and $43751+38=43789$ the *Dividend*, &c.

There are ſeveral uſeful Contractions, both in Diviſion *and* Multiplication, *which I have purpoſely omitted until I come to treat of* Decimal Arithmetick. Alſo I have omitted the Buſineſs of Evolution or Extracting of *Roots*, until further on; and ſo ſhall conclude this *Chapter* with a few *Examples* of *Diviſion* unwrought at large, leaving them for the Learner's Practice.

579) 43800771 (75649.
Or 75649) 43800771 (579.

45007)

45007) 23884044718 (530674.
Or 530674) 23884044718 (45007.
356) 244572000 (687000.
59600) 57659066400 (967434.
10000) 679543820000 (67954382.
79) 282016 (3569$\frac{5}{79}$.

CHAP. III.

Concerning Addition *and* Subſtraction *of* Numbers *of different* Denominations, *and how to* reduce *them from one* Denomination *to another.*

SECT. 1.

1. *Of* Engliſh *Coin.*

THE leaſt Piece of *Money* uſed in *England* is a *Farthing*, and from thence ariſeth the reſt, as in this *Table.*

Farth.				
4 =	1 *d. Pen.*			
48 =	12 =	1 *s. Shill.*		
960 =	240 =	20 =	1 *l. Pound Sterling.*	

And { 5 *s.* is a *Crown.* / 10 *s.* is an *Angel.* / 6 *s.* 8 *d.* a *Noble.* / 13 *s.* 4 *d.* a *Mark.*

Note, When *l. s. d. q.* are placed over *(or to the Right-hand of) Numbers*, they denote thoſe *Numbers* to ſignify *Pounds*, *Shillings*, *Pence*, and *Farthings.*

l. s. d. q.

As 35 10 6 2. Or 35 *l.* 10 *s.* 6$\frac{1}{2}$ *d.* Either of theſe do ſignify 35 *Pounds*, 10 *Shillings*, 6 *Pence*, 2 *Farthings.*

The ſame muſt be underſtood of all the following *Characters*, belonging to their reſpective *Tables*, *viz.* Of *Weights*, *Meaſures*, &c.

2. *Troy Weight.*

The Original of all *Weights* uſed in *England*, was a *Corn* of *Wheat* gathered out of the *middle* of the *Ear*, and being well dried, 32 of them were to make one *Penny Weight*, 20 *Penny Weights* one *Ounce*, and 12 *Ounces* one *Pound Troy.* *Vide Statutes* of 51 *Hen.* III. 31 *Edw.* I. 12 *Hen.* VII.

But

But in later Times it was thought ſufficient to *divide* the aforeſaid *Penny Weight* into 24 equal *Parts*, called *Grains*, being the leaſt *Weight* now in common Uſe; and from thence the reſt are computed, as in this *Table*.

Gr. Grain.			
24 =	1 *P. W. Penny Weight.*		
480 =	20 =	1 *Oz. Ounce.*	
5760 =	240 =	12 =	1 lb *Pound.*

Note, By *Troy Weight* are weighed *Jewels*, *Gold*, *Silver*, *Corn*, *Bread*, and all *Liquors*.

Beſides the common *Diviſions* of *Troy Weight*, I find in *Angliæ Notitia*, or, *The Preſent State of England*, Printed in the Year 1699, that the *Moneyers (as that Author calls them)* do *ſubdivide* the *Grain*.

Thus
- 24 *Blanks* = 1 *Periot.*
- 20 *Periots* = 1 *Droite.*
- 24 *Droites* = 1 *Mite.*
- 20 *Mites* = 1 *Grain*, &c. as before.

3. *Apothecaries Weights.*

The *Apothecaries divide* a *Pound Troy*, as in this *Table*.

Gr. Grain.				
20 =	1 ℈ *Scruple.*			
60 =	3 =	1 ʒ *Dram.*		
480 =	24 =	8 =	1 ℥ *Ounce.*	
5760 =	288 =	96 =	12 =	1 ℔ *Troy*, the ſame as before.

By theſe *Weights* the *Apothecaries* compound their Medicines: but buy and ſell their Drugs by *Averdupois Weight*.

4. *Averdupois Weight.*

When *Averdupois Weight* became firſt *in Uſe*, or by what *Law* it was firſt ſettled, I cannot find out in the *Statute Books*; but on the contrary, I find that there ſhould be but one *Weight (and one Meaſure)* uſed throughout this *Realm*, *viz.* that of *Troy*, *(Vide* 14 *Ed.* III. and 17 *Ed.* III.) So that it ſeems *(to me)* to be firſt introduced by *Chance*, and ſettled by *Cuſtom*, *viz.* from giving good or large *Weight* to thoſe *Commoditities* uſually weighed by it, which are ſuch as are either very *Courſe* and *Droſſy*, or very

very ſubject to waſte; as all kind of Grocery Wares. And *Pitch*, *Tar*, *Roſin*, *Wax*, *Tallow*, *Flax*, *Hemp*, &c. *Copper*, *Tin*, *Steel*, *Iron*, *Lead*, &c. Alſo *Fleſh*, *Butter*, *Cheeſe*, *Salt*, &c. To theſe and the like (I preſume) it was thought convenient to allow a greater *Weight* than the Laws had provided, which happen'd to be about a Sixth part more: For I found by a very nice Experiment, that one *Pound Averdupois* is equal to 14 *Ounces*, 11 *Penny Weights*, and $15\frac{1}{2}$ *Grains Troy*. And it is now computed as in the following Table.

Drams.				
16 = 1	*Oz. Ounces.*			
256 =	16 = 1	lb *Pounds.*		
28672 =	1792 =	112 = 1	*C. Hundred.*	
573440 =	35840 =	2240 =	20 = 1	*Tun.*

lb

And { 14 = a *Stone*; 28 = $\frac{1}{4}$ *of C.*; 56 = $\frac{1}{2}$ *of C.*; 84 = $\frac{3}{4}$ *of C.*

5. *Long Meaſure.*

As the leaſt part of *Weight* came at firſt from a *Wheat Corn*; ſo (it is generally ſaid) the leaſt part of a *Long Meaſure* was at firſt a *Barley Corn*, taken out of the middle of the Ear, and being well dried, three of them in length were to make one *Inch*; and thence the reſt, as in this Table.

Barley Corns.					
3 = 1	*In. Inches.*				
36 =	12 = 1	*F. Feet.*			
108 =	36 =	3 = 1	*Y. Yards.*		
594 =	198 =	$16\frac{1}{2}$ =	$5\frac{1}{2}$ = 1	*P. Poles.*	
23760 =	7920 =	660 =	220 =	40 = 1	*Furlongs.*
190080 =	63360 =	5280 =	1760 =	320 = 8 = 1	*Mile.*

And { 4 *Nails* = $\frac{1}{4}$ of a *Yard*; $1\frac{1}{4}$ *Yard* = 1 *Ell*; 2 *Yards* = 1 *Fathom*.

Note, That forty *Poles* (or *Perches*) in Length, and four in Breadth, do make a Statute *Acre* of *Land*.

That is, 220 *Yards*, *multiplied* into 22 *Yards* = 4840 Square *Yards* are a Statute *Acre*.

And according to the Tranſactions of the *French Academy*, *Anno* 1687, a *Paris Foot Royal* is = 12,8 *Inches Engliſh*; Six of thoſe Feet make a *Toiſe*; and 57060 *Toiſes* = 365184 *Engliſh Feet*, are the *Meaſure* of one Degree of a great Circle upon the Surface of the Earth. So that one Degree is 69 *Miles* and 288 *Yards*, which is very near to our Countryman Mr *Norwood*'s Experiment made betwixt *London* and *York*, *Anno* 1635; who found that 367196 *Feet* = 69 *Miles*, and 958 *Yards* do make a

Degree. And not 60 *Miles*, according to the common received Opinion and Practice of the *Navigators* or *Seamen*.

Hence, according to the *French* Account, the Circumference of the Earth (ſuppoſing it to be a true *Spherical Figure*) is 24899 *Engliſh Miles*.

6. *Of Liquid Meaſures.*

All *Meaſures* of Capacity, both Liquid and Dry, were at firſt made from *Troy Weight*, *Vide Statutes* 9 *H.* III. 51 *H.* III. 12 *H.* VII. *&c.* wherein it is enacted, that eight *Pound Troy Weight*, of *Wheat*, gathered out of the middle of the Ear, and well dried, ſhould make one *Gallon* of *Wine Meaſure:* And that there ſhould be but one *Meaſure* for *Wine*, *Ale*, and *Corn*, throughout this Realm, (*Vid. Stat.* 14 *Ed.* III. 15 *Rich.* II.) But Time and Cuſtom hath altered *Meaſures*, as they have done *Weights* (and perhaps for one and the ſame Reaſon) for now we have three different *Meaſures*, *viz.* one for *Wine*, one for *Ale* or *Beer*, and one for *Corn*.

I have inſerted *Tables* of each, as they are now computed by *Cubick Inches*, and practiſed in the Art of *Gauging*, &c.

The common *Wine Gallon* ſealed at *Guild-Hall* in *London*; by which all *Wines*, *Brandies*, *Spirits*, *Strong-waters*, *Mead*, *Perry*, *Cyder*, *Vinegar*, *Oil*, and *Honey*, &c. are meaſured and ſold; is ſuppoſed to contain 231 *Cubick Inches*, and from thence the reſt are computed, as in this Table.

Cubick Inches.							
231	= 1	*G. Gallons.*					
9702	= 42	= 1	*Terce.*				
14553	= 63	= 1½	= 1	*Hog ſhead.*			
19404	= 84	= 2	= 1⅓	= 1	*Puncion.*		
29106	= 126	= 3	= 2	= 1½	= 1	*Butt or Pipe.*	
58212	= 252	= 6	= 4	= 3	= 2	= 1	*Tun.*

Note, { 18 = 1 *Rundlet*, and 31½ makes a *Wine* or *Vinegar Barrel*. (*Vide* 1 *R.* III. *Gallons.*

But Dr *Wybard* in his Tectometry, *Page* 289, doth ſuppoſe the *Wine Gallon* to contain but 224, or 225 *Cubick Inches* at the moſt, and purſuant to this Account an Experiment was made by Mr *Richard Walker* and Mr *Philip Shales*, two General Officers in the Exciſe. They cauſed a Veſſel to be very exactly made of Braſs, in Form of a *Parallelopipedon*, each Side of its Baſe was 4 *Inches*, and its Depth 14 *Inches*; ſo that its juſt Content was 224 *Cubick Inches*. This Veſſel was produced at *Guild-Hall* in *London* (*May* 25, 1688) before the *Lord-Mayor*, the *Commiſſioners* of *Exciſe*, the Reverend Mr *Flamſtead*, Aſtr. Reg. Mr

Mr *Halley*, and ſeveral other ingenious Gentlemen, in whoſe Preſence Mr *Shales* did exactly fill the aforeſaid Brazen Veſſel with clear Water, and very carefully emptied it into the old Standard *Wine Gallon* kept in *Guild-Hall*, which did ſo exactly fill it, that all then preſent were fully ſatisfied the *Wine Gallon* doth contain but 224 *Cubick Inches. (This notable Experiment I ſaw tried.)* However, for ſeveral Reaſons, it was at that time thought convenient to continue the former ſuppoſed Content of 231 *Cubick Inches* to be the *Wine Gallon*, and that all Computations in *Gauging* ſhould be made from thence, as above.

The *Beer* or *Ale Gallon* (which are both one) is much larger than the *Wine Gallon*; it being (as I preſume) made at firſt to correſpond with *Averdupois Weight*, as the *Wine Gallon* did with *Troy Weight*: For (as I ſaid before, *Page* 33.) one *Pound Averdupois* is equal to 14 *Ounces* 12 *Penny Weights Troy*, very near.

And, as one *Pound Troy* is in proportion to the *Cubick Inches* in a *Wine Gallon*, ſo is one *Pound Averdupois* to the *Cubick Inches* in an *Ale Gallon*. That is, $12 : 231 :: 14\frac{12}{20} : 281\frac{1}{2}$, very near the *Cubick Inches* contained in an *Ale Gallon*, as appears from an Experiment made by one *Nicholas Gunton*, General *Gauger* in the *Exciſe*, about 41 Years ago, who, by ſuch a Veſſel mentioned before in the laſt *Page*, did find the Standard *Ale-Quart* (kept in the *Exchequer*, *Vid.* 12 *Car.* II) to contain juſt $70\frac{1}{2}$ *Cubick Inches*, conſequently the *Ale Gallon* muſt contain 282 *Cubick Inches*, and from thence the following *Tables* are computed.

Ale-Meaſure.

Cubick Inches.						
282	= 1	*Gallon.*				
2256	= 8	= 1	*Firkin.*			
4512	= 16	= 2	= 1	*Kilderkin.*		
9024	= 32	= 4	= 2	= 1	*Barrel.*	
13536	= 48	= 6	= 3	= $1\frac{1}{2}$	= 1	*Hogſhead.*

Note, { *A Firkin of Soap and of Herrings are the ſame with that of Ale.*

Beer-Meaſure.

Cub. Inches.						
282	= 1	*Gallon.*				
2538	= 9	= 1	*Firkin.*			
5076	= 18	= 2	= 1	*Kilderkin.*		
10152	= 36	= 4	= 2	= 1	*Barrel.*	
15228	= 54	= 6	= 3	= $1\frac{1}{2}$	= 1	*Hogſhead.*

N. B. This Diſtinction or Difference betwixt *Ale* and *Beer-Meaſure*, is now only uſed in *London*. But in all other Places of *England* the following *Table* of *Beer* or *Ale*, whether it be ſtrong or ſmall, is to be obſerved, according to a Statute of *Exciſe* made in the Year 1689.

Cub. Inches.	
282 = 1	*Gallon.*
2397 = 8½ = 1	*Firkin.*
4794 = 17 = 2 = 1	*Kilderkin.*
9588 = 34 = 4 = 2 = 1	*Barrel.*
14382 = 51 = 6 = 3 = 1½ = 1	*Hog ſhead.*

7. Of *Dry Meaſure*.

Dry Meaſure is different both from *Wine* and *Ale Meaſure*, being as it were a Mean betwixt both, tho' not exactly ſo; which upon Examination I find to be in proportion to the aforeſaid old Standard *Wine Gallon*, as *Averdupois Weight* is to *Troy Weight*; That is, As one *Pound Troy* is to one *Pound Averdupois*, ſo is the *Cubick Inches* contained in the old *Wine Gallon:* To the *Cubick Inches* contained in the *Dry* or *Corn Gallon*.

Viz. 12 : 14$\frac{12}{25}$ :: 224 : 272½, which is very near to 272¼, the common received Content of a *Corn Gallon:* Altho' now it is otherwiſe ſettled by an Act of Parliament made in *April* 1697, the Words of that Act are theſe:

Every round Buſhel *with a plain and even Bottom, being made eighteen* Inches *and a half wide throughout, and eight* Inches *deep, ſhould be eſteemed a Legal* Wincheſter Buſhel, *according to the Standard in his* Majeſty's Exchequer.

Now a Veſſel being thus made will contain 2150,42 *Cubick Inches*, conſequently the *Corn Gallon* doth contain but 268$\frac{4}{5}$ *Cubick Inches*.

Cub. Inches.	
268,8 = 1	*Gallon.*
537,6 = 2 = 1	*Peck.*
2150,4 = 8 = 4 = 1	*Buſhel.*
17203,2 = 64 = 32 = 8 = 1	*Quarter.*

Note, { 4 *Buſhels* = a *Comb.* / 10 *Quarters* = a *Wey*, and / 12 *Weys* = a *Laſt of Corn.* }

I obſerv'd amongſt the Lead-Mines in *Derbyſhire*, (*Anno* 1692) that the *Miners* bought and ſold their Lead Ore, by a *Meaſure* which they call'd an Ore Diſh; whoſe Dimenſions I carefully took, and found it

Thus { Length 21.3. / Breadth 6. / Depth 8.4. } *Inches.*

Conſe-

Consequently its Content is 1073,52 *Cubick Inches*, which is very near equal to 4 *Corn Gallons*, according to the abovementioned Settlement.

Nine of those Dishes they call a Load of Ore, which if it be pretty good, will produce about 3 hundred Weight of Lead.

8. *Of* Time.

It is not an easy Thing to give a true *Definition* of *Time*; for (according to the *Philosophick Poet*)

> Time *of itself is nothing, but from Thought*
> *Receives its Rise, by labouring Fancy wrought*
> *From Things consider'd, whilst we think on some*
> *As present, some as past, or yet to come.*
> *No Thought can think on* Time, *that's still confest,*
> *But thinks on Things in Motion or at Rest.*
>
> And so on, Vide *Lucretius*, Book I.

That is, *Time* only shews the *Duration* or *Mutation* of Things, a Year being the *Standard* or *Integer*, by which such Continuance or Change is computed. And a *Year* is that *Space* of *Time* in which the *Sun* (apparently) compleats its *Revolution* from any one *Point* in the *Ecliptick* (an imaginary *Circle* in the *Heavens*) to the same *Point* again, which, according to modern *Observations*, is perform'd in 365 *Days*, 5 *Hours*, 48 *Minutes*, 57 *Seconds*, 21 *Thirds*, &c. But a *Second* being the least part of *Time* that can be truly *measured* by the *Motion* of any *Mechanical Engine*, as a *Clock*, &c. (a *Third* being less than the Twinkling of an Eye) I begin the following Table with *Seconds*.

Seconds.″							
60 =	1′ *Minute.*						
3600 =	60 =	1° *Hour.*					
86400 =	1440 =	24 =	1 *Day.*	°	′	″	
31556937 =	525949 =	8765 =	365 +	5 +	48 +	57 =	1 *Year, called (a Solar Year.*

But the common *Year*, usually call'd the *Julian Year*, doth consist of 365 *Days* and 6 *Hours*, and is divided into twelve unequal *Months*, called *Calendar Months*, whose Names and *Number* of *Days* are the Subject of every *Almanack*.

To

To these *Tables* it may not be amiss to give a brief Account of such *Coins*, *Weights*, and *Measures*, as are frequently mentioned in the Scriptures. As I have deduced them from those which seem to be the most Correct, inserted in the *Index* to the large *Bible*, Printed *Anno* 1702, and compared with those used in *England*, by the Lord Bishop of *Peterborough*.

The *Hebrew Weights*, compared with	*Troy Weight.* Oz.	Pw.	Grains.
A Gerah=	0 .	0 .	$10\frac{12}{20}$
10 *Gerahs*=*a Bekah*=	0 .	4 .	$13\frac{1}{2}$
2 *Bekahs*=*a Shekel*=	0 .	9 .	3
100 *Shekels*=*a Menah*=	45 .	12 .	12

Note, A *Shekel* is said to be their Original *Weight*.

Their *Coin*	*English Coin.* *l.*	*s.*	*d.*	
A *Silver Menah*=	7 .	1 .	$5\frac{1}{4}$	Weight 60 *Shekels*.
Talent of *Silver*=	357 .	11 .	$10\frac{1}{2}$	Weight is 300 *Shekels*.
Talent of *Gold*=	5075 .	15 .	$7\frac{1}{2}$	The same Weight men-
The *Gold Dram*=	1 .	0 .	4	tioned *Ez.* ii. 19.

The *Roman Money* mentioned in the *New-Testament*.

A *Denarius*, or *Silver Penny*=	7 *d.*	3 *Farthings*.	
Asses of *Copper*=	0 .	3 *Farthings*.	
Assarium=	0 .	$1\frac{1}{2}$ *Farthing*.	
Quadrans=	0 .	$\frac{3}{4}$ of a *Farthing*.	
A *Mite*=	0 .	$\frac{1}{3}$ of a *Farthing*.	

Their *Long Measure*, compared with	*English Measure.* Yar.	Feet.	In. Pts.
A Finger's Breadth=	0.	0 .	0,912
4 *Fingers*=*a Hand's Breadth*=	0.	0 .	3,648
2 *Hands*=*the least Span*=	0.	0 .	7,296
3 *Hands Breadth*=*the longest Span*=	0.	0 .	10,944
2 *Spans*=*the longest Cubit*=	0.	1 .	9,888
4 *Cubits*=*a Fathom*=	2.	1 .	3,552
6 *Cubits*=*Ezekiel's Reed*=	3.	1 .	11,328
400 *Cubits*=*a Stadium*=	243.	0 .	7,2
10 *Stadiums*=*a Mile*=	2432.	0 .	0
3 *Miles*=*a Parasang*=	7296.	0 .	0
Which is 4 *English Miles* and	256.		

Their

Their *Meaſures* of *Capacity*, compared with { *Engliſh Wine.* Gal. Pints. Inch.

A Cotyla =	0 .	0½ .	3,037
A Log =	0 .	0½ .	9,83
4 *Logs* = *a Cab* =	0 .	3 .	10,458
10 *Catyla's* = *an Omer* =	0 .	6 .	1,5
3 *Cabs* = *a Hin* =	1 .	2 .	2,5
2 *Hins* = *a Seah* =	2 .	4 .	5,
3 *Seahs* = *an Epha* =	7 .	4 .	15,
10 *Epha's* = *a Chomer* =	75 .	5 .	5,625

Sect. 2. Addition of *Weights*, &c.

The foregoing *Tables* being ſo well underſtood, as that you can readily tell (without pauſing) how many *Units* of any one *Denomination*, do make one of the next *Superior Denomination (eſpecially in thoſe* Tables *which are moſt uſeful for your Buſineſs)* it will then be as eaſy to *add* or *ſubſtract* them, as to *add* or *ſubſtract whole Numbers*, due Care being taken in placing all *Numbers* that are of one *Denomination* exactly underneath each other. That is to ſay, in *Money*, place *Pounds* under *Pounds*, *Shillings* under *Shillings*, *Pence* under *Pence*, &c. Underſtand the like in *Weights* and *Meaſures*, &c. according to their ſeveral *Denominations:* Then in *Addition* obſerve this *Rule.*

RULE.

Always begin with thoſe Figures *of the loweſt or leaſt* Denomination, *and* add *them all together into one* Sum, *then conſider how many of the next* Superior Denomination *are contained in that* Sum, *ſo many Units you muſt carry to the ſaid next* Superior Denomination *to be* added *together with thoſe* Figures *that ſtand there; and if any thing remain over or above thoſe* Units *ſo carried, that* Overplus *muſt be ſet down underneath its own* Denomination: *And ſo proceed on from one* Denomination *to another until all be finiſhed.*

Example in Coin.

Let it be required to *add* 35 *l.* 14 *s.* 06 *d.* and 27 *l.* 02 *s.* 10 *d.* and 54 *l.* 13 *s.* 04 *d.* and 10 *l.* 17 *s.* 09 *d.* into one *Sum.*

The particular *Sums* being placed, as before directed, will ſtand as in the Margin following.

Then according to the *Rule*, I begin with the *Pence* (being here the loweſt or leaſt *Denomination)* and *adding* them all together, I find their *Sum* to be 29 *d.* that is 2 *s.* and 5 *d.* (for

24=2 *s.* and 29—24=5) the 5 *d.* I ſet down underneath its own *Denomination*, and carry the 2 *s.* to the Place of *Shillings*, *adding* them and all the *Shillings* together, I find the *Sum* to be 48 *s. viz.* 2 *l.* 8 *s.* I ſet down the 8 *s.* underneath its own place of *Shillings*, and carry the 2 *l.* to the Place of *Pounds*, *adding* them and all the *Pounds* together, I find their *Sum* is 128 *l.* conſequently the *Total Sum* required is 128 *l.* 08 *s.* 05 *d.*

l.	*s.*	*d.*
35	14	06
27	02	10
54	13	04
10	17	09
128	08	05

Now, for as much as it often happens in keeping Books of *Accounts*, (and in other Buſineſs) that it is required to *add* up large *Sums* of Money, conſiſting of 30, 40, or more ſeveral particular *Sums*, nay, perhaps filling up the whole length of a Sheet of Paper, I humbly conceive in thoſe Caſes the beſt and eaſieſt way will be to part them into Parcels, not exceeding above 10 or 12 particular *Sums* in each Parcel; that done, add together all the *Sums* of thoſe Parcels into one *Sum*, and that will be the *Total Sum* required.

Alſo to avoid the making of *Points*, or other *Marks* amongſt your *Figures*, it will be convenient to get the following *Tables* by heart.

The Pence Table.

d.	*s.*	*d.*	*s.*
12	1	72	6
24	2	84	7
36	3	96	8
48	4	108	9
60	5	120	10

The Shillings Table.

s.	*l.*	*s.*	*l.*
20	1	120	6
40	2	140	7
60	3	160	8
80	4	180	9
100	5	200	10

The Uſe of theſe *Tables* is ſo obvious, that I preſume it is needleſs to explain them.

Examples in Addition *of* Weights.

Troy Weight.

lb.	*Oz.*	*Pw.*	*Gr.*
3	09	00	10
5	08	15	21
10	10	12	22
0	11	19	23
Sum 21	04	09	04

Averdupois Weight.

Tun.	*C.*	*Q.*	lb.	*Oz.*
12	15	2	24	12
7	10	3	21	15
0	18	1	14	11
1	19	3	27	15
Sum 23	05	0	05	05

Examples

Examples in *Addition* of *Long-Meaſure.*

Yards	*Qrs.*	*Nails*	*Miles*	*Fur.*	*Poles*	*Yards*	*Feet*	*Inch.*
35	2	3	2	6	32	4	2	9
17	3	1	0	7	27	3	1	10
129	1	2	1	3	39	1	2	11
182	3	2 Sum	5	2	19	5	0	0

I think it needleſs to ſet down more *Examples* of this kind, for if theſe 5 (eſpecially the laſt) be well underſtood, they will be ſufficient to ſhew how any other may be performed.

Sect. 3. Subſtraction *of Weights*, &c.

Subſtraction is but the Converſe of the precedent Work, and may be performed by obſerving this *Rule.*

RULE.

Begin with the Loweſt or Leaſt Denomination *(as before in* Addition) *and Take or* Subſtract *the* Figure *(or* Figures) *in that place of the* Subtrahend, *from the* Figure *(or* Figures) *that ſtand over them of the ſame* Denomination; *ſetting down the* Remainder. *(as in* Page 12.) *But if that cannot be done, then you muſt increaſe the upper* Figure *(or* Figures) *with one of the next Superior* Denomination, *and from that* Sum *make* Subſtraction; *and ſo proceed to the next Superior* Denomination, *where you muſt pay the one borrowed, by adding* Unity *to the* Subtrahend *in that place,* &c. *as in whole* Numbers.

Examples in Coin.

	l.	*s.*	*d.*
From	386	09	08
Take	173	04	06
Remains	213	05	02

	l.	*s.*	*d.*
From	569	10	06
Subſt.	389	15	08
	179	14	10

The Firſt of theſe *Examples* is ſelf-evident. In the Second *Example*, beginning at the place of *Pence* (being here the Leaſt *Denomination)* I am to take 8 *d.* from 6 *d.* but becauſe that cannot be done, I muſt (according to the *Rule)* borrow one of the next *Denomination*, viz. 1 *s.* and *add* it to the 6 *d.* which makes it 18 *d.* (for 1 *s.*=12 *d.* and 12 *d.*+6 *d.*=18 *d.* then I take 8 *d.* from that 18 *d.* and there remains 10 *d.* to be ſet down underneath the place of *Pence*; that done, I proceed to the place of *Shillings*, where I muſt now pay the 1 *s.* ſaying one borrowed and 15 makes 16 from 10 cannot be, but

16 from 30 and there *remains* 14. That is, I borrow one of the next *Denomination*, viz, 1 *l.* and add to it the 10 *s.* which makes it 30 *s.* for 1 *l.*=20 *s.* and 20 *s.*+10=30) having ſet down the *Remaining* 14 *s.* underneath its own place of *Shillings*, I proceed to the place of *Pounds*, where paying the 1 *l.* borrowed, it will be 1 borrowed and 9 is 10 from 9 cannot be, but 10 from 19 and there *Remains* 9, and ſo on as in whole *Numbers* until all be finiſhed; and the *Remainder* will be 179 *l.* 14 *s.* 10 *d.*

This *Example* being a little conſidered will render all others in this *Rule* eaſy.

Examples in Weights.

	Troy Weight.				*Averdupois Weight.*			
	lb.	*oz.*	*pwt.*	*gr.*	*c.*	*qr.*	*lb.*	*oz.*
From	9	10	16	18	17	2	15	10
Take	5	09	18	22	14	3	18	12
Reſts	4	00	17	20	2	2	24	14

Examples in Long Meaſure.

	yds.	*qrs.*	*nails*	*miles*	*fur.*	*pol.*	*yds.*	*feet*	*inches*
From	78	3	2	22	3	26	3½	0	9
Take	29	3	3	18	6	29	4	2	11
Reſts	48	3	3	3	4	36	4	0	10

Example in Time.

	days	♁	′	″
From	27	18	35	21
Subſtract	16	21	46	36
Remains	10	20	48	45

The Proof of *Addition* and *Subſtraction* in theſe *Numbers* of different *Denominations*, is the very ſame with that of whole *Numbers* in *Page* 13. I ſhall therefore refer you to that place, and omit repeating it here.

Sect. 4. *Of* Reduction.

BY *Reduction*, *Numbers* of different *Denominations* are brought into one *Denomination*.

That is, it alters or changes any Superior *Denomination* propoſed, into any Inferior or Leſſer *Denomination* Required; ſtill

ſtill keeping them equivalent in value. And by that means they become fitly prepared for *Multiplication* and *Diviſion*; which otherwiſe could not ſo conveniently be performed. Therefore the Buſineſs of *Reduction* is very uſeful in the *Rule of Proportion*, (commonly called the *Golden Rule*, or *Rule of Three*) eſpecially to thoſe who do not underſtand either *Vulgar* or *Decimal Fractions*. And it is thus performed:

RULE.

Conſider how many Units *of the* Denomination *Required, make one of that* Denomination *propoſed to be Reduced (which is eaſily known by its reſpective Table) and with that* Number *of* Units, Multiply *the* Denomination *propoſed, and their* Product *will be the* Number *Required.*

Example in Coin.

Let it be Required to Reduce or Change 357 *l.* into *Shillings*, and thoſe *Shillings* into *Pence*, which ſhall ſtill be equal in value with the 357 *l.*

357
Multiply with 20 the *Shillings* in one *Pound.*

7140=the *Shillings* in 357 *l.*
Multiply with 12 the *Pence* in one *Shilling.*

1428
714

85680=the *Pence* in 357 *l.* as was Required.

Or 357 *l.* may be reduced into *Pence*, at one Operation; Thus,

357 *l.*
Multiply with 240 the *Pence* contained in one *Pound.*

1428
714

85680=the *Pence* in 357 *l.* as before.

But when the *Numbers* propoſed to be Reduced are of ſeveral *Denominations*, and it is required to bring them all to the Loweſt; you muſt Reduce the higheſt or greateſt *Denomination* to the next leſs, Adding the *Numbers* that are of that leſs *Denomination* together, then *Reduce* their *Sum* to the next lower *Denomination*, Adding together all the *Numbers* that are of that *Denomination*, and ſo proceed gradually on till all is done.

EXAMPLE.

Let it be required to Reduce 375 *l.* 17 *s.* 10 *d.* 3 *q.* into one *Denomination*, viz. into *Farthings*.

375 *l.* 17 *s.* 10 *d.* 3 *q.*
20

7500=the *Shillings* in 375 *l.*
+ 17 *s.*

7517=the *Shillings* in 375 *l.* 17 *s.*
12

15034
7517

90204=the *Pence* in 375 *l.* 17 *s.*
+ 10 *d.*

90214=the *Pence* in 375 *l.* 17 *s.* 10 *d.*
4

360856=the *Farthings* in 375 *l.* 17 *s.* 10 *d.*
+ 3 *q.*

360859 *Farth*=375 *l.* 17 *s.* 10 *d.* 3 *q.* as was required.

The Work of this *Example*, and all other Operations of this kind, may be ſomewhat ſhortened by obſerving the following Method.

375 *l.* 17 *s.* 10 *d.* 3 *q.*
20 *Multiply* and *Add* in the 17 *s.*

7517
12 *Multiply* and *Add* in the 10 *d.*

15044
7517

90214
4 *Multiply* and *Add* in the 3 *qrs.*

360859 the *Farthings* as before.

Example in Troy Weight.

Suppoſe it be Required to *Reduce* 29 *lb.* 8 *oz.* 18 *pwt.* 21 *gr.* into the leaſt *Denomination*, viz. into *Grains*.

Thus

Thus 29 *lb.* 8 *oz.* 18 *pwt.* 21 *gr.*
Multiply with 12 the *oz.* in 1 *lb.* and *add* in the 8 *oz.*

66
29

356 = the *oz.* in 29 *lb.* 8 *oz.*
Multiply with 20 the *pwts* in 1 *oz.* and *add* in the 18 *pwt.*

7138 = the *pwts* in 29 *lb.* 8 *oz.* 18 *pwt.*
Multiply with 24 the *grs* in 1 *pwt.* and *add* in the 21 *grs.*

28553
14278

171333 the *grs* = 29 *lb.* 8 *oz.* 18 *pwts.* 21 *grs.*

These two Examples at large being well underſtood, may ſuffice to ſhew how all Operations of this kind are performed; either in *Weights*, *Meaſures*, or *Time*. I ſhall only inſert a few Examples of each ſort for the Learner's Practice.

1. In 23 *C.* 3 *qrs.* 21 *lb.* 9 *oz.* Averdupois Weight; How many *Ounces?* Anſw. 42905 *Ounces.*

2. In 252 *Eng. Miles*, How many *Yards*, *Feet*, and *Inches?* Anſw. 443520 *yds.* = 1330560 *feet* = 15966720 *inches.*

3. In 1692 common *Years*, How many *Days*, *Hours*, and *Minutes?* Anſw. 618003 *days*, 14832072 *hours*, 889924320 *minutes.*

Note, a common *Year* = 365 *Days*, 6 *Hours*, ſee *Page* 37.

4. In 5786 *Pounds*, 17 *Shillings*, 9 *Pence*, Sterling; How many *Shillings*, *Pence*, and *Farthings?* Anſw. 115737 *s.* 1388853 *d.* or 5555412 *farthings.* That is, 5786 *l.* 17 *s.* 9 *d.* = 115737 *s.* 9 *d.* = 1388853 *d.* &c.

The next thing will be to ſhew how to bring Numbers from a leſſer to a greater Denomination, which by moſt Authors is called (tho' very improperly)

Reduction *aſcending.*

This Work is the Converſe of the laſt, and is performed by *Diviſion.* Thus,

RULE.

Conſider how many of the Denomination *propoſed make one of the* Denomination *required, and make that* Number *your* Diviſor, *by which divide the* Denomination *propoſed; and the* Quotient *will be the* Number *required.*

EXAMPLE.

EXAMPLE.

Let it be required to find how many *Shillings* and *Pounds* are contained in 85680 *Pence*.

The *Pence* in 1 *s.* are 12) 85680 (7140 *s.*=85680 *d.*

Again the *Shillings* in 1 *l.* are 20) 7140 (357 *l.* the Anſwer required.

Another Example in Coin.

How many *Pence*, *Shillings*, and *Pounds*, are contained in 264859 *Farthings*.

```
                12)        20)
4) 264859 (66214 d. (5517 s. (275 l.
   24        62        151
    08        21        117
     05        94       (17) s.
      19      (10) d.
```

Remains (3) *q.* { *Note*, the *Remainder* is always of the ſame Denomination with the *Dividend*.

The laſt *Quotient* 275 *l.* together with the ſeveral *Remainders*, give the Anſwer required.

Viz. 275 *l.* 17 *s.* 10 *d.* 3 *q.* =264859 *Farthings*.

Example in Troy Weight.

Suppoſe it were required to find how many *Pwts. Ozs.* and *lbs.* are contained in 171333 *Grains*.

```
                20)          12)
24) 171333 gr. (7138 pw.  (356 (29 lb.
    168...      113          24
      33        138         116
      24        ---         108
      ---      (18) pws.    ---
       93                   (8) oz.
       72
      ---
      213
      192
      ---
Remains (21) grs.
```

Anſw. 29 *lb.* 8 *oz.* 18 *pwt.* 21 *grs.* This and the laſt Example are the Reverſe or Proof of thoſe in Pages 43, 45.

1. In 42905 *Ounces*, *Averdupois Weight*; How many *Pounds*, &c.

Thus

Thus 16) 42905 (2681 *lb.* 28) (95 *qrs.* 4) (23 *C.*

109 252 15

130 161 (3)

25 140

(9) (21) Anſw. 23 *C.* 3 *qrs.* 21 *lb.* 9 *oz.*

2. In 15966720 *Inches*; How many *Engliſh* Miles, &c.

Anſw. 252 *Miles*, &c. as Occaſion requires.

There are many uſeful Queſtions may be anſwered by the help of *Reduction* only: As the changing of one ſort of Coin for another; and comparing one ſort of Meaſure with another, *&c.*

For Inſtance: Suppoſe one had 347 *Rixdollars*, at 4 *s.* 6 *d. per* Dollar; and deſired to know how many *Pounds Sterling* they make,

347

54=the Pence in one Dollar, *viz.* 4 *s.* 6 *d.*=54 *d.*

1388

1735

12) 18738 *d.* 20) (1561 *s.* (78 *l.*

67 161

73 (1) *s.*

18

(6) *d.*

Anſw. 78 *l.* 1 *s.* 6 *d. Sterl.* are=347 *Rixdollars.*

Queſt 2. In 645 *Flemiſh* Ells; How many Ells *Engliſh?*

Note, 3 *Quarters of a Yard Engliſh make one Ell Flemiſh, and* $1\frac{1}{4}$, *or* 5 *Quarters of a Yard, is an Engliſh Ell.*

Therefore, 645

3 = the *qrs* of a Yard in 1 Ell *Flemiſh.*

qrs in 1 Ell=5) 1935 (387 *Engliſh* Ells for the Anſwer.

Queſt. 3. Suppoſe a Bill of Exchange were accepted at *London*, for the Payment of 400 *l. Sterl.* for the Value deliver'd at *Amſterdam* in *Flemiſh* Money at 1 *l.* 13 *s.* 6 *d.* for 1 Pound *Sterl.* How much *Flemiſh* Money was deliver'd at *Amſterdam?*

Firſt, 1 *l.* 13 *s.* 6 *d.* = 402 *d.* the Value of one Pound *Sterl.* at *Amſterdam.*

Then, 402 *d.* × 400= 160800 *d.* = 670 *l. Flemiſh*, and ſo much was delivered at *Amſterdam.*

CHAP.

CHAP. IV.

Of Vulgar Fractions.

Sect. I. Of Notation.

A *Fraction*, or *Broken Number*, is that which repreſents a *Part* or *Parts* of any thing propoſed, (*vide* Page 3.) and is expreſſed by two Numbers placed one above the other with a Line drawn betwixt them:

Thus, $\left\{\begin{matrix}3 & \textit{Numerator.}\\ \overline{4} & \textit{Denominator.}\end{matrix}\right.$

The Denominator, or Number placed underneath the Line, denotes how many equal Parts the thing is ſuppoſed to be divided into (being only the Diviſor in Diviſion.) And the Numerator, or Number placed above the Line, ſhews how many of thoſe Parts are contained in the Fraction (it being the Remainder after Diviſion). (*See* Page 29.) And theſe admit of three Diſtinctions:

Viz. $\left\{\begin{matrix}\textit{Proper} \text{ or } \textit{Simple}\\ \textit{Improper}\\ \textit{Compound}\end{matrix}\right\}$ *Fractions.*

A proper, pure, or *Simple Fraction*, is that which is leſs than an Unit. That is, it repreſents the immediate Part or Parts of any thing leſs than the whole, and therefore it's Numerator is always leſs than the Denominator.

As $\left\{\begin{matrix}\frac{1}{4} \text{ is one } \textit{Fourth Part.}\\ \frac{1}{3} \text{ is one } \textit{Third Part.}\end{matrix}\right.$ $\left\{\begin{matrix}\frac{1}{2} \text{ is one } \textit{Half.}\\ \frac{2}{3} \text{ is two } \textit{Thirds}, \text{ \&c.}\end{matrix}\right.$

An *Improper Fraction* is that which is greater than an Unit. That is, it repreſents ſome Number of Parts greater than the whole thing; and it's Numerator is always greater than the Denominator.

As $\frac{5}{3}$ or $\frac{9}{7}$ or $\frac{41}{15}$ &c.

A *Compound Fraction* is a Part of a Part, conſiſting of ſeveral Numerators and Denominators connected together with the Word [of].

As $\frac{1}{3}$ of $\frac{3}{4}$ of $\frac{2}{5}$, *&c.* and are thus read, The *one Third* of the *three Fourths* of the *two Fifths* of an Unit.

That is, when an Unit (or whole thing) is firſt divided into any Number of equal Parts, and each of thoſe Parts are ſubdivided

ſubdivided into other Parts, and ſo on: Then thoſe laſt Parts are called *Compound Fractions*, or *Fractions of Fractions*.

As for inſtance, ſuppoſe a Pound Sterling (or 20 *s.*) be the Unit or Whole; then is 8 *s.* the $\frac{2}{5}$ of it, and 6 *s.* the $\frac{3}{4}$ of thoſe two Fifths, and 2 *s.* is the $\frac{1}{3}$ of thoſe three Fourths; *viz.* 2 *s.*$=$ $\frac{1}{2}$ of $\frac{3}{4}$ of $\frac{2}{5}$ of one Pound Sterling.

All *Compound Fractions* are reduced into Single ones, Thus,

RULE.

Multiply all the Numerators *into one another for a* Numerator, *and all the* Denominators *into one another for the* Denominator.

Thus the $\frac{1}{3}$ of $\frac{3}{4}$ of $\frac{2}{5}$ will become $\frac{6}{60}$. Or $\frac{1}{10}$.

For $1 \times 3 \times 2 = 6$ the Numerator, and $3 \times 4 \times 5 = 60$ the Denominator, but $\frac{6}{60}$ or $\frac{1}{10}$ of a *l.* Sterl. is 2 *s.* As above.

Sect. 2. *To* Alter *or* Change *different* Fractions *into one* Denomination *retaining the ſame Value.*

IN order to gain a clear Underſtanding of this Section, it will be convenient to premiſe this Propoſition, *viz.* If a Number multiplying two Numbers produce other Numbers, the Numbers produced of them ſhall be in the ſame Proportion that the Numbers multiplied are, 17 *Euclid* 7.

That is to ſay, If both the Numerator and Denominator of any *Fraction* be equally multiplied into any Number, their Products will retain the ſame Value with that Fraction.

As in theſe, $\frac{2 \times 2}{3 \times 2} = \frac{4}{6}$. Or $\frac{2 \times 3}{3 \times 3} = \frac{6}{9}$. Or $\frac{2 \times 5}{3 \times 5} = \frac{10}{15}$, *&c.*

That is, $\frac{2}{3}$ and $\frac{4}{6}$. Or $\frac{2}{3}$ and $\frac{6}{9}$. Or $\frac{2}{3}$ and $\frac{10}{15}$ are of the ſame Value, in reſpect to the Whole or Unit.

From hence it will be eaſy to conceive, how two or more Fractions that are of different Denominations, may be altered or changed into others that ſhall have one common Denominator, and ſtill retain the ſame Value.

Example. Let it be required to change $\frac{2}{3}$ and $\frac{3}{7}$ into two other Fractions that ſhall have one common Denominator, and yet retain the ſame Value.

According to the foregoing Propoſition, if $\frac{2}{3}$ be equally multiplied with 7, it will become $\frac{14}{21}$, *viz.* $\frac{2 \times 7}{3 \times 7} = \frac{14}{21}$. Again, if $\frac{3}{7}$ be equally multiplied with 3, it will become $\frac{9}{21}$, *viz.* $\frac{3 \times 3}{7 \times 3} = \frac{9}{21}$.

And by this means I have obtained two new Fractions, $\frac{12}{20}$ and $\frac{21}{28}$, that are of one Denomination, and of the ſame Value with the two firſt propoſed, *viz.* $\frac{12}{20} = \frac{3}{5}$ and $\frac{21}{28} = \frac{3}{7}$.

And from hence doth ariſe the general Rule for bringing all Fractions into one Denomination.

RULE.

Multiply all the Denominators *into each other for a new (and common)* Denominator. *And each* Numerator *into all the* Denominators *but it's own, for new* Numerators.

Example. Let the propoſed Fractions be $\frac{1}{3}$, $\frac{2}{5}$, $\frac{3}{4}$, and $\frac{6}{7}$.

Then, by the Rule,

A new Denominator will be thus fonnd.	And the new Numerators will be thus found.			
3	1.	2.	3.	6
5	5	3	3	3
15	5	6	9	18
4	4	4	5	5
60	20	24	45	90
7	7	7	7	4
420	140.	168.	315.	360

Hence 420 is the common *Denominator*; and 140. 168. 315. 360, are the new *Numerators*, which being placed Fraction-wiſe are $\frac{140}{420}$. $\frac{168}{420}$. $\frac{315}{420}$. $\frac{360}{420}$ the New Fractions required.

That is, $\frac{140}{420} = \frac{1}{3}$. $\frac{168}{420} = \frac{2}{5}$. $\frac{315}{420} = \frac{3}{4}$. and $\frac{360}{420} = \frac{6}{7}$.

Sect. 3. *To bring mix'd* Numbers *into* Fractions, *and the contrary.*

MIX'D *Numbers* are brought into improper *Fractions* by the following Rule.

RULE.

Multiply the Integers, *or whole Numbers, with the* Denominator *of the given Fraction, and to their Product add the* Numerator, *the Sum will be the Numerator of the Fraction required.*

Example. $9\frac{4}{5}$ by the Rule will become $\frac{49}{5}$. For $9 \times 5 = \frac{45}{5}$.

And, $\frac{45}{5} + \frac{4}{5} = \frac{49}{5}$ the improper Fraction required.

Again, $13\frac{14}{15}$ will become $\frac{209}{15}$. For $13 \times 15 = \frac{195}{15}$.

And $\frac{195}{15} + \frac{14}{15} = \frac{209}{15}$. And ſo for any other as Occaſion requires.

To find the true Value of any improper Fraction given is only the Converſe of this Rule. For if $\frac{49}{5} = 9\frac{4}{5}$, as before is evident:

Then

Then it follows that if 49 be divided by 5, the Quotient will give $9\frac{4}{5}$. And if 206 be divided by 15, it will give $13\frac{11}{15}$, *&c.* conſequently it follows, that

If the Numerator of any improper Fraction be divided by it's Denominator, the Quotient will diſcover the true Value of that Fraction.

EXAMPLES.

$\frac{35}{7}=5$. And $\frac{41}{9}=4\frac{5}{9}$. And $\frac{121}{20}=6\frac{1}{20}$. Or $\frac{15}{4}=3\frac{3}{4}$, *&c.*

When whole Numbers are to be expreſſed Fraction-wiſe, it is but giving them an Unit for a Denominator. Thus 45 is $\frac{45}{1}$, 9 is $\frac{9}{1}$, and 25 is $\frac{25}{1}$, *&c.*

Sect. 4. *To* Abbreviate *or* Reduce Fractions *into their Loweſt or Leaſt Denomination.*

THIS is done, not out of any neceſſity, but for the more convenient managing of ſuch *Fractions* as are either propoſed in large Terms; or ſwell into ſuch either by *Addition* or otherwiſe: beſides it is moſt like an Artiſt to expreſs or ſet down all *Fractions* in the loweſt Terms poſſible; and to perform that, it will be neceſſary to conſider theſe following Propoſitions.

Numbers are either Prime or Compoſed.

1. A Prime Number is that which can only be meaſured by an Unit. *Euclid* 7. *Defin.* 11.

That is, 3, 5, 7, 11, 13, 17, *&c.* are ſaid to be Prime Numbers, becauſe it is not poſſible to divide them into equal Parts by any other Number but Unity or 1.

2. Numbers Prime the one to the other, are ſuch as only an Unit doth Meaſure, being their common Meaſure. *Euclid* 7. *Defin.* 12.

For inſtance, 7 and 13 are Prime Numbers to each other, becauſe they cannot be divided by any Number but an Unit. And 9 and 14 are alſo Prime Numbers to each other, for altho' 3 will meaſure or divide 9 without leaving a Remainder; yet 3 will not meaſure 14 without leaving a Remainder: Again, altho' 2 will meaſure 14 without any Remainder, yet 2 will not Meaſure 9 without leaving a Remainder, *&c.*

3. A compoſed Number is that which ſome certain Number meaſureth. *Euclid.* 7. *Defin.* 13.

For inſtance, 15 is a compoſed Number of 3 and 5, for $5 \times 3 = 15$, conſequently 3 or 5 will juſtly meaſure 15. Alſo 20

is composed of 5 and 4, *viz.* $5 \times 4 = 20$, therefore 5 and 4 will each justly measure 20.

4. Numbers composed the one to the other, are they which some Number being a common Measure to them both doth measure. *Euclid* 7. *Defin.* 14.

That is, if two or more Numbers can be divided by one and the same Divisor; then are those Numbers said to be composed one to another.

For Instance, 14 and 21 are Numbers composed the one to the other, because they can both be measured or divided by 7. For $7 \times 2 = 14$, and $7 \times 3 = 21$; therefore 7 is a common Measure to 14 and 21. So that if $\frac{14}{21}$ were proposed to be abbreviated, it will become $\frac{2}{3}$.

$$\text{Thus} \begin{cases} 7)\ 14 = 2 \\ 7)\ 21 = 3 \end{cases}$$

And how those greatest common Measures may be found comes from *Euclid* 7. *Prob.* 1, 2, 3. and is thus:

RULE.

Divide the gaeater Number by the lesser, and that Divisor by the Remainder (if there be any) and so on continually until there be no Remainder left: Then will that last Divisor be the greatest common Measure (and if it happen to be 1, *then are those Numbers Prime Numbers, and are already in their lowest Terms; but if otherwise) Divide the Numbers by that last Divisor, and their Quotients will be their least Terms required.*

EXAMPLE.

Let it be required to find the greatest common Measure of 72 and 108, *viz.* of $\frac{72}{108}$.

```
72) 108 (1
    72
    ---
    36) 72 (2
        72
        ---
        (0)
```

Here because there is no Remainder; 36 is the greatest common Measure.

$$\text{Therefore,} \begin{cases} 36)\ 72 = 2 \\ 36)\ 108 = 3 \end{cases}$$

Hence $\frac{72}{108}$ is abbreviated to $\frac{2}{3}$ the lowest Terms.

Again, to find the greatest common Measure of 744 and 899.

Thus,

Thus, 744) 899 (1
744
155) 744 (4
620
124) 155 (1
124
31) 124 (4
124
(0)

Here 31 is found to be the greateſt common Meaſure by which 744 and 899 may be abbreviated to 24 and 29 their loweſt Terms.

Thus, $\frac{31}{31}$) $\frac{744}{899}$ ($=\frac{24}{29}$, *&c.*

Note, If the propoſed Numbers be even, they may be brought lower by a continued halfing of them, ſo long as they can be halfed, *viz*, divided by 2.

EXAMPLE.

It is required to Reduce $\frac{56}{84}$ to it's leaſt Terms.

Firſt, $\frac{2}{2}$) $\frac{56}{84}$ ($=\frac{28}{42}$. Again, $\frac{2}{2}$) $\frac{28}{42}$ ($=\frac{14}{21}$.

This done, you eaſily perceive that 7 will be the common Meaſure to 14 and 21, *viz.* $\frac{7}{7}$) $\frac{14}{21}$ ($=\frac{2}{3}$, *&c.*

If the Numbers propoſed to be reduced have each a Cypher, or Cyphers annexed to them, they will be abbreviated by cutting off a like Number of Cyphers from both.

Thus, $\frac{150}{300}$ will be $\frac{15}{30}$. And $\frac{200}{300}$ will be $\frac{2}{3}$, *&c.*

That is, $\frac{150}{300}=\frac{15}{30}=\frac{1}{2}$. And $\frac{200}{300}=\frac{2}{3}$. And $\frac{3600}{4000}=\frac{36}{40}=\frac{18}{20}=\frac{9}{10}$.

Sect. 5. Addition *of* Fractions.

WHAT hath been done by the Rules in this Chapter, is chiefly to prepare and fit *Fractions* of different Denominations for *Addition* or *Subſtraction*, as Occaſion requires, *viz.* If they are *Compound Fractions*, they muſt be reduced to Simple or Pure *Fractions*, *per Rule*, *Sect.* 1.

If they are of different Denominations, they muſt be altered or changed, *per Rule*, *Sect.* 2.

That is, all *Fractions* muſt be brought into one Denomination before they can either be added or ſubſtracted; and that being done, *Addition* is thus performed.

RULE.

Add together all the Numerators, and their Sum will be a New Numerator, under which ſubſcribe the Common Denominator.

Examples

Examples in Simple Fractions.

Let it be propoſed to add $\frac{1}{3}$, $\frac{2}{5}$, and $\frac{3}{4}$ together. Firſt, $\frac{1}{3}=\frac{20}{60}$, $\frac{2}{5}=\frac{24}{60}$, and $\frac{3}{4}=\frac{45}{60}$, *per Sect.* 2.

Then $\frac{20}{60}+\frac{24}{60}+\frac{45}{60}=\frac{89}{60}$, the ſum required, which according to *Section* 3, is $1\ \frac{29}{60}$, *viz.* $\frac{89}{60}=1\ \frac{29}{60}$.

Examples in Compound Fractions.

Let it be required to add $\frac{3}{7}$ and $\frac{2}{3}$ of $\frac{3}{4}$ into one Sum. Firſt $\frac{2}{3}$ of $\frac{3}{4}$ becomes $\frac{6}{12}$ or $\frac{1}{2}$ per Sect. 1. And (per Sect. 2.) $\frac{3}{7}$ and $\frac{1}{2}$ is $\frac{6}{14}$ and $\frac{7}{14}$, *viz.* $\frac{3}{7}=\frac{6}{14}$, and $\frac{1}{2}=\frac{7}{14}$; but $\frac{7}{14}+\frac{6}{14}=\frac{13}{14}$ the Sum required, *viz.* $\frac{3}{7}+\frac{2}{3}$ of $\frac{3}{4}=\frac{13}{14}$.

Examples in Mixed Numbers.

It is required to add $5\ \frac{2}{3}$ to $7\ \frac{3}{4}$, theſe per Sect 3. will be $\frac{17}{3}$ and $\frac{31}{4}$. But $\frac{17}{3}$ and $\frac{31}{4}$ will become $\frac{68}{12}$ and $\frac{93}{12}$ per Sect. 2. Then $\frac{68}{12}+\frac{93}{12}=\frac{161}{12}$, and $\frac{161}{12}=13\ \frac{5}{12}$ the Sum required.

Or you may bring only the *Fractions* to one Denomination.

Thus, $5\ \frac{2}{3}$ and $7\ \frac{3}{4}$ will become $5\ \frac{8}{12}$ and $7\ \frac{9}{12}$.

Then $5\ \frac{8}{12}+7\ \frac{9}{12}=12\ \frac{17}{12}$. That is, $13\ \frac{5}{12}$. As before.

Sect. 6. Subſtraction *of* Fractions.

RULE.

SUBSTRACT one Numerator from the other, (according as the Queſtion requires) and their Difference will be a new Numerator, under which Subſcribe the Common Denominator, as in Addition.

EXAMPLE 1.

Let it be required to take $\frac{2}{9}$ out of $\frac{3}{7}$. Firſt $\frac{2}{9}$ and $\frac{3}{7}$ per Sect. 2. will become $\frac{14}{63}$ and $\frac{27}{63}$; then $\frac{27}{63}-\frac{14}{63}=\frac{13}{63}$, that is, $\frac{3}{7}-\frac{2}{9}=\frac{13}{63}$. As was required.

EXAMPLE 2.

It is required to Subſtract $\frac{2}{3}$ of $\frac{8}{9}$ from $1\ \frac{3}{14}$. Firſt, $\frac{2}{3}$ of $\frac{8}{9}=\frac{16}{27}$ per Sect. 1. Again $\frac{16}{27}$ and $\frac{13}{14}$ will become $\frac{224}{378}$ and $\frac{351}{378}$. per Sect. 2. Then $\frac{351}{378}-\frac{224}{378}=\frac{127}{378}$.

EXAMPLE 3.

From $6\ \frac{1}{8}$ Subſtract $3\ \frac{19}{48}$. Firſt, $6\ \frac{1}{8}=\frac{49}{8}$, and $3\ \frac{19}{48}=\frac{163}{48}$ per Rule Sect. 3. Again, $\frac{49}{8}=\frac{2352}{384}$, and $\frac{163}{48}=\frac{1304}{384}$, per Rule Sect. 2. Then, $\frac{2352}{384}-\frac{1304}{384}=\frac{1048}{384}=2\ \frac{280}{384}=2\ \frac{35}{48}$. Or otherwiſe thus:

First,

First, $6\frac{1}{8} = 5\frac{9}{8}$, then bring $\frac{2}{8}$ and $\frac{19}{48}$ into one Denomination, *viz.* $5\frac{2}{8} = 5\frac{42}{384}$ and $3\frac{19}{48} = 3\frac{152}{384}$.
Then $5\frac{432}{384} - 3\frac{152}{384} = 2\frac{280}{384} = 2\frac{35}{48}$. As before.

EXAMPLE.

Let it be required to Subſtract $\frac{3}{7}$ of $\frac{5}{9}$ of $\frac{2}{5}$ from 7.
First, $\frac{3}{7}$ of $\frac{5}{9}$ of $\frac{2}{5} = \frac{30}{189}$. And $7 = 6\frac{189}{189}$.
Then $6\frac{189}{189} - \frac{30}{189} = 6\frac{159}{189} = 6\frac{53}{63} = 7 - \frac{3}{7}$ of $\frac{5}{9}$ of $\frac{2}{5}$. As was required.

If theſe few *Examples* be well underſtood, the whole Buſineſs of adding and ſubſtracting *Vulgar Fractions* will be eaſy; which is really much more difficult to perform than either *Multiplication* or *Diviſion*; as will appear in the next *Section*.

Sect. 7. Multiplication *of* Fractions.

IN order to perform either *Multiplication* or *Diviſion*, you muſt prepare the Terms to be multiplied (or divided) thus; reduce Compound *Fractions* to Simple ones, *per Sect.* 1. Bring mixed Numbers into improper *Fractions*, and expreſs whole Numbers *Fraction-wiſe*, *per Sect.* 3. Alſo it will be convenient to abbreviate them to their ſmalleſt Terms, when it can be done. Then *Multiplication* may be thus performed.

Rule. *Multiply the Numerators one into another for a new Numerator; and the Denominators one into another for a new Denominator. As in theſe*

EXAMPLES.

1. The Product of $\frac{2}{5}$ into $\frac{3}{7} = \frac{6}{35}$. That is, $\frac{2 \times 3}{5 \times 7} = \frac{6}{35}$.

2. And the Product of $\frac{9}{19}$ into $\frac{20}{27} = \frac{180}{342}$. Or $\frac{5}{19}$.

3. Again, the Product of $\frac{7}{11}$ into $\frac{2}{5}$ of $\frac{5}{7} = \frac{70}{385}$. Or $\frac{2}{11}$.
For $\frac{2}{5}$ of $\frac{5}{7} = \frac{10}{35}$. Then $\frac{7}{11} \times \frac{10}{35} = \frac{70}{385} = \frac{2}{11}$.

4. Let it be required to multiply 6 with $3\frac{2}{5}$. Theſe prepared for the Work will ſtand thus. $\frac{6}{1} \times \frac{17}{5}$.
Viz. $6 = \frac{6}{1}$ and $3\frac{2}{5} = \frac{17}{5}$. Then $\frac{6}{1} \times \frac{17}{5} = \frac{102}{5}$, or $20\frac{2}{5}$.
Or, otherwiſe thus $6 \times 3 = 18$. And $\frac{2}{5} \times 6 = \frac{12}{5} = 2\frac{2}{5}$.
Then $18 + 2\frac{2}{5} = 20\frac{2}{5}$. As before.

5. Let it be required to multiply $7\frac{4}{9}$ with $5\frac{3}{7}$.
Firſt $7\frac{4}{9} = \frac{67}{9}$, and $5\frac{3}{7} = \frac{38}{7}$. Then $\frac{67}{9} \times \frac{38}{7} = \frac{2546}{63} = 40\frac{26}{63}$.

Now the Reaſon of this Rule for Multiplying of *Fractions*, and conſequently of theſe Operations, and all others performed by it; will be evident from this following.

Viz.

Viz. If $\frac{4}{2}$ be multiplied with $\frac{12}{3}$ according to the Rule, their Product will be $\frac{48}{6}$. But $\frac{48}{6} = 8$.

Now $\frac{4}{2} = 2$, and $\frac{12}{3} = 4$ per Sect. 3. But $4 \times 2 = 8$. *Ergo*, &c.

Sect. 8. Diviſion *of* Fractions.

THE *Fractions* being firſt prepared as before directed, *Diviſion* may be thus performed:

Rule. *Multiply the Numerator of the Dividend into the Denominator of the dividing Fraction for a New Numerator: and multiply the other Numerator and Denominator together for a new Denominator.*

EXAMPLES.

1. Let $\frac{6}{35}$ be divided by $\frac{3}{7}$, *viz.* $\frac{3}{7}$) $\frac{6}{35}$ ($\frac{42}{105} = \frac{2}{5}$ the Quotient.

That is, according to the Rule $6 \times 7 = 42$ the new Numerator, and $35 \times 3 = 105$, the new Denominator, &c. as above.

2. Let it be required to divide $\frac{20}{27}$ by $\frac{5}{12}$, *viz.* $\frac{5}{12}$) $\frac{20}{27}$ ($\frac{240}{135} = 1\frac{7}{9}$.

For $12 \times 20 = 240$ the new Numerator, and $27 \times 5 = 135$ the new Denominator.

3. Suppoſe it were required to divide $\frac{2}{11}$ by $\frac{2}{5}$ of $\frac{5}{7}$.

Firſt, $\frac{2}{5}$ of $\frac{5}{7} = \frac{10}{35}$. Then $\frac{10}{35}$) $\frac{2}{11}$ ($\frac{70}{110} = \frac{7}{11}$.

4. Let $20\frac{2}{5}$ be divided by $3\frac{2}{5}$; *viz.* $\frac{102}{5}$ by $\frac{17}{5}$:

For $20\frac{2}{5} = \frac{102}{5}$, and $3\frac{2}{5} = \frac{17}{5}$. Then $\frac{17}{5}$) $\frac{102}{5}$ ($= 6$ the Quotient.

5. Let it be required to divide $40\frac{26}{63}$ by $5\frac{3}{7}$.

Firſt, $40\frac{26}{63} = \frac{2546}{63}$, and $5\frac{3}{7} = \frac{38}{7}$. Then $\frac{38}{7}$) $\frac{2546}{63}$ ($\frac{178822}{2394}$.

But $\frac{178822}{2394} = 7\frac{4}{9}$ the true Quotient required.

6. Suppoſe it were required to divide 13 by $\frac{5}{7}$.

Firſt, $13 = \frac{13}{1}$. Then $\frac{5}{7}$) $\frac{13}{1}$ ($\frac{91}{5} = 18\frac{1}{5}$ the Quotient.

7. Again, let it be required to divide $\frac{5}{7}$ by 6.

Viz. $\frac{6}{1}$) $\frac{5}{7}$ ($\frac{5}{42}$ for the Quotient required.

N. B. From hence you may obſerve, that when any whole Number is divided by a Fraction leſs than Unity or 1, the Quotient will be greater than the Number propoſed to be divided: But if any Fraction be divided by a whole Number, greater than 1, then the Quotient will be leſs than the Dividend: As in the two laſt *Examples*.

As to the *Reaſon* (or *Proof*) of this *Rule* for *dividing Fractions* It is only the *Converſe* to that of *Multiplication*, and will be very evident from this following.

Let $\frac{4}{6}^{8}$ be *divided* by $\frac{4}{2}$. Which according to the *Rule* is thus, $\frac{4}{2}) \frac{4}{6}^{8} (\frac{2}{2}^{6} = 4$. The true *Quotient*. Now $\frac{4}{6}^{8} = 8$. And $\frac{4}{2} = 2$. *per Sect.* 3. Conſequently $\frac{4}{6}^{8}$ *divided* by $\frac{4}{2}$ is but the ſame with 8 *divided* by 2. *viz.* 2.) 8 (4. The *Quotient* as before.

I could have inſerted *Geometrical Demonſtrations*, for the *Rules* of *Multiplication* and *Diviſion* of *Fractions*; but ſuppoſing the Learner purely unacquainted with thoſe kind of Demonſtrations, I thought theſe might be more intelligible to him, eſpecially in this place.

CHAP. V.

Of Decimal Fractions.

WHEN, or by whom, this excellent Invention of Decimal Arithmetick, *was firſt introduced is uncertain; but doubtleſs it's Improvements, and the Perfections it is now in, is owing to latter Years.*

Sect. 1. *Of* Notation.

IN *Decimal Fractions*, the *Integer* or *whole Thing* (whether it be *Coin*, *Weight*, *Meaſure*, or *Time*, &c.) is ſuppoſed to be *divided* into *Ten equal Parts*; and every one of thoſe *Ten Parts* are ſuppoſed to be *ſubdivided* into other *Ten equal Parts*, &c. *ad infinitum*.

The *Integer* being thus *divided (by Imagination)* into 10, 100, 1000, 10000, *&c. equal Parts*, becomes the *Denominator* to the *Decimal Fractions*.

Thus, $\frac{2}{10}$. $\frac{3}{100}$. $\frac{7}{1000}$. $\frac{53}{10000}$. $\frac{754}{100000}$, &c.

Now theſe *Denominators* are ſeldom or never ſet down, but only the *Numerators*; and thoſe are either diſtinguiſhed, or ſeparated from *whole Numbers* by a *Point* or a *Comma*.

Thus, 5,4 is $5\frac{4}{10}$. and 0,7 is $\frac{7}{10}$. 35,05 is $35\frac{5}{100}$, &c.

But before we proceed further in *Notation*, it will be convenient for the Learner to conſider the following *Table*, (taken out of the learned Mr *Oughtred*'s *Clavis Mathematica*) which ſhews the very Foundation of *Decimal Fractions*.

Whole Numbers,						*Decimal Parts.*					
5	4	3	2	1	0,	1	2	3	4	5	6
&c.	*Tens of Thousands.*	*Thousands.*	*Hundreds.*	*Tens.*	*Units Place.*	*Parts of Ten, or* $\frac{1}{10}$	*Parts of a Hundred.*	*Parts of a Thousand.*	*Parts of Ten Thousand.*	*Parts of 100 Thousand.*	*Parts of a Million.*
	&c.										*&c.*

By this *Table* it is evident, that as in whole *Numbers* or *Integers*, every *Degree* from the *Unit's* Place increaſes towards the left-hand by a Ten-fold *Proportion:* So in *Decimal Parts* every *Degree* is decreaſed towards the right-hand by the ſame *Proportion*, viz. by *Tens.*

Therefore theſe *Decimal Parts* or *Fractions*, are really more *Homogeneal*, or agreeing with *whole Numbers*, than *Vulgar Fractions*; for indeed all plain *Numbers* are in effect but *Decimal Parts* one to another.

That is, ſuppoſe any *Series* of *equal Numbers*, as 444, *&c.* The firſt 4 towards the Left is *Ten* times the *Value* of the 4 in the middle, and that 4 in the middle is *Ten* times the *Value* of the laſt 4 to the Right of it, and but the *Tenth Part* of that 4 on the Left, *&c.*

Therefore all or any of them may be taken either as *Integers*, or *Parts* of an *Integer:* If *Integers*, then they muſt be ſet down without any *Comma* or ſeparating *Point* betwixt them thus, 444 But if *Integers*, and one *Part* or *Fraction*, put a *Comma* betwixt them thus, 44,4 which ſignifies 44 *whole Numbers*, and 4 *Tenths* of an *Unit:* Again, if two Places of *Parts* be required, ſeparate them with a *Comma* thus, 4,44 *viz.* 4 *Units*, and 44 *hundred Parts* of an *Unit*, &c.

From hence (duly compared with the *Table*) it will be eaſy to conceive that *Decimal Parts* take their *Denomination* from the Place of their laſt *Figure.*

That is, $\left\{\begin{array}{l} ,5 = \frac{5}{10} \\ ,56 = \frac{56}{100} \\ ,056 = \frac{56}{1000} \end{array}\right\}$ *Parts* of an *Unit*, &c.

Cyphers

Cyphers annexed to *Decimal Parts*, alter not their *Value*. As ,50, and ,500, or ,5000, &c. are each but 5 *Tenths* of an *Unit*, For $\frac{50}{100} = \frac{5}{10}$. And $\frac{500}{1000} = \frac{5}{10}$. Or $\frac{5000}{10000} = \frac{5}{10}$. *Per Sect.* 4. of the laſt *Chapter*.

But *Cyphers* prefixed to *Decimal Parts* decreaſe their *Value*, by removing them further from the *Comma*.

Thus,
- ,5 = 5 *Tenth Parts.*
- ,05 = 5 *Parts* of a *Hundred.*
- ,005 = 5 *Parts* of a *Thouſand.*
- ,0005 = 5 *Parts* of *Ten Thouſand*, &c.

Conſequently the true *Value* of all *Decimal Parts* are known by their Diſtance from the *Units* Place; the which being once rightly underſtood, the reſt will be eaſy.

Sect. 2. Addition *and* Subſtraction *of* Decimals.

IN ſetting down the propoſed Numbers to be added, or ſubſtracted, great care muſt be taken in placing every Figure directly underneath thoſe of the ſame Value, whether they be mix'd Numbers, or pure Decimal Parts, and to perform that you muſt have a due regard to the Comma's, or ſeparating Points, which ought always to ſtand in a direct Line one under another; and to the Right-hand of them carefully place the Decimal Parts, according to their reſpective Values, or Diſtances from Unity. Then

Rule — *Add or ſubſtract them, as if they were all whole Numbers; and from their Sum, or Difference, cut off ſo many Decimal Parts as are the moſt in any of the given Numbers.*

EXAMPLES in Addition.

Let it be required to find the Sum of theſe following Numbers, *viz.* 34,5 + 65,3 + 128,7 + 95 + 87,8 + 7,9, which being truly placed, will ſtand

Thus,

34,5
65,3
128,7
95,0
87,8
7,9
419,2

Their Sum required,

EXAMPLE. 2.

Let it be required to find the Sum of 25,854+34,578+9,076 +13,907.

25,854
34,578
9,076
13,907

83,415 The Sum required.

When the Decimal Parts propoſed to be added (or ſubſtracted) have not the ſame Number of Places, you may for convenience of Operation ſupply or fill up the void Places, by annexing Cyphers. As in theſe *Examples.*

EXAMPLE 3.	*EXAMPLE* 4.	*EXAMPLE* 5.
45,0700	574,678953	0,975642
50,7580	95,796430	,745257
123,0057	78,054600	,000598
74,7020	54,789000	,800700
24,8000	8,900000	,640530
318,3357	Sum 812,218983	3,162727

EXAMPLES in Subſtraction.

Let it be required to find the Difference between 45,375 and 74,284.

EXAMPLE 1.	*EXAMPLE* 2.	*EXAMPLE* 3.
That is, From 74,284	From 437,5	From 75,0034
Take 45,375	Take 80,657	Take 57,875
Remains 28,909	347,043	17,1284

EXAMPLE 4.

Let it be required to find the Exceſs between 562 and 93,5784.

EXAMPLE 4.	*EXAMPLE* 5.
That is, From 562,	From 345,7578
Take 93,5784	Take 157,
The Exceſs 468,4216	188,7578

Note, The two laſt Examples are ſuppoſed to be ſupplied with Cyphers, which if actually done would ſtand thus,

562,0000	345,7578
93,5784	157,0000
Remains 468,4216 As before,	188,7578

EXAMPLE

EXAMPLE 6.

From 0,547893
Take 0,439758
0,108135

EXAMPLE 7.

From 1,000000
Take 0,997543
0,002457

The Proof of Addition and Subſtraction in Decimals, is the ſame with that of whole Numbers, page 13, *&c.*

Sect. 3. Multiplication of Decimals.

WHETHER the Factors or Numbers to be multiplied are pure Decimals, or mixed, multiply them as if they were all whole Numbers, and for the true Value of their Product obſerve this

Rule. { *Cut off* (viz. *ſeparate with a Comma*) *ſo many Places of Decimal Parts in the Product, as there are in both the Factors accounted together. As in theſe.*

EXAMPLE 1.

3,024
2,23
9072
6048
6048
6,74352

EXAMPLE 2.

32,12
24,3
9636
12848
6424
780,516

The Reaſon why ſuch a Number of Decimal Parts muſt be cut off in the Product, may be eaſily deduced from theſe Examples. Thus,

In Example 1. It is evident, that 3, the whole Number in the Multiplicand, being multiplied with 2, the whole Number in the Multiplier; can produce but 6 (*viz.* $3 \times 2 = 6$.) So that of neceſſity all the other Figures in the Product muſt be Decimal Parts; according as the Rule directs.

Or, the Rule is evident from the Multiplication of whole Numbers only: Thus, ſuppoſe 3000 were to be multiplied with 200, their Product will be 600000; That is, there will be ſo many Cyphers in the Product, as are in both the Factors, (*Vide p.* 18.) Now if, inſtead of thoſe Cyphers in the Factors, we ſuppoſe the like Number of Decimal Parts; then it follows, that there ought to be the ſame Number of Decimal Parts in the Product, as there were Cyphers in the Factors.

Again, the Rule may be otherwiſe made evident from Vulgar Fractions, thus: Let 32,12 be multiplied with 24,3, and

and their Product will be 780,516 as in *Example* 2, above. Now $32,12 = 32\frac{12}{100}$. and $24,3 = 24\frac{3}{10}$. which being brought into Improper Fractions (*per Sect.* 3. *page* 50.) will become $32\frac{12}{100} = \frac{3212}{100}$. and $24\frac{3}{10} = \frac{243}{10}$.

Then $\frac{3212}{100} \times \frac{243}{10} = \frac{780516}{1000}$. *per Sect.* 7. *page* 55.

But $\frac{780516}{1000} = 780\frac{516}{1000}$. *viz.* 780,516, as before.

Any of these three Ways do, I presume, sufficiently prove the Truth of the aboveſaid Rule, *&c.*

EXAMPLE 3.

78,546
436
471276
235638
314184
34246,056

EXAMPLE 4.

5745
,0675
28725
40215
34470
387,7875

N. B. *It sometimes falls out in multiplying Parts with Parts, that there will not be so many Figures in the Product, as there ought to be places of Decimal Parts by the Rule: In that Case you must supply their Defect by prefixing Cyphers to the Product; as in these Examples.*

EXAMPLE 5.

,2365
,2435
11825
7095
9460
4730
,05758775

EXAMPLE 6.

,0347
,0236
2082
1041
694
,00081892

When any proposed Number of Decimals is to be multiplied with 10 . 100 . 1000 . 10000, *&c.* it is only removing the separating Point in the Multiplicand, so many places towards the Right-hand, as there are Cyphers in the Multiplier.

Thus, ,578 × 10 = 5,78. And, ,578 × 100 = 57,8.
Again, ,578 × 1000 = 578. Or, ,578 × 10000 = 5780.

These

Theſe things being conſidered, it will be eaſy to multiply Decimals, and determine their true products. As in theſe following Examples.

57,056 multiplied into 0,578 will produce 32,978368
7,6543 into 5,4246 will produce 41,52151578
0,56879 × 0,05674 = 0,0322731446
0,03246 × 0,02364 = 0,0007672544
87649 × 0,03687 = 3231,61863
94,35786 × 6,57869 = 620,7511100034
3,141592 × 52,7438 = 165,6995001296

Now it oftentimes happens, that it will be needleſs to expreſs all the Figures of the Product at large, (eſpecially, when the Factors have each of them many places of Decimal Parts, as in the two laſt Examples) only ſo many of them as may ſuffice for the intended Deſign; and yet the Product may be as true to ſo many Figures as are retained, as if the Factors had been multiplied at large. And ſuch compendious Contractions are not only of Curioſity, but may alſo be found of great Eaſe and Uſe to the ingenious Practitioner; eſpecially in reſolving adfected Equations, or in calculating of Trigonometrical Problems by the Natural Signs and Tangents, &c. All which may be thus performed.

Viz. *Set the Unit's Place of the Multiplier directly underneath that Figure of the Multiplicand, whoſe Place you intend to keep in the Product; and place all the other Figures of the Multiplier in a quite contrary order to the uſual way. Then in multiplying always begin at that Figure of the Multiplicand which ſtands over the Figure wherewith you are then a multiplying, ſetting down the firſt Figure of each particular Product, directly underneath one another; yet herein you muſt have a due regard to the Increaſe which would ariſe out of the two next Figures to the Right-hand of that Figure in the Multiplicand which you then begin with.*

EXAMPLE.

Let it be required to multiply 3,141592 with 52,7438 and let there be only four Places of Decimal Parts retained in the Product.

If the propoſed Numbers were to be multiplied at large they muſt ſtand in a direct Order as uſual.

Thus { 3,141592 / 52,7438 } And would produce ten Places of Parts, as in the laſt Example.

But

and their Product will be 780,516 as in *Example* 2, above. Now $32,12 = 32\frac{12}{100}$. and $24,3 = 24\frac{3}{10}$. which being brought into Improper Fractions (*per Sect.* 3. *page* 50.) will become $32\frac{12}{100} = \frac{3212}{100}$. and $24\frac{3}{10} = \frac{243}{10}$.

Then $\frac{3212}{100} \times \frac{243}{10} = \frac{780516}{1000}$. *per Sect.* 7. *page* 55.

But $\frac{780516}{1000} = 780\frac{516}{1000}$. *viz.* 780,516, as before.

Any of these three Ways do, I presume, sufficiently prove the Truth of the abovesaid Rule, *&c.*

EXAMPLE 3.

```
  78,546
     436
  ------
  471276
 235638
 314184
--------
34246,056
```

EXAMPLE 4.

```
   5745
  ,0675
  -----
  28725
 40215
 34470
-------
387,7875
```

N. B. *It sometimes falls out in multiplying Parts with Parts, that there will not be so many Figures in the Product, as there ought to be places of Decimal Parts by the Rule: In that Case you must supply their Defect by prefixing Cyphers to the Product; as in these Examples.*

EXAMPLE 5.

```
  ,2365
  ,2435
  -----
  11825
  7095
 9460
 4730
---------
,05758775
```

EXAMPLE 6.

```
  ,0347
  ,0236
  -----
   2082
  1041
  694
--------
,00081892
```

When any proposed Number of Decimals is to be multiplied with 10 . 100 . 1000 . 10000, *&c.* it is only removing the separating Point in the Multiplicand, so many places towards the Right-hand, as there are Cyphers in the Multiplier.

Thus, $,578 \times 10 = 5,78$. And, $,578 \times 100 = 57,8$.
Again, $,578 \times 1000 = 578$. Or, $,578 \times 10000 = 5780$.

These

Theſe things being conſidered, it will be eaſy to multiply Decimals, and determine their true products. As in theſe following Examples.

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7,6543 into 5,4246 will produce 41,52151578
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Viz. *Set the Unit's Place of the Multiplier directly underneath that Figure of the Multiplicand, whoſe Place you intend to keep in the Product; and place all the other Figures of the Multiplier in a quite contrary order to the uſual way. Then in multiplying always begin at that Figure of the Multiplicand which ſtands over the Figure wherewith you are then a multiplying, ſetting down the firſt Figure of each particular Product, directly underneath one another; yet herein you muſt have a due regard to the Increaſe which would ariſe out of the two next Figures to the Right-hand of that Figure in the Multiplicand which you then begin with.*

EXAMPLE.

Let it be required to multiply 3,141592 with 52,7438 and let there be only four Places of Decimal Parts retained in the Product.

If the propoſed Numbers were to be multiplied at large they muſt ſtand in a direct Order as uſual.

Thus { 3,141592 / 52,7438 } { And would produce ten Places of Parts, as in the laſt Example.

But

But seeing it is required to have only four Places of those Parts in the product, set them down as before directed, and they will stand

Thus	3,141592	The Multiplicand placed as before.
	8347,25	The Multiplier in a Reverse Order.
	1570796	The Product with 5, regard had to 5 times 2.
	62832	The Product with 2, increased with 9×2.
	21991	Product with 7, increased with 5×7+9×7.
	1257	Product with 4, increased with 1×4+5×4.
	94	Product with 3, increased with 4×3.
	25	Product with 8, increased with 4×8+1×8.
	165,6995	The true Product as was required.

The Reason of this Contraction is very obvious from the whole Operation wrought at large.

Thus { 3,141592 / 52,7438

25	132736
94	24776
1256	6368
2 1991	144
6 2831	84
157 0796	0
165,6995	001296

From hence it is evident, that all the Figures in the Square to the Right-hand, are wholly omitted in the former Contraction; and that the last single Product *here, is the first there; consequently the Reason of placing the* Multiplier *in a Reverse Order, must needs appear very plain.*

EXAMPLE 3.

Suppose it were required to multiply 257,356 with 76,48 and to have only the entire Product of Integers.

257,356			257,356
84,67	The same at large.	{	76,48
18015			20 58848
1544			102 9424
103			1544 136
20			18014 92
19682			19682,58688

The chiefest Care and Difficulty that attends these Contractions, is the true setting down of the Units place in the Multiplier underneath the proper Figure of the Multiplicand, according to the designed Product.

Viz.

Viz. In Example 1. It was required to have four Places of Decimal Parts in the Product; therefore the Unit's Place of the Multiplier was ſet under the fourth Place of Decimals in the Multiplicand: And in Example 2, becauſe it was required to have an entire Product of Integers only; therefore the Unit's Place of the Multiplier was ſet under the Unit's Place of the Multiplicand. This, I ſay, being once rightly underſtood, will render the Method eaſy in Practice.

Sect. 4. Diviſion *of* Decimals.

DIVISION is accounted the moſt difficult Part of Decimal Arithmetick: In order therefore to make it plain and eaſy, it will be convenient to reſume what has been ſaid in *page* 25.

Viz. *The Quotient Figure is always of the ſame Value or Degree with that Figure of the Dividend, under which the Unit's Place of it's Product ſtands.*

As for Inſtance, Let 294 be divided by 4.

4) 294 (7 — This is not 7 but 70, becauſe the Units Place of 4×7 ſtands under the Tens Place of the Dividend.
28

14 (3 — But this is only 3.
12

Remains (2) Hence $73\frac{2}{4}$ is the Quotient.

Now if to the Remainder 2 there be annexed a Cypher (thus, 2,0) and then divided on, it muſt needs follow that the Units Place of the Product ariſing from the Diviſor into the Quotient, will ſtand under the annexed Cypher; conſequently the Quotient Figure will be of the ſame Value or Degree with the Place of that Cypher: But that is the next below the Units Place, therefore the Quotient Figure is of the next Degree or Place below Unity; That is, in the firſt Place of Decimal Parts.

Thus 4) 2,0 (,5

So that 4) 294,0 (73,5 the true Quotient required.

This being well underſtood; Diviſion of Decimals may (in all the various Caſes) be eaſily performed. However, that it may be rendered plain and eaſy even to the meaneſt Capacity, if poſſible; Let Diviſion be again defined, as in *page* 21.

Viz. *If that Number which Divides another, be multiplied with the Number which is quoted, their Product will be the Number divided.*

This Definition alone (if compared with the Rule, *page* 61.) will afford a general Rule for discovering the true Value of the Quotient Figure in Division of Decimals.

Rule { *The Place of Decimal Parts in the Divisor and Quotient being counted together, must always be equal in Number with those in the Dividend. And from this general Rule ariseth four Cases.*

Case 1. When the Places of Parts in the Divisor and Dividend are equal, the Quotient will be whole Numbers.

As in these *Examples.*

8,45) 295,75 (35	0,0078) ,4368 (56
2535	390
4225	468
4225	468
(0)	(0)

Case 2. When the Places of Parts in the Dividend exceed those in the Divisor; cut off the Excess for Decimal Parts in the Quotient. As in these *Examples.*

24,3) 780,516 (32,12		436) 34246,056 (78,546
729		3052
515		3726
486		3488
291		2380
243		2180
486	,534) ,30438 (57,	2005
486	2670	1744
(0)	3738	2616
	3738	2616
	(0)	(0)

Case 3. When there are not so many Places of Parts in the Dividend, as are in the Divisor; annex Cyphers to the Dividend to make them equal. Then will the Quotient be whole Numbers, as in Case 1.

EXAMPLES.

EXAMPLES.

Let it be required to divide 192,1 by 7,684, and 441 by ,7875

```
7,684) 192,100 (25          ,7875) 441,0000 (560
       153 68                      393 75
       ------                      ------
        38 420                      47 250
        38 420                      47 250
       ------                      ------
           (0)                         (0)
```

Case 4. If after Division is finished, there are not so many Figures in the Quotient, as there ought to be Places of Parts by the general Rule; supply their defect by prefixing Cyphers to it.

EXAMPLES.

Let it be required to divide 7,25406 by 957.

```
957) 7,25406 (,00758 the true quotient required.
     6 699
     -----
      5550
      4785
      ----
      7656
      7656
      ----
       (0)
```

Again

```
,575) ,0007475 (,0013
        575
        ---
        1725
        1725
        ----
          (0)
```

Note, When Decimal Numbers are to be divided by 10. 100. 1000. 10000. *&c.* that is, when the Divisor is an Unit with Cyphers; Division is performed by removing or placing the separating Point in the Dividend, so many Places towards the Left-hand, as there are Cyphers in the Divisor.

EXAMPLE.

```
10) 5784 (578,4          100 (578,4 (57,84
1000) 5784 (5,784      10000 (578,4 (,05784
```

Note, *These Operations are the direct Converse to those in* page 62.

I presume it needless to give more Examples at large; only I shall insert a few Dividends, and Divisors, with their Quotients, wherein are contained all the Varieties that can happen in Division of Decimals.

```
,574) 493,066 ( 859        5,74) 49,3066 (8,59
 574) 493,066 (,859        5,74) 493066,00 (85900
 574) 49,3066 (,0859       ,0574) 493,0660 (8590
5,74) 4930,66 ( 859        ,0574) ,493066 (8,59
```

There is alſo a compendious Way of contracting Diviſion, like that of Multiplication, *page* 64, by which much Labour may be ſaved; eſpecially when the Diviſor hath many Places of Decimal Parts in it: And it is thus performed.

Having determined how many Places of whole Numbers there will be in the Quotient, if any at all; or if none, of what Value or Place the firſt Figure in the Quotient will be: Then omit, or prick off one Figure of the Diviſor at each Operation; *viz.* for every Figure you place in the Quotient, prick off one in the Diviſor; having a due Regard to the Increaſe which would ariſe from the Figure ſo omitted.

EXAMPLE.

Let it be required to divide 70,23 by 7,9863.

The Work contracted.	The ſame at Length.
7,9863) 70,2300 (8,7938	7,9863) 70,2300 (8,7938
.... 63 8904	63 8904
6 3396	6 3396 | 0
5 5904	5 5904 | 1
749 2	749 1 | 90
718 7	718 7 | 67
30 5	304 | 230
23 9	239 | 589
06	64 | 6410
64	63 | 8904
(2)	0 | 7506

The Work contracted I preſume is ſo obvious (if compared with the ſame at large) that it is needleſs to give any farther Explanation of it.

Sect. 5. *To* Reduce Vulgar Fractions *into* Decimals, *and the contrary.*

ANY Vulgar Fraction being given, it may be reduced, or rather changed into Decimal Parts equivalent to it. Thus,

Rule { *Annex Cyphers to the Numerator, and then divide it by the Denominator, the Quotient will be the Decimal Parts equivalent to the given Fraction; or at leaſt ſo near it as may be thought neceſſary to approach.*

EXAMPLE.

EXAMPLE.

It is required to change or reduce $\frac{3}{4}$ into Decimals.

4) 3,00 (,75 The Decimal Parts required.

That is, $\frac{3}{4} = \frac{75}{100} = ,75$.

Again $\frac{1}{2} = ,5$; thus 2) 1,0 (,5. And $\frac{1}{4} = ,25$; 4) 1,00 (,25

Suppoſe it were required to change $\frac{4}{7}$ into Decimals.

7) 4,0000000000 (,5714285714 *&c.* $= \frac{4}{7}$.

Note, When the laſt Figure of the Diviſor (that is, the Denominator of the propoſed Fraction) happens to be one of theſe Figures; *viz.* 1 . 3 . 7 . or 9 . (as in the Example) then the Decimal Parts can never be preciſely equal to the given Fraction; yet by continuing the Diviſion on, you may bring them to be very near the Truth. As in this Example; Suppoſe it was required to Change $\frac{1}{13}$ into Decimal Parts.

13) 1,0000 (,076923076923 *&c. ad infinitum.*

91..

90

78

120

117

30

26

40

39

10

&c.

That is, 0,076923076923 $= \frac{1}{13}$ *ferè.*

And from hence it may be farther obſerved; that in theſe imperfect Quotients, the Figures do return again and circulate in the ſame Order as before: as you may eaſily perceive they begin to do in the ſeventh Place of both theſe laſt Examples.

As at firſt.

Theſe being underſtood, it will be eaſy to find the Decimal Parts equivalent to any known Part or Parts of Coin, Weights, Meaſures, Time, *&c.* If you firſt reduce the given Parts of Coin, *&c.* into a Vulgar Fraction, whoſe Denominator is the Number of thoſe known Parts contained in the Integer, and the given Parts it's Numerator.

Examples in Coin, &c.

1. Let it be required to find the Decimals of 16 *s.* 6 *d.* Firſt 16 *s.* $= \frac{16}{20}$ of one Pound, and 6 *d.* $= \frac{1}{40}$ of 1 *l.*

But $\frac{16}{20} + \frac{1}{40} = \frac{33}{40}$. Then 40) 33,000 (,825 the Decimal Parts required: That is ,825 = 16 *s.* 6 *d.*

Again, Suppoſe it were required to find the Decimals equal to 3 *l.* 13 *s.* 4 *d.*

Here

Here 3 *l.* is 3 Integers, and 13 *s.* = $\frac{13}{20}$ of 1 *l.* and 4 *d.* = $\frac{4}{240}$. But $\frac{13}{20} + \frac{4}{240} = \frac{160}{240}$. Then 240) 160,000 (0,666666 *&c.* Hence 3 *l.* 13 *s.* 4 *d.* = 3,666666 *&c.* As was required.

2. What are the Decimals equal to 7 $\frac{3}{4}$ Inches, one Foot being made the Integer.

First, 7 Inches are $\frac{7}{12}$ of 1 Foot, and $\frac{3}{4}$ of 1 Inch are $\frac{3}{48}$. But $\frac{7}{12} + \frac{3}{48} = \frac{31}{48}$. Then 48) 31,000 (,64583 *&c.* = 7 $\frac{3}{4}$ Inch.

3. Let it be required to change 8 *Oz.* 19 *Pwt.* 8 *Grains* into Decimals; one Pound Troy being the Integer.

Theſe being reduced into the leaſt Terms, and added together, will become $\frac{4304}{5760}$ of 1 Pound.

Then 5760) 4304,000 (,74722 *&c.* The Decimals required.

And thus may any propoſed Parts of Coin, Weights, Meaſures, *&c.* be reduced or changed into Decimal Parts; which perhaps may at firſt ſeem ſomewhat tedious in Practice, but being a little acquainted with them it will be found very eaſy; and the ingenious Practitioner will (with a little Conſideration) ſoon find how to reduce them almoſt mentally; or with the help of a very few Figures, without the Uſe of ſuch large Tables as are uſually inſerted in Books of Decimal Arithmetick; or at moſt they may be contracted into ſuch as theſe following, which if duly applied to thoſe Tables in Chap. 3. will be found very uſeful.

Decimal Tables.

In Engliſh Coin.	*Averdupois Weight.*
0,05 ..... = 1 *s.*	0,0625 .... = 1 *Ounce.*
0,0046667 = 1 *d.*	0,00390625 = 1 *Dram.*
0,00104167 = 1 *Farthing.*	1 lb being the *Integer.*
1 *l.* being the *Integer.*	
Troy Weight.	*Averdupois Great Weight.*
0,05 ..... = 1 *Pwt.*	0,25 ..... = $\frac{1}{4}$ *C.*
0,00208333 = 1 *Grain.*	0,00892857 = 1 lb.
1 *Oz.* being the *Integer.*	0,00055803 = 1 *Ounce.*
	1 *C.* being the *Integer.*
Apothecaries Weight.	*Time.*
0,125 ..... = 1 *Dram.*	0,04166667 = 1 *Hour.*
0,04166667 = 1 *Scruple.*	0,00069444 = 1 *Minute.*
0,00208333 = 1 *Grain.*	0,00001157 = 1 *Second.*
1 *Oz.* being the *Integer.*	1 *Day*, or 24 *Hours*, being made the *Integer.*

The Uſe of theſe Tables will be evident by the following

EXAMPLE.

EXAMPLE.

Let it be required to find the Decimal Parts equivalent to 17 *s.* 9 *d.* 2 *Farthings.*

First 0,05 = 1 *s.* Therefore 17 × ,05 = ,85 .... = 17 *s.*

And ,004166 = 1 *d.* Therefore ,004166 × 9 = ,037494 = 9 *d.*

Alſo 2),004166 (= ,002083 = $\frac{1}{2}$ *d.*

Conſequently their Sum, *viz.* 0,889577 = 17 *s.* 9 $\frac{1}{2}$ *d.*

Now to find the Value of Decimals in known Parts of Coin or Weights, *&c.* is only the Converſe of the former Work, and is thus performed.

Multiply the given Decimals with the Denominator of the Vulgar Fraction required: That is multiply the Decimals with ſuch a Number of Units as are contained in the next lower Denomination of that Kind or Species which your Decimal is of; and the Product will be the Number required.

EXAMPLE.

1. What is the Value of 0,825 Decimals of 1 Pound Sterling. That is, how many Shillings, Pence, *&c.* = ,825. Firſt, the next lower Denomination is 20, becauſe 20 *s.* make one Pound.

Therefore 0,825
20
Shillings 16,500 and Parts of 1 Shilling.
12
Pence 6,000 Anſwer 0,825 = 16 *s.* 6 *d.*

Again, What are the known Parts of Engliſh Coin equal to 3,666666 Decimals.

Here the 3 Integers are 3 Pounds. Then ,666666
20
Shillings 13,333320
12
Anſwer 3,666666 = 3 *l.* 13 *s.* 4 *d.* 006640
3 3332
Pence 3,999840 = 4 near.

What is the Value of 0,74722 Parts of 1 lb *Troy.*

First, ,74722	Then, ,96664	Again, 33280
12	20	24
1 49444	*Pwts.* 19,33280	1 3312
7 4722		6 656
Oz. 8,96664	*Oz.* *Pwt.* *Gr.*	7,98720

These collected are 8. 19. 8. very near.

And

And thus any propoſed Number of Decimals may be turned or changed into the known Parts of what they repreſent, *viz.* Whether they be Parts of Coin, Weights, Meaſures, or Time, &c.

I have omitted inſerting more Examples of this kind, becauſe I take the Excellency, and indeed the chief Uſe of Decimal Fractions to conſiſt more in Geometrical Computations than in the common or practical Parts of Arithmetick, as will appear further on; although even in thoſe they are very uſeful upon ſeveral Accounts; eſpecially in the Computations of Intereſt and Annuities, *&c.* But of that more in it's proper Place. I ſhall therefore conclude this Chapter with a Remark or two upon the Nature and Properties of Fractions in general.

If any given Number (whether it be whole or mixed) be multiplied with a Fraction either Vulgar or Decimal, the Product will be leſs than the Multiplicand, in ſuch a Proportion as the multiplying Fraction is leſs than an Unit or 1.

That is; as the Denominator of the Fraction is to it's Numerator, ſo will the given Number be to the Product.

Therefore, whenever any Number is to be multiplied with a Fraction, whoſe Numerator is an Unit: Divide that Number by the Denominator of the Fraction, and the Quotient will be the Product required. Thus $12 \times \frac{1}{4} = 3$. And $12 \div 4 = 3$. Again, $12 \times \frac{1}{2} = 6$. And $12 \div 2 = 6$, *&c.*

From hence it follows, that if any Number be divided by a Fraction, the Quotient will be greater than the Dividend, by ſuch a Proportion as Unity is greater than the dividing Fraction.

Thus $12 \div \frac{1}{4} = 48$, *viz.* $\frac{1}{4} : 1 :: 12 : 48$, *&c.* But the Truth of theſe will be beſt underſtood after the next Chapter.

CHAP. VI.

Of Continued Proportions, *and how to change or vary the Order of Things.*

Sect. 1. *Concerning* Arithmetical Progreſſion, *uſually called* Arithmetical Proportion Continued.

WHEN any Rank or Series of Numbers do either increaſe or decreaſe by an equal interval or common Difference; thoſe Numbers are ſaid to be in Arithmetical Progreſſion.

As

As { 1 . 2 . 3 . 4 . 5 . 6 . 7 &c. } { Here the Interval or
{ 7 . 6 . 5 . 4 . 3 . 2 . 1 } { common Difference is 1.

Or { 2 . 4 . 6 . 8 . 10 . 12 . 14 . &c. } { Here the common
{ 1 . 3 . 5 . 7 . 9 . 11 . 13 . &c. } { Difference is 2.

And so of any other Series, whose common Difference is 3 . 4 . 5 . &c.

Lemma 1.

If any three Numbers be in Arithmetical Progression, the Sum of the two Extreams (*viz.* the first and last) will be equal to the Double of the Mean or middle Number.

As in these, 2 . 4 . 6 . Or 3 . 6 . 9 . Or 3 . 7 . 11.
Viz. 2+6=4+4. Or 3+9=6+6. And 3+11=7+7. &c.

Lemma 2.

If any four Numbers are in Arithmetical Progression, the Sum of the two Extreams will be equal to the Sum of the two Means.

As in these, 2 . 4 . 6 . 8 . Or 3 . 6 . 9 . 12.
Viz. 2 + 8 = 4 + 6. And 3 + 12 = 6 + 9. &c.

Corollary 1.

From these two Lemma's it is easy to conceive, that if never so many Numbers be in Arithmetical Progression, the Sum of the two Extreams will be equal to the Sum of any two Means, that are equally distant from those Extreams.

As in these, 2 . 4 . 6 . 8 . 10 . 12 . 14 . 16.
Then 2 + 16 = 4 + 14 = 6 + 12 = 8 + 10.

Or if the Number of Terms be odd, as these,
2 . 4 . 6 . 8 . 10 . 12 . 14 . 16 . 18. &c.
Then 2 + 18 = 4 + 16 = 6 + 14 = 8 + 12 = 10 + 10.

Lemma 3.

Every Series of Numbers in Arithmetical Progression is composed of the Interval or common Difference, so often repeated as there are Terms in the Progression, except the first.

As in these, 1. 3. 5. 7. 9. 11. 13. 15. 17. &c.
Here the Interval or common Difference being two, it will be 1+2=3. 3+2=5. 5+2=7. 7+2=9. 9+2=11. 11+2=13. 13+2=15. 15+2=17. &c.

Corollary 2.

Hence it is evident, that the Difference betwixt the two Extreams (viz. 1 *and* 17) *is composed of the common Difference, multiplied into the Number of all the Terms, excepting the first.*

As in the aforesaid Progression, 1. 3. 5. 7. 9. 11. 15. 17.

The Number of Terms without the firſt is 8 } Multiply
The common Difference is 2 }

The Difference betwixt the two Extreams 16

Propoſition 1.

In any Series of Numbers in Arithmetical Progreſſion, the two Extreams, and the Number of Terms being given, thence to find the Sum of all the Series.

Theorem. *Multiply the Sum of the two Extreams into the Number of all the Terms; and divide the Product by* 2. *The Quotient will be the Sum of all that Series. Per Corol.* 1.

EXAMPLE 1.

It is required to find the Number of all the Strokes a Clock ſtrikes in one whole Revolution of the Index, *viz.* 12 Hours.

Here 1 + 12 = 13 the Sum of the two Extreams.
12 the Number of all the Terms.

26
13

Then 2) 156 (78. The Number of Strokes required.

EXAMPLE. 2.

Suppoſe one Hundred Eggs were placed in a Right Line a Yard diſtant from one another, and the firſt Egg were a Yard from a Baſket; whether or no may a Man gather up theſe 100 Eggs ſingly one after another, ſtill returning with every Egg to the Baſket and putting it in, before another Man can run four Miles. That is, which will run the greater Number of Yards.

In this Queſtion 200+2=202 Is the Sum of the two Extr.
And 100 Is the Number of all the Terms.

Then 2) 20200 (10100 { The Number of Yards he runs that takes up the Eggs.

Now 4 Miles = 7040 Yards } The Yards he runs that takes up
But 10100 − 7040 = 3060 } the Eggs more than the other.

Propoſition 2.

In any Series of Numbers in Arithmetical Progreſſion, the two Extreams and Number of Terms being given; thence to find the common Difference of all the Terms in that Series.

Theorem 2. *The Difference betwixt the two Extreams, being divided by the Number of Terms leſs than an Unit or* 1. *The Quotient will be the common Difference of the Series. Per Corol.* 2.

EXAMPLE

EXAMPLE 1.

One had Twelve Children that differed alike in all their Ages; the youngeſt was Nine Years old, the eldeſt was Thirty-ſix and a half; what was the Difference of their Ages, and the Age of each?

Here 36,5—9=27,5 The Difference of the two Extreams.
And 12—1=11. The Number of Terms leſs than an Unit.
Then 11) 27,5 (2,5 The common Difference required.
Conſequently 9+2,5=11,5 The Age of the youngeſt but one.
And 11,5+2,5=14 The Age of the youngeſt but two. And ſo on for the reſt. *Per Corol.* 2.

EXAMPLE 2.

A Debt is to be diſcharged at eleven ſeveral Payments to be made in Arithmetical Progreſſion. The firſt Payment to be Twelve Pounds Ten Shillings, and the laſt to be Sixty-three Pounds. What is the whole Debt, and what muſt each Payment be?

Per Theorem 1. Find the whole Debt thus:

12,5+63=75,5 The Sum of the Extreams.
1 1 The Number of Terms.
75 5
755
2) 830,5 (415,25=415 *l.* 5 *s.* The whole Debt.

Then, *per Theorem* 2. find the common Difference of each Payment.

Thus 63—12,5=50,5 The Difference of the Extreams.
And 11—1=10 The Number of Terms leſs 1.
Then 10) 50,5 (5,05=5 *l.* 1 *s.* The common Difference.

l. *s.* *l.* *s.* *l.* *s.*
Conſequently 12 . 11 + 5 . 1 = 17 . 11 The ſecond Payment.
l. *s.* *l.* *s.* *l.* *s.*
And 17 . 11 + 5 . 1 = 22 . 12 The third Payment, &c.

EXAMPLE 3.

A Man is to travel from *London* to a certain Place in ten Days, and to go but two Miles the firſt Day, increaſing every Day's Journey by an equal Exceſs; ſo that the laſt Day's Journey may be Twenty-nine Miles; what will each Day's Journey be, and how many Miles is the Place he goes to diſtant from *London*?

First 29—2=27 The Difference of the Extreams.
And 10—1=9 The Number of Terms leſs 1.
Then 9) 27 (3 The common Difference.
Conſequently 2+3=5 The ſecond Day's Journey.
And 5+3=8 The third Day's Journey, *&c.*
Again 29+2=31 The Sum of the Extreams.
10 The Number of Terms.
2) 310 (155 The Diſtance required.

There are eighteen Theorems more relating to Queſtions in Arithmetical Progreſſion; but becauſe they would require a great many Words to ſhew the Reaſon of them; I therefore refer the Reader to the Second Part, *viz.* That of *Algebra*, where he may find their *Analytical Inveſtigation*.

Sect. 2. Concerning Geometrical Proportion *continued*; ſometimes called *Geometrical Progreſſion.*

WHEN a Rank or ſeries of Numbers do either increaſe by one common Multiplicator, or decreaſe by one common Diviſor; Thoſe Numbers are ſaid to be in Geometrical Proportion continued.

As { 2 . 4 . 8 . 16 . 32 . *&c.* here 2 is the common Multiplier.
{ 64 . 32 . 16 . 8 . 4 . *&c.* here 2 is the common Diviſor.
Or { 2 . 6 . 18 . 54 . 162 . *&c.* here 3 is the common Multiplier
{ 162 . 54 . 18 . 6 . 2. here 3 is the common Diviſor.

Note, The common Multiplier (or Diviſor) is called the Ratio; and it ſhews the Habitude or Relation the Numbers have to one another, *viz.* whether they are Double, Triple, Quadruple, *&c.* which *Euclid* thus defines.

Ratio (*or* Rate) *is the mutual Habitude or Reſpect of two Magnitudes (conſequently two Numbers) of the ſame kind each to other, according to Quantity*, Euc. 5. Def. 3.

Proportion (rather Proportionality) is a Similitude of *Ratio*'s, Euc. 5. Def. 4.

So that there cannot be leſs than three Terms to form a Proportionality or Similitude of Ratio's; and if but three Terms, the ſecond muſt ſupply the Place of two, As in theſe 2 . 4 . 8 . That is, 2 : 4 :: 4 : 8 . (of :: ſee *page* 5.)

Here 4 the middle Term ſupplies the Place of two Terms, to wit, of the ſecond and third; 8 bearing the ſame Reaſon, Likeneſs,

Likeneſs, or Proportion to 4, as 4 doth to 2. *viz.* As 2 : is to 4 : : So is 4 : to 8.

Lemma 1.

If three Numbers are proportional, the Rectangle or Product of the two Extreams; *viz.* of the firſt and laſt Terms will be equal to the Square of the Mean or middle Term. (20 *Eucl.* 7.)

As in theſe 2 : 4 : : 4 : 8. Here 8 × 2 = 16 the Product of the Extreams.

And 4 × 4 = 16 the Square of the Mean. *Ergo* 8×2=4×4.

Corol. 1.

Hence it follows, that if the Product of any two Numbers be equal to the Square of a third Number; thoſe three Numbers will be in Proportion.

Lemma 2.

If four Numbers are proportional, the Product of the two Extreams will be equal to the Product of the two Means (19 *Euclid* 7.)

As in theſe, 2 : 4 : : 8 : 16. Here 16 × 2 = 32.

And 8 × 4 = 32. Conſequently 16 × 2 = 8 × 4.

Corol. 2.

From hence it follows, that if the Product of any two Numbers, be equal to the Product of any other two Numbers, thoſe four Numbers are Proportionals.

And from theſe two *Lemma's* it will be eaſy to conceive, that if never ſo many Numbers are in continued Proportion; the Product of the two Extreams, will be equal to the Product of any two Means, that are equally diſtant from the Extreams.

As in theſe 2 . 4 . 8 . 16 . 32 . 64 . *&c.*

Here 64 × 2 = 32 × 4 = 16 × 8. *&c.* And if the Number of Terms be odd,

As in theſe 2 . 4 . 8 . 16 . 32 . 64 . 128 . *&c.*

Then 128 × 2 = 64 × 4 = 32 × 8 = 16 × 16.

Note, *The Character made Uſe of to ſignify continued Proportionals is* ∺.

In

In every Series of ∴ (*viz. of continued Proportionals*) that Number which is compared to another, is called the Antecedent of the Ratio; and that Number to which it is compared, is called it's Conſequent.

As in theſe, 2 : 4 :: 4 : 8. Here 2 is the Antecedent, and 4 is the Conſequent; and 4 the middle Term is an Antecedent to 8 it's Conſequent: whence it follows, that in every Series of ∴ all the middle Terms between the firſt and laſt are both Antecedents and Conſequents.

As in theſe, 2 . 4 . 8 . 16 . 32 . 64 &c. Here 4 . 8 . 16 . 32. are both Conſequents and Antecedents.

For 2 : 4 :: 4 : 8 :: 8 : 16 :: 16 : 32 :: 32 : 64 &c.

So that all the Terms except the laſt are Antecedents. And all the Terms except the firſt are Conſequents.

Lemma 3.

If never ſo many Numbers are proportional, it will be: As any one of the Antecedents is to it's Conſequent, So will the Sum of all the Antecedents be; to the Sum of all the Conſequents. (12 *Euclid* 5.)

That is, in the foregoing Series,

2 : 4 :: 2+4+8+16+32 : 4+8+16+32+64.

For it is evident, that 4+8+16+32+64 the Sum of all the Conſequents, is double to 2+4+8+16+32 the Sum of all the Antecedents; as 4 is to 2, according to the Ratio, and would have been Triple, or Quadruple, &c. had the Ratio been 3 or 4, &c.

Note, *In every Series of ∴ the Ratio is found by dividing any of the Conſequents by it's Antecedents.*

As in theſe 2 : 6 :: 6 : 18 :: 18 : 54 :: 54 : 162.

Here 2) 6 (3 the Ratio. Or 6) 18 (3 &c.

From the ſecond and third Lemma's may be raiſed two general Theorems or Rules, for finding the Sum of any Series in ∴ without a continued Addition of all the Terms.

Let the Series 2 . 4 . 8 . 16 . 32 . 64 . 128 . be given, to find it's Sum.

Suppoſe z = the Sum of all the Terms.

Then will $z - 128$ = the Sum of all the Antecedents.

And $z - 2$ = the Sum of all the Conſequents.

But 2 : 4 :: $z - 128$: $z - 2$. *per Lemma* 3.

Ergo $4z - 512 = 2z - 4$. *per Lemma* 2.

Conſequently

Consequently $4z - 2z = 512 - 4$.

Theorem. $\left\{ z = \frac{512-4}{4-2} \right.$ In Words at Length thus,

Theorem 1. *From the Product of the second and last Terms subtract the Square of the first Term; and that Remainder being divided by the second Term less the first will give the Sum of all the Series.*

Or if the first Term, the common Ratio, and the last Term be only given. Then,

Theorem 2. *Multiply the last Term into the Ratio, and from their Product subtract the first Term; divide that Remainder by the Ratio less Unity or 1, and it will give the Sum of all the Series.*

For $4z - 2z = 512 - 4$. As above.
Consequently $2z - z = 256 - 2$ *viz.* the last divided by 2.

Then $z = \frac{256-2}{2-1}$ *Theorem* 2.

EXAMPLE.

Let 2 . 6 . 18 . 54 . 162 . 486. be the given Series. Here 2 is the first Term, 3 is the Ratio, and 486 the last Term.

But $486 \times 3 = 1458$. And $1458 - 2 = 1456$.
Then $3 - 1 = 2)$ 1456 (728 the Sum required.
That is, $728 = 2 + 6 + 18 + 54 + 162 + 486$.

Since in either of these Theorems it is required to have the last Term known (the which in a long Series of ÷÷ will be very tedious to come at by a continued Multiplication; it will therefore be convenient to shew how to obtain either the last Term or any other Term, whose Place is assigned, without producing all the Terms.

In order to that, it will be necessary to premise the Coherence or Similitude that is betwixt Numbers in Arithmetical Progression and those in Geometrical Proportion.

If to any Series of Numbers in ÷÷ when the first Term is not an Unit or 1, there be assigned a Series of Numbers in Arithmetical Progression, beginning with an Unit or 1, and whose common Difference is 1. called Indices or Exponents.

Thus, $\left\{ \begin{matrix} 1 . 2 . 3 . 4 . 5 . 6 . 7 & \text{Indices} \\ 2 . 4 . 8 . 16 . 32 . 64 . 128 \ \&c. \div\div \end{matrix} \right.$

Then

Then will the Addition or Subſtraction of any two of thoſe Indices (or Numbers in Arithmetical Progreſſion) directly correſpond with the Product, or Quotient of their reſpective Terms in the Series of ∺.

That is, { As 3 + 4 = 7 / So 8 × 16 = 128 the ſeventh Term in ∺

Again, { As 6 + 4 = 10 / So 64 × 16 = 1024. the tenth Term in ∺

Or, { As 7 − 3 = 8. / So 128 ÷ 8 = 16. } Or, { As 6 − 2 = 4 / So 64 ÷ 4 = 16 } &c.

But if the Series of ∺ begin with an Unit, the Indices muſt begin with a Cypher.

As in theſe, { 0 . 1 . 2 . 3 . 4 . 5 . 6. &c. / 1 . 2 . 4 . 8 . 16 . 32 . 64.

Now by the help of the Indices, and a few of the firſt Terms in any Series of ∺ It is plain that any Term whoſe Place or Diſtance from the firſt Term is aſſigned, may be ſpeedily obtained without producing the whole Series.

EXAMPLE 1.

A Man bought a Horſe, and was to give a Farthing for the firſt Nail, two for the ſecond, four for the third, &c. in ∺, the Number of Nails was to be 7 in every Shoe, *viz.* 28 Nails in all. What muſt he have paid for the Horſe?

Firſt { 0 . 1 . 2 . 3 . 4 . 5. Indices / 1 . 2 . 4 . 8 . 16 . 32. Farthings in ∺

Then, { 5 + 5 = 10 / 32 × 32 = 1024 } And { 10 + 10 = 20 / 1024 × 1024 = 1048576

Again, { 4 + 3 = 7 / 16 × 8 = 128 } Laſtly, { 20 + 7 = 27 / 1048576 × 128 = 134217728

Which is here to be accounted the 28 and laſt Term. Becauſe the firſt Term in the Series is 1, which doth neither multiply nor divide.

Now this 134217728 being the Number of Farthings to be paid for the laſt Nail, by it the common Ratio which is 2, and the firſt Term which is 1, may be found the Sum of all the Series, *per Theorem* 2.

134217728
2
268435456 From this Product ſubſtract 1.

Viz. 268435456—1=268435455. Then 2—1=1 the Diviſor.

Conſequently 268435455 is the Sum of all the Series, or Price of the Horſe in Farthings, which being brought into Pounds, (ſee *Page* 46.) will be 279620 *l.* 5 *s.* 3 *d.* 3 *qrs.*

EXAMPLE 2.

A cunning Servant agreed with a Maſter (unſkilled in Numbers) to ſerve him Eleven Years without any other Reward for his Service but the Produce of one Wheat Corn for the firſt Year; and that Product to be ſowed the ſecond Year, and ſo on from Year to Year until the end of the Time, allowing the Increaſe to be but in a ten-fold Proportion.

It is required to find the Sum of the whole Produce.

First { 1 . 2 . 3 . 4 . 5 . Indices or Years.
10 . 100 . 1000 . 10000 . 100000 Wheat Corns in ÷÷

Then { As 4 + 2 = 6
So 10000 × 100 = 1000000 the 6th Year's Produce

And { 6 + 5 = 11
1000000 × 100000 = 100000000000. The eleventh or laſt Year's Produce.

Then (either by *Theorem* 1, or 2) the Sum of all the Series will be 11111111111110 Corns. Now it may be computed from Page 31 and 34, that 7680 Wheat Corns, round and dry out of the middle of the Ear, will fill a Statute Pint. If ſo,

Then 7680) 11111111111110 (14467592 Pints, but 64 Pints are contained in a Buſhel.

Therefore 64) 14467592 (226056 $\frac{1}{8}$ Buſhels. Suppoſe it to be ſold for 3 Shillings the Buſhel;

Then { 226056 $\frac{1}{8}$
3
Shillings 678168 $\frac{3}{8}$ = 33908 *l.* 8 *s.* 4 $\frac{1}{2}$ *d.* A very good Recompence for Eleven Years Service.

There are ſeveral pretty Queſtions reſolved by Numbers in Arithmetical Progreſſion; and by thoſe in ÷÷ which the ingenious Learner will eaſily perceive hereafter; *viz.* When we come to the Solution of Queſtions relating to Intereſt and Annuities, &c.

There is alſo a third Kind of Proportion, called Muſical, which being but of little or no common Uſe, I ſhall therefore give but a ſhort Account of it.

Muſical Proportion or Habitude is, when of three Numbers; the firſt hath the ſame Proportion to the third, as the Difference between the firſt and ſecond hath to the Difference between the ſecond and third.

As in theſe, 6 . 8 . 12. *viz.* 6 : 12 :: 8 — 6 : 12 — 8

If there are four Numbers in Muſical Proportion; The firſt will have the ſame Proportion to the fourth, as the Difference between the firſt and ſecond hath to the Difference between the third and fourth.

As in theſe, 8 . 14 . 21 . 84.

Here 8 : 84 :: 14 — 8 = 6 : 84 — 21 = 63.

That is, 8 : 84 :: 6 : 63.

The Method of finding out Numbers in Muſical Proportion, is beſt expreſſed by Letters; as ſhall be ſhewed in the *Algebraick Part.*

Sect. 3. *How to* Change *or* Vary *the Order of Things*, &c.

THIS being a Thing not treated of in any common Books of Arithmetick (that I have had the opportunity of peruſing) made me think it would be acceptable to the young Learner to know how often it is poſſible to vary or change the Order or Poſition of any propoſed Number of Things.

As how many ſeveral Changes may be rung upon any propoſed Number of Bells; or how many ſeveral Variations may be made of any determined Number of Letters, or any other Things propoſed to be varied.

The Method of finding out the Number of Changes is by a continual Multiplication of all the Terms in a Series of Arithmetical Progreſſions, whoſe firſt Term and common Difference is Unity or 1. *And the laſt Term the Number of Things propoſed to be varied,* viz. 1×2×3×4×5×6×7, *&c.* As will appear from what follows.

1. If the Things propoſed to be varied are only two, they admit of a double Poſition (as to Order of Place) and no more.

Thus, $\left\{\begin{matrix} 1 . 2 \\ 2 . 1 \end{matrix}\right\} = 2 = 1 \times 2$

2. And if three Things are propoſed to be varied, they may be

be changed ſix ſeveral Ways (as to their Order of Place) and no more.

For, beginning with 1, there will be { 1 . 2 . 3 / 1 . 3 . 2

Next, beginning with 2, there will be { 2 . 1 . 3 / 2 . 3 . 1

Again, beginning with 3, it will be { 3 . 1 . 2 / 3 . 2 . 1

Which in all make 6 or 3 Times 2, *viz.* $1 \times 2 \times 3 = 6$

Suppoſe four Things are propoſed to be varied; Then they will admit of 24 ſeveral Changes, as to their Order of different Places.

For beginning the Order with 1 it will be { 1 . 2 . 3 . 4 / 1 . 2 . 4 . 3 / 1 . 3 . 2 . 4 / 1 . 3 . 4 . 2 / 1 . 4 . 2 . 3 / 1 . 4 . 3 . 2

Here are ſix different Changes.

And for the ſame Reaſon there will be 6 different Changes, when 2 begins the Order, and as many when 3 and 4 begins the Order; which in all is $24 = 1 \times 2 \times 3 \times 4$. And by this Method of proceeding, it may be made evident, that 5 Things admit of 120 ſeveral Variations or Changes; and 6 Things of 720, *&c.* As in this following Table.

The Number of Things propoſed to be varied.	*The manner how their ſeveral Variations are produced.*	*The different Changes or Variations every one of the propoſed Numbers can admit of.*
1	1	= 1
2	1 × 2	= 2
3	2 × 3	= 6
4	6 × 4	= 24
5	24 × 5	= 120
6	120 × 6	= 720
7	720 × 7	= 5040
8	5040 × 8	= 40320
9	40320 × 9	= 362880
10	362880 × 10	= 3628800
11	3628800 × 11	= 39916800
12	39916800 × 12	= 479001600
&c.	*&c.*	*&c.*

Theſe may be thus continued on to any aſſigned Number. Suppoſe to 24 the Number of Letters in the Alphabet, which will admit of 620448401733239439360000 ſeveral Variations.

From theſe Computations may be ſtarted ſeveral pretty, and indeed, very ſtrange Queſtions.

EXAMPLES.

Six Gentlemen, that were travelling, met together by Chance at a certain Inn upon the Road, where they were ſo pleaſed with their Hoſt, and each other's Company, that in a Frolick they made a Contract to ſtay at that Place, ſo long as they, together with their Hoſt, could ſit every Day in a different Order or Poſition at Dinner; which by the foregoing Computations will be found near 14 Years. For they being made 7 with their Hoſt, will admit of 5040 different Poſitions; but 5040 being divided by 365 $\frac{1}{4}$ (the Number of the Days in one Year) will give 13 Years and 291 Days. A very pretty Frolick indeed.

I have been told, that before the Fire of *London* (which happened *Anno* 1666) there were 12 Bells in St *Mary Le Bow*'s Church in *Cheapſide*, *London*. Suppoſe it were required to tell how many ſeveral Changes might have been rung upon thoſe 12 Bells; and at a moderate Computation how long all thoſe Changes would have been ringing but once over.

Firſt, $1\times2\times3\times4\times5\times6\times7\times8\times9\times10\times11\times12=479001600$, the Number of Changes.

Then ſuppoſing there might be rung 10 Changes in 1 Minute: *viz.* $12\times10=120$ Strokes in a Minute, which is 2 Strokes in a Second of Time: Now according to that Rate there muſt be allowed 47900160 Minutes to ring them once over in all their different Changes; *viz.* 10) 479001600 (47900160.

In one Year there are 365 Days, 5 Hours, and 49 Minutes; which, being reduced into Minutes, is 525949.

Then 525949) 47900160 (91 Years and 26 Days.

So long would thoſe 12 Bells have been continually ringing without any Intermiſſion, before all their different Changes could have been truly rung but once over. It is ſtrange, and ſeems almoſt incredible, that a few Things ſhould produce ſuch Varieties.

But that which ſeems yet more ſtrange and ſurpriſing (yea, even impoſſible to thoſe who are not verſed in the Power of Numbers) is,

is, that if two Bells more had been added to the aforeſaid 12 they would have advanced the Number of Changes (and conſequently the Time) beyond common Belief. For 14 Bells would require (at the ſame rate of ringing as before) about 16575 Years to ring all their different Changes but once over.

And if it were poſſible to ring 24 Bells in Changes (and at the ſame rate of 10 Changes in a Minute, which is 2 Strokes in one Second) they would require more than 1170000000000000 Years to ring them but once over in all their different Changes; as may eaſily be computed from the precedent Table.

CHAP. VII.

Of Proportion Disjunct; *commonly called the* Golden Rule.

Proportion Disjunct or the **Golden Rule**, is either Direct or Reciprocal, called Inverſe. And thoſe are both Simple and Compound.

SECT. 1.

Direct Proportion is, when of four Numbers, the firſt bearing the ſame Ratio or Proportion to the ſecond; as the third doth to the fourth.

As in theſe 2 : 8 :: 6 : 24.

Conſequently, the greater the ſecond Term is, in reſpect to the firſt; the greater will the fourth Term be, in reſpect to the third.

That is, as 8 the ſecond Term is 4 Times greater than 2 the firſt Term: So is 24 the fourth Term, 4 Times greater than 6 the third Term.

Whence it follows, that if four Numbers are in Direct Proportion, the Product of the two Extreams will always be equal to the Product of the two Means, as well in Disjunct as in continued Proportion; according to *Lemma* 2. *page* 77.

For As 2 : 2 × 4 :: 6 : 6 × 4. Or As 3 : 3 × 5 :: 6 : 6 × 5.
But 2 × 6 × 4 = 2 × 4 × 6. Or 3 × 6 × 5 = 3 × 5 × 6.

That is, the Product of the Extreams is equal to that of the Means.

Again,

Again, the leſs the ſecond Term is, in reſpect to the firſt; the leſs will the fourth Term be in reſpect to the third.

As in theſe 18 : 6 :: 12 : 4.

That is, 18 : 18 ÷ 3 :: 12 : 12 ÷ 3.

But 18 × 12 ÷ 3 = 18 ÷ 3 × 12. *Viz.* 18 × 4 = 6 × 12.

Conſequently 2 . 8 . 6 . 24. And 18 . 6 . 12 . 4 . are true Proportionals, *per Corol.* 2. *page* 77.

From theſe Conſiderations, comes the Invention of finding a fourth Number in Proportion to any three given Numbers. *Whence it is called the Rule of Three.*

For if the ſecond Number multiplied into the third, be equal to the firſt multiplied into the fourth, it is eaſy to conceive, that if the Product of the ſecond and third be divided by the firſt, the Quotient muſt needs be the fourth Number. For if that Number, which divides another, be multiplied into the Quotient produced by that Diviſion; their Product will be equal to the Number divided. *See page* 21.

As in theſe 2 : 8 :: 6 : 24. Here 8 × 6 = 48 = 24 × 2.

But if 24×2=48, then will 48÷2=24. Or 48÷24=2.

Note, Any four Numbers in direct Proportion may be varied ſeveral ways. As in theſe.

Viz. If 2 : 8 :: 6 : 24. Then 2 : 6 :: 8 : 24.

And 6 : 24 :: 2 : 8. Or 24 : 6 :: 8 : 2. *&c.*

Theſe Variations being well underſtood, will be of no ſmall Uſe in the ſtating of any Queſtion in this Rule of Three.

When three Numbers are given, and it is required to find a fourth Proportional; the greateſt Difficulty (if there be any) will be in the right ſtating the Queſtion, or abſtracting the Numbers out of the Words in the Queſtion, and placing them down in their proper Order.

Now this will be very eaſy, if it be truly conſidered, that always two of the three given Terms, are only ſuppoſed, and aſſign or limit the Ratio or Proportion. The third moves the Queſtion; and the fourth gives the Anſwer.

As for inſtance; if 3 Yards of Cloth coſt 9 Shillings: What will 6 Yards coſt at the ſame Rate or Proportion?

Here 3 Yards, and 9 Shillings, are two ſuppoſed Numbers that imply the Rate; as appears by the Word [if] *viz.* If 3 Yards coſt 9 Shillings (then comes the Queſtion) What will 6 Yards coſt?

N. B.

N. B. The Term, which moves the Queſtion, hath generally ſome of thoſe Words before it; *viz.* **What will? How many? How long? How far?** or **How much?** &c.

Then (carefully obſerve this; *viz.*) The firſt Term in the Suppoſition muſt always be of the ſame kind and Denomination with that Term which moves the Queſtion. And the Term ſought will always be of the ſame kind and Denomination with the ſecond Term in the Suppoſition.

Thus, { *yds. ſhil. yds. ſhil.* 3 : 9 :: 6 : ——— } Then

All Queſtions in direct Proportion may be anſwered by three ſeveral *Theorems*.

Theorem 1. { *Multiply the ſecond and third Terms together, and divide their Product by the firſt Term; the Quotient will be the Anſwer required.*

yds ſhil. yds ſhil.
Thus 3 : 9 :: 6 : 18. The Anſwer.
6
——
3) 54 (18 Shillings, { becauſe the ſecond Term was Shillings.

Theorem 2. { *Divide the ſecond Term by the firſt, then multiply the Quotient into the third Term; and their Product will be the Anſwer required.*

yds ſhil. yds. ſhil.
3 : 9 :: 6 : 18.
Thus 3) 9 (= 3. Then 3 × 6 = 18, as before.

Theorem 3. { *Divide the third Term by the firſt, then multiply the Quotient into the ſecond Term, and their Product will be the Anſwer.*

yds ſhil. yds ſhil.
3 : 9 :: 6 : 18.
Thus) 6 (= 2. And 9 × 2 = 18, as before.

Here you ſee that all the three *Theorems* are equally true; but the firſt is moſt general, and uſually practiſed. Yet the two laſt may be readily performed, when either the ſecond or third Term can be divided by the firſt; and will be found of ſingular Uſe in the *Rules of Fellowſhip*, &c. as will appear further on.

Queſt.

Queſt. 2. If 8 Pounds of Tobacco coſt 14 *Shillings*; what will half a hundred Weight (*viz.* 56 Pounds) coſt at the ſame Rate?

Thus 8 lb : 14 *s.* :: 56 lb : 4 *l.* 18 *s.* The Anſwer.

14

224
56

8) 784 (= 98 *s.* = 4 *l.* 18 *s.*

Or thus 8) 56 (= 7 Then 14 × 7 = 98 *s.* as before.

Queſt. 3. If 14 *Shillings* will buy 8 Pounds of *Tobacco*; how much will 4 *l.* 18 *s.* buy after the ſame Rate?

Stated thus, 14 : 8 lb :: 4 *l.* 18 *s.* = 98 *s.* : ——
Then 98 × 8 = 784. And 14) 784 (56 lb. The Anſwer.

Queſt. 4. If half a hundred Weight of *Tobacco* be worth 4 *l.* 18 *s.* How much may I buy for 14 *Shillings* at that Rate?

Stated thus, 4 *l.* 18 *s.* = 98 *s.* : 56 lb :: 14 *s.* : ——
Then 56 × 14 = 784. And 98) 784 (8 lb. The Anſwer.

Queſt. 5. Suppoſe 4 *l.* 18 *s.* will buy 56 Pounds of *Tobacco*; what will 8 Pounds of the ſame *Tobacco* coſt?

This Queſtion is thus ſtated, 56 lb : 4 *l.* 18 *s.* = 98 *s.* :: 8 lb :—
Then 98 × 8 = 784. And 56) 784 (= 14 *s.* The Anſwer.

Note, The three laſt Queſtions are only the ſecond varied, being propoſed purely to give an Inſtance how any Queſtion in this *Rule of Three* may be varied, according to *page* 86.

Queſt. 6. What will three quarters of a Yard of *Velvet* coſt, when the Price of 21 Yards and a half is worth 22 *l.* 10 *s.* 6 *d.* This Queſtion truly ſtated will ſtand

Thus, 21 $\frac{1}{2}$ *yds* : 22 *l.* 10 *s.* 6 *d.* :: $\frac{3}{4}$ to the Anſwer.

Which may be found three ſeveral Ways; *viz.* by *Reduction*; by *Vulgar Fractions*; and by *Decimals*.

1. By *Reduction*. Bring the firſt and third Terms into one Denomination; *viz.* into Quarters, and reduce the ſecond Term into it's leaſt Denomination, *per Sect.* 4. *page* 42.

Thus 21 $\frac{1}{2}$ = 86 Quarters. And 22 *l.* 10 *s.* 6 *d.* = 5406 *Pence.*
Then 86 : 5406 :: 3 : 15 *s.* 8 $\frac{10}{86}$ *d.* For 5406 × 3 = 16218.

And

And 86) 162 18 (=188 $\frac{50}{86}$ *d.* Then 188 $\frac{50}{86}$ *Pence* = 15 *s.* 8 *d.* 2 $\frac{14}{43}$ *Farthings*; the Anſwer required.

2. The ſame Queſtion ſtated in *Vulgar Fractions* will ſtand Thus 21 $\frac{1}{2}$ = $\frac{43}{2}$: 22 $\frac{21}{40}$ = $\frac{901}{40}$ :: $\frac{3}{4}$: See *Sect.* 3. *(page* 50.) Then $\frac{901}{40} \times \frac{3}{4} = \frac{2703}{160}$. And $\frac{43}{2}$) $\frac{2703}{160}$ (= $\frac{5406}{6880}$ *page* 55, 56.

Theſe $\frac{5406}{6880}$ Parts of a Pound are brought into *Shillings* by Multiplying the Numerator with 20, and dividing the Product by it's Denominator, *&c.*

Thus 5406 x 20 = 108120. And 6880) 108120 (15 *s.* And there remains 4920. Again 4920 x 12 = 59040. Then 6880) 59040 (8 *d.* and $\frac{560}{860}$, as before.

3. The ſame wrought by *Decimal Fractions* will be thus; 21 $\frac{1}{2}$ = 21,5; 22 *l.* 10 *s.* 6 *d.* = 22,525, and $\frac{3}{4}$ = 0,75 Therefore 21,5 : 22,525 :: 0,75 : to the Anſwer. Then 22,525 x 0,75 = 16,89375 And 21,5) 16,89375 (0,7857 *l.* = 15 *s.* 8 *d.* 2 *far.* $\frac{2772}{10000}$

Queſt. 7. If 2 *C.* 3 *qrs.* 21 lb. of *Sugar* coſt 6 *l.* 1 *s.* 8 *d.* What will 12 *C.* 2 *qrs.* coſt at the ſame Rate? That is, 2 *C.* 3 *qrs.* 21 lb : 6 *l.* 1 *s.* 8 *d.* :: 12 *C.* 2 *qrs.* To what?

4	20	4
11 *qrs.*	121 *s.*	50 *qrs.*
28	12	28
88	250	1400 lb.
22	121	

Viz. 308+21=329lb : 1460 *d.* :: 1400 lb : ——

Then 1460x1400=2044000. And 329) 2044000 (6212 $\frac{3}{4}$ *d.* = 25 *l.* 17 *s.* 8 $\frac{3}{4}$ *d.* the Anſwer required.

The ſame Queſtion ſtated in Decimals will ſtand

Thus 2,9375 : 6,0833 :: 12,5 : To the Anſwer.

Then 6,0833 x 12,5 = 76,04125 which being divided by 2,9375 will give 25,8863, *&c.* the Anſwer in Decimals, which brought into Coin, will be 25 *l.* 17 *s.* 8 $\frac{3}{4}$ *d.* as before.

Note, *When the firſt Term is an Unit or* 1, *the Queſtion is anſwered by Multiplication only.*

Example. Suppoſe I give 5 *Shillings* 4 *Pence* for one Ounce of Silver, What muſt I pay for 32 $\frac{1}{2}$ Ounces at the ſame Rate?

That is 1 Ounce : 5 *s.* 4 *d.* :: 32 $\frac{1}{2}$ Ounces : To, *&c.* Which is beſt ſtated thus 1 : 64 *d.* :: 32,5 :

Then $32,5 \times 64 = 2080\ d. = 8\ l.\ 13\ s.\ 4\ d.$ the Anſwer required. For 1 neither multiplies nor divides.

When the ſecond or third Term is an Unit or 1, then the Queſtion is anſwered by Diviſion only. As in this *Example.*

If a Silver Tankard weighing 21 Ounces, coſt 5 *l.* 19 *s.* What is that an Ounce?

Thus 21 *oz.* : 5 *l.* 19 *s.* = 119 *s.* : : 1 : 5 *s.* 8 *d.* the Anſwer.
That is 21) 119 (= 5 *s.* $\frac{14}{21}$ = 5 *s.* 8 *d.*

The Proof of all Queſtions in the *Rule of Three Direct*, may be eaſily conceived from what hath been already ſaid; *viz.* That the Product of the firſt and fourth Terms, muſt always be equal to the Product of the ſecond and third Terms.

Or otherwiſe, by varying the Queſtion, as in the ſecond, third, fourth, and fifth Queſtions.

I ſhall conclude this Section with inſerting a few Queſtions and their Anſwers; leaving their Work for the Learner's Practice.

Queſt. 1. What will the Carriage of 17 *C.* 3 *qrs.* 11 lb. come to, at the Rate of 7 *s.* the Hundred?

Anſwer 6 *l.* 4 *s.* 11 $\frac{1}{4}$ *d.*

Queſt. 2. If 6 *l.* 4 *s.* 11 $\frac{1}{4}$ *d.* be paid for the Carriage of 17 *C.* 3 *qrs.* 11 lb; What was paid for the Carriage of 1 lb.

Anſwer 3 Farthings.

Queſt. 3. A *Grocer* bought 3 *C.* 1 *qr.* 14 lb Weight of *Cloves*, at the Rate of 2 *s.* 4 *d.* *per Pound*, and ſold them for 52 *l.* 14 *s.* Whether did he gain or loſe by the Bargain, and how much?

Anſwer, he gained 8 *l.* 12 *s.*

Queſt. 4. A *Draper* bought of a Merchant eight Packs of *Cloth*; every Pack had four Parcels in it; and each Parcel contained ten Pieces; every Piece was Twenty-ſix Yards; he gave after the Rate of four Pounds ſixteen Shillings for 6 Yards. What came the eight Packs to, and what were they worth *per Yard?*

Anſw. They came to 6656 *l.* And were worth 16 *s.* *per Yard.*

Queſt. 5. A Merchant bought 436 Yards of Broad *Cloth* for 8 *s.* 6 *d.* *per Yard*; and ſold it again for 10 *s.* 4 *d.* *per Yard.* What did he gain by the 436 *Yards?*

Anſwer, he gained 39 *l.* 19 *s.* 4 *d.*

Queſt.

Queſt. 6. A *Goldſmith* bought a Wedge of *Gold*, which weighed 14 lb. 3 *oz.* 8 *pw.* for 514 *l.* 4 *s.* What did he pay *per Ounce?*

Anſw. 3 *l per Ounce.*

Queſt. 7. What will 48 *oz.* 17 *pw.* 20 *Grains of Silver* Plate come to, at the Rate of 5 *s.* 6 *d. per Ounce?*

Anſw. 13 *l.* 1 *s.* 10 $\frac{3}{4}$ *d.*

Queſt. 8. If in four Weeks one ſpend 13 *s.* 4 *d.* How long will 53 *l.* 6 *s.* laſt at that Rate?

Anſw. 6 Years, 47 Days, 2 Hours, 24 *′.*

Queſt. 9. What will the one eighth Part of a Ship be worth, when the half is valued at 1015 *l.* 10 *s.*

Anſw. 253 *l.* 17 *s.* 6 *d.*

Queſt. 10. The Sun is ſaid to perform one entire Revolution (or 360 Degrees) in the Space of 365 Days, 5 Hours, 48 Minutes, and 57 Seconds of Time, called a *Tropical* or *Solar Year*; How much doth it move in one Day?

Anſw. 59′, 8″, 19‴, *&c.*

Queſt. 11. If $\frac{5}{8}$ of a Yard of *Velvet* coſt $\frac{2}{3}$ of a *Pound Sterling*, What will $\frac{1}{16}$ of a Yard coſt of the ſame *Velvet* at that Rate?

Anſw. $\frac{16}{240} = 1$ *s.* 4 *d.*

Queſt. 12. Suppoſe 2 *l.* and $\frac{3}{8}$ of $\frac{1}{3}$ of a *Pound Sterling* will buy 3 Yards and $\frac{2}{3}$ of $\frac{3}{5}$ of a Yard of *Cloth*, How much will $\frac{3}{4}$ of a Yard coſt at that Rate?

Anſw. $\frac{225}{480}$ of a Pound $= 9$ *s.* 4 $\frac{1}{2}$ *d.*

Sect. 2. *Of* Reciprocal Proportion; *uſually called* The Rule *of* Three Inverſe.

Reciprocal Proportion is, when of four Numbers the third (*viz.* that which moves the Queſtion) beareth the ſame Ratio to the firſt: As the ſecond does to the fourth.

Therefore, the leſs the third Term is, in reſpect to the firſt; the greater will the fourth Term be, in reſpect to the ſecond.

EXAMPLE 1.

If ſixteen Men can do a Piece of Work in ſix Days; How many Days muſt eight Men require to do the ſame Work, at the ſame Rate of Working?

Here it is plain that eight Men muſt needs have more Time than 16 Men to do the ſame Work. Conſequently the greater

the third Term is, in reſpect to the firſt, the leſſer will the fourth Term be in reſpect to the ſecond.

Example 2. If 8 Men can do a Piece of Work in 12 Days, How many Days will 16 Men require to do the ſame Work? Here it is plain the fourth Term muſt be leſs than the ſecond, becauſe 16 Men undoubtedly can do the ſame Work in leſs Time than 8 Men can.

From theſe Conſiderations, compared with thoſe in page 85. *it will be eaſy to perceive, whether the Terms of any propoſed Queſtion are in Direct or Reciprocal Proportion.*

For when, according to the true Meaning and Deſign of any Queſtion in Proportion, More requires More, or Leſs requires Leſs, the Terms are in Direct Proportion, as in the laſt Section.

But if More requires Leſs, or Leſs requires More (as above), then the Terms will be in Reciprocal Proportion.

The Manner of placing down the propoſed Terms is the ſame in both Rules, *viz.* The firſt Term in the Suppoſition muſt be of the ſame Kind and Denomination with the third Term which moves the Queſtion; and the Term ſought muſt be of the ſame Kind and Denomination with the ſecond Term in the Suppoſition. As in the two laſt *Examples*.

		Men	*Days*	*Men*	*Days*
Thus, in	*Example* 1.	16 :	6 ::	8 :	——
	Example 2.	8 :	12 ::	16 :	——

The Queſtion being truly ſtated, obſerve this *Theorem*.

Theorem. *Multiply the firſt and ſecond Terms together, and divide the Product by the third Term, the Quotient will be the Anſwer required.*

Thus in the ſecond *Example* $12 \times 8 = 96$.

Then 16) 96 (= 6 Days the Anſwer required.

That is, 16 Men may do the ſame Work in 6 Days as 8 Men can do in 12 Days.

Now the Reaſon of this Operation (and conſequently of the *Theorem*) is grounded upon this Conſideration; *viz.* If 8 Men require 12 Days to do the Work, it is plain that one Man would require 8 Times 12 Days = 96 Days to do the ſame Work; but if 1 Man can do it in 96 Days, moſt certainly 16 Men can do it in one 16th Part of that Time. Therefore 96 divided by 16 will give the Anſwer required, *viz.* 16) 96 (6 as before, *&c.*

Queſt. 3. Suppoſe 800 Soldiers were beſieged in a Town, and their Victuals were computed to ſerve them two Months (or 56 Days) How many of thoſe Soldiers muſt depart the Garriſon, that the ſame Victuals may ſerve the remaining Soldiers 5 Months.

The

The Queſtion truly ſtated will ſtand

Months Soldiers Months Soldiers

Thus, 2 : 800 :: 5 : ——

2

5) 1600 (320 : So many Soldiers may ſtay in the Garriſon.

Conſequently, 800 — 320 = 480 Soldiers that muſt go out of the Garriſon, which is the Anſwer required.

Queſtion 4. *A* borrowed of his Friend *B* 250 *l.* for ſix Months, promiſing to do him the like Kindneſs upon Demand: Some Time after *B* deſires *A* to lend him 400 *l.* the Queſtion is, how long *B* muſt keep the 400 *l.* to be fully ſatisfied for his former Kindneſs to *A.*

Thus, 250 *l.* : 6 Months :: 400 *l.* : ——

6

400) 1500 (3 Months.

12

3

28 Days in one Month.

4) 84 (21 Days. Anſw. 3 Months, 21 Days.

Queſtion 5. If a Penny White Loaf ought to weigh eight Ounces *Troy Weight*, when Wheat is ſold for Six Shillings Six-Pence the Buſhel, what muſt it weigh when Wheat is ſold for four Shillings the Buſhel?

Thus 6 *s.* 6 *d.* = 78 *d.* : 8 *oz.* :: 4 *s.* = 48 *d.* : to the Anſwer

8

48) 624 (13 *oz.* the Anſwer required.

48

144

144

(0)

The Proof of this Inverſe Rule is eaſily deduced from it's Operations; *viz.* The Product of the firſt and ſecond Terms, muſt be equal to the Product of the third and fourth Terms.

Note, Any Queſtion that falls under this Inverſe Rule or Reciprocal Proportion, may be ſo ſtated as to have it's Terms in Direct Proportion; by only changing the Places of the firſt and third Terms in the Queſtion. Thus,

Queſtion

Queſtion 6. If a Field will feed eighteen Horſes for ſeven Weeks: How long will it feed Forty-two Horſes at the ſame Rate of feeding?

Firſt, 18 Horſes : 7 Weeks : : 42 Horſes : 3 Weeks.

Here the Terms are ſtated inverſely, as before.

Otherwiſe thus, 42 Horſes : 7 Weeks : : 18 Horſes : 3 Weeks. Then $18 \times 7 = 126$. And $126 \div 42 = 3$ Weeks. The Anſwer required.

Sect. 3. *Of* Compound Proportion; *commonly called* The Double Rule of Three.

Compound Proportion (as it is here meant) is, when there are five Numbers given to find out a ſixth Proportional; and this is generally performed by a Double Poſition; that is, by ſtating and working the Queſtion at two Operations, either in Direct or Reciprocal Proportion, according as the Queſtion requires.

And therefore it is called, The Double Golden Rule, or Double Rule of Three.

The Double Rule Direct is, when the ſixth Term or Number ſought, is found by two Operations, both of them in Direct Proportion.

Example 1. If a Hundred Pounds gain ſix Pounds Intereſt in twelve Months; how much will three Hundred Pounds gain in nine Months, at the ſame Rate?

Firſt 100 *l.* : 6 *l.* : : 300 *l.* : 18 *l.*

6

100) 1800 (18 *l.* The Intereſt of 300 *l.* for 12 Months.

Months. *Months.*

Then, 12 : 18 *l.* : : 9 : 13 *l.* 10 *s.*

9

12) 162 (13 *l.* 10 *s.* The Anſwer required.

I ſuppoſe the Learner will eaſily conceive the Reaſon of theſe two Operations. For, firſt it is plain by Direct Proportion, that if 100 *l.* gain 6 *l.* in twelve Months, 300 *l.* will gain 18 *l.* in the ſame Time, and at the ſame Rate.

And

And by the ſame Rule it is plain, that if 12 Months will produce or give 18 *l.* Intereſt for 300 *l.* then 9 Months muſt needs give 13 $\frac{1}{2}$ for the ſame Sum, *viz.* 300 *l.*

The Double Rule of Three Inverſe is, when the ſixth Term or Number ſought is found at two Operations (as before). But one of them requires an Anſwer in Reciprocal Proportion.

Queſtion 2. If 6 Buſhels of Oats will ſerve 4 Horſes 8 Days, How many Days will 21 Buſhels ſerve 16 Horſes, at the ſame Rate of feeding?

This *Queſtion* being parted into two Poſitions, the firſt will be thus:

If 6 Buſhels of Oats will ſerve 4 Horſes 8 Days, How many Days will 21 Buſhels ſerve them?

Here it is plain, that 21 Buſhels will ſerve them longer than 6 Buſhels; therefore the firſt Poſition falls in Direct Proportion.

Buſh. Days. Buſh. Days.

Thus, 6 : 8 :: 21 : 28

8

6) 168 (28 Days

That is, if 6 Buſhels will ſerve 4 Horſes 8 Days, 21 Buſhels will ſerve them 28 Days.

The next Poſition muſt be to find how long the ſaid 21 Buſhels will ſerve 16 Horſes at the ſame Rate of feeding: it is plain, that 21 Buſhels cannot ſerve 16 Horſes ſo many Days as they will ſerve 4 Horſes; therefore this ſecond Poſition falls in Reciprocal Proportion.

Horſes. Days. Horſes. Days.

Thus, 4 : 28 :: 16 : 7 the Anſwer required.

After the like manner any Queſtion in the Double Rule of Three may be anſwered by two ſingle Poſitions, if Care be taken in ſtating them right, *viz.* Whether their Operation muſt be performed by the ſingle Rule Direct, or Inverſe.

But all Queſtions in this Double Rule, where five Numbers are propoſed to find a ſixth, may more eaſily and readily be anſwered by one general Theorem; which compriſeth both the Direct and Inverſe Rules; without conſidering either of them being deduced from the ſingle Operations beforegoing.

But firſt you muſt carefully note, that in all Queſtions of this Nature, three of the five propoſed Terms are always conditional and

and ſuppoſed; and that the other two move the Queſtion. As for Inſtance in *Example* 1.

Viz. If 100 *l.* will gain 6 *l.* in twelve Months: theſe 3 Terms are only ſuppoſed or conditional. Then comes the Queſtion; What will 300 *l.* gain in 9 Months? Now, in order to raiſe the general Theorem, let us ſuppoſe, inſtead of Numbers, theſe Letters.

Viz. Let $\left\{\begin{array}{lll} P = 100. & \text{The } \textit{Principal.} \\ T = 12. & \text{The } \textit{Time.} \\ G = 6. & \text{The } \textit{Gain.} \end{array}\right\}$ In the Suppoſition of any propoſed Queſtion.

And, $\left\{\begin{array}{lll} p = 300. & \text{The } \textit{Principal.} \\ t = 9. & \text{The } \textit{Time.} \\ g = 13,5 & \text{The } \textit{Gain.} \end{array}\right\}$ The three Terms wherein the Queſtion lies.

Then $P : G :: p : \frac{Gp}{P} =$ The Product of the two Means divided by the firſt Extream.

That is, $100 : 6 :: 300 : \frac{300 \times 6}{100} = 18$ Which is the firſt Part of the Queſtion.

Then $T : \frac{Gp}{P} :: t : g$ Which is the ſecond Part of the Queſtion.

Viz. $12 : 18 :: 9 : 13,5$

Ergo $Tg = \frac{Gpt}{P}$ That is, the Product of the Extreams is equal to that of the Means.

Conſequently, $Tg\,P = Gpt$ is the *Theorem*.

This *Theorem* affords two Rules, by which all Queſtions in this Double Rule of Three, or rather of five Numbers, may be reſolved; due Regard being had to the true placing down of the propoſed Terms, which muſt be thus:

Always place the three conditional Terms in this Order; let that Number which is the principal Cauſe of Gain, Loſs, or Action, *&c. (viz. P.)* be put in the firſt Place; that Number which denotes the Space of Time, or Diſtance of Place, *&c. (viz. T.)* be put in the ſecond Place. And that Number which is the Gain, Loſs, or Action, *&c. (viz. G.)* be put in the third Place. Now according to theſe Directions, the conditional Terms of the laſt Queſtion will ſtand thus; *P. T. G.*

That done, place the other two Terms which move the Queſtion, underneath thoſe of the ſame Name,

Thus, $\left\{\begin{array}{l} P.\ T.\ G. \\ p.\ t. \end{array}\right.$

Then

Then if the Blank or Term ſought, fall under the third Place, as in this Queſtion,

It will be $\left\{\frac{G\,p\,t}{T\,P} = g.\right.$ Which gives this Rule.

Rule 1. { *Multiply the three laſt Terms together for a Dividend, and the two firſt together for a Diviſor; the Quotient ariſing from them will be the ſixth Term.*

That is, in our propoſed *Example* 1.
Thus $6 \times 300 \times 9 = 16200$ the Dividend.
And $100 \times 12 = 1200$ the Diviſor.
Then 1200) 16200 (13 $\frac{1}{2}$ the Anſwer, as before.

But if the Blank or Term ſought fall under the firſt Place, then

It will be $\left\{\frac{T\,g\,P}{t\,G} = p.\right.$

Or if the Blank fall under the ſecond Place,

It will be $\left\{\frac{T\,g\,P}{G\,p} = t.\right.$ Either of theſe gives this Rule.

Rule 2. { *Multiply the firſt, ſecond, and laſt Terms together for a Dividend, and the other two together for a Diviſor; the Quotient ariſing from them will be the ſixth Term.*

And becauſe our *Example* 2. falls under the Conſideration both of Direct and Reciprocal Proportion, let it be here propoſed again. *Viz.* If 6 Buſhels of Oats will ſerve 4 Horſes 8 Days; how many Days will 21 Buſhels ſerve 16 Horſes, *&c.*

If the Terms of this Queſtion be placed down as before directed they will ſtand

Thus { *Horſes. Days. Buſhels.*
4 . 8 . 6. Terms in the Suppoſition.
16 21

Here the Blank falls under the ſecond Place, therefore it muſt be found by the ſecond Rule.

Thus $4 \times 8 \times 21 = 672$ the Dividend.
And $16 \times 6 = 96$ the Diviſor.
Then 96) 672 (7 the Anſwer, as before.

Queſt. 3. What Principal or Stock will gain 20 *l.* in 8 Months at 6 *per Cent. per Annum.*

Prin.	*Time.*	*Gain.*	
100	12	6	Terms in the Suppoſition.
	8	20	

In this Queſtion the Blank falls under the firſt Place, therefore it muſt be found by the ſecond Rule.

Thus $100 \times 12 \times 20 = 24000$ the Dividend.
And $8 \times 6 = 48$ the Diviſor,
Then 48) 24000 (500 *l.* the Anſw. required.

The Proof of all Queſtions in this Double Rule of five Numbers, is beſt performed by varying the Queſtion; *viz.* by ſtating it in another Order, as in the laſt *Example:* Thus,

If 100 *l.* gain 6 *l.* in 12 Months, what will 500 *l.* gain in 8 Months?

The Anſwer to this Queſtion muſt be 20 *l.* if the Work of the laſt *Example* be true.

	Prin.	*Time.*	*Gain.*	
Stated thus	100	12	6	then, *per* Rule 1,
	500	8		

$500 \times 8 \times 6 = 24000$. And $100 \times 12 = 1200$.
Then 1200) 24000 (20 *l.* the Anſwer, *&c.*

Queſt. 4. If two Men can do 12 Rods of Ditching in 6 Days, How many Rods may be done by 8 Men in 24 Days, at the ſame Rate of working?

Anſw. 192 Rods.

Queſt. 5. If the Carriage of 5 *C.* 3 *qrs. Weight*, 150 Miles, coſt 3 *l.* 7 *s.* 4 *d.* What muſt be paid for the Carriage of 7 *C.* 2 *qrs.* 25 lb *Weight*, 64 Miles, at the ſame Rate?

Anſw. 1 *l.* 18 *s.* 7 $\frac{1}{2}$ *d.*

Queſt. 6. If 8 Men deſerve 2 *l.* Wages for 5 Days Work, How much will 32 Men deſerve for 24 Days, at the ſame Rate?

Anſw. 38 *l.* 8 *s.*

Queſt. 7. Suppoſe a Hundred Pounds would Defray the Expences of five Men for Twenty-two Weeks and ſix Days, How long would twelve Men be in ſpending of one Hundred and Fifty Pounds, at the ſame Rate?

Anſw. 14 Weeks and 2 Days.

CHAP.

CHAP. VIII.

Of Trading in Company, usually called the Rule of Fellowship; *also* Bartering, *and* Exchanging *of Coins*, &c.

THE *Rule of Fellowship* is that by which the Accompts of several Partners trading in a Company, are so adjusted or made up, that every Partner may have his just Part of the Gain, or sustain his just Part of the Loss; according to the Proportion or Share of Money he hath in the Joint-Stock: Now this falls under two Considerations, called the *Single* and *Double Rules of Fellowship*.

Sect. 1. *The* Single Rule of Fellowship; viz. *That without* Time.

BY the *Single Rule of Fellowship* is adjusted the Accompts of those Partners that put all their several and perhaps different Sums of Money, into a common Stock at one and the same Time; and therefore it is usually called the *Rule of Fellowship* without Time: Now all Questions of this Nature are answered by so many several Operations in the *Rule of Three Direct*, as there are Partners in the Stock.

For, as the Total Sum of Money in the Stock is in Proportion to the whole Gain, or Loss: so is every Man's particular Part of that Stock; to his particular Share of that Gain, or Loss.

Quest. 1. Three Partners, suppose *A*, *B*, and *C*, make a Joint-Stock of 96 *l.* in this manner.

A puts in 24 *l.* *B* puts in 32 *l.* and *C* puts in 40 *l.* with this 96 *l.* they trade and gain 12 *l.* It is required to find each Man's true Part of that Gain.

The Operation will stand thus

$$96\,l. : 12\,l. :: \left\{\begin{array}{l} 24\,l. : 3\,l. = A\text{'s} \\ 32\,l. : 4\,l. = B\text{'s} \\ 40\,l. : 5\,l. = C\text{'s} \end{array}\right\} \text{Part of the Gain.}$$

Proof $3\,l. + 4\,l. + 5\,l. = 12\,l.$ the whole Gain.

That is, if the Sum of each Man's particular Gain, amounts to the whole Gain, the Work is true; if not, some Error is committed which must be found out.

Note, These Operations will be very much abbreviated, if you work them by *Theorem* 2. *page* 87. For here 96 is a common Antecedent, and 12 is the common Consequent in all the three Proportions.

Therefore 96 : 12 :: 1 : 0,125 a common Multiplicator.

Then $\left.\begin{matrix}24\\32\\40\end{matrix}\right\} \times 0{,}125 = \left\{\begin{matrix}3\ l.\\4\ l.\\5\ l.\end{matrix}\right\}$ for $\left\{\begin{matrix}A,\\B,\\C,\end{matrix}\right\}$ as before.

Now this Method is more readily performed than the other, eſpecially when the Partners are many; becauſe one ſingle Diviſion ſerves for all the Work.

Queſt. 2. Three Merchants, *A*, *B*, and *C*, freight a Ship with 248 Tun of Wine: Thus, *A* loaded 98 Tun, *B* 86 Tun, and *C* 64 Tun. By Extremity of Weather the Seamen were forced to caſt or throw 93 Tun of it over-board. How much of this Loſs muſt each Merchant ſuſtain?

First 248 : 93 :: 1 : 0,375 the common Multiplier.

Then $\left.\begin{matrix}98\\86\\64\end{matrix}\right\} \times 0{,}375 = \left\{\begin{matrix}36{,}75 \text{ for } A\text{'s}\\32{,}25 \text{ for } B\text{'s}\\24{,}00 \text{ for } C\text{'s}\end{matrix}\right\}$ Loſs.

Proof 93,00 = the whole Loſs.

Now if the Queſtion were to find how much of the remaining Wine that was ſaved, belongs to *A*, to *B*, and to *C*.

Then $\left.\begin{matrix}98 - 36{,}75 = 61{,}25\\86 - 32{,}25 = 53{,}75\\64 - 24{,}00 = 40{,}00\end{matrix}\right\}$ belongs to $\left\{\begin{matrix}A.\\B.\\C.\end{matrix}\right.$

That is, *A* ought to have 61 Tun and 63 Gallons. *B* ought to have 53 Tun and 189 Gallons. And *C* ought to have 40 Tun of what was left.

Queſt. 3. Suppoſe ſix Men, *viz.* *A*, *B*, *C*, *D*, *E*, and *F*, make a Joint-Stock of 2558 *l.*

		l. *s.* *Decimals.*
Thus *A*		654 . 10 = 654,50
B		543 . 15 = 543,75
C	puts in	480 . 00 = 480,00
D		254 . 10 = 254,50
E		365 . 05 = 365,25
F		260 . 00 = 260,00

The whole Stock 2558 . 00 = 2558,00 according to the *Queſt.*

With

With this Stock of 2558 *l.* they trade eighteen Months, and gain 831 *l.* 7 *s.* It is required to find every Man's Part or Share of that Gain.

Note, *Although the Time of Trading, viz. eighteen Months, be mentioned in the Question, yet it is no Way concerned in answering of it; as you may observe in the following Work.*

First, 2558 *l.* : 831,35 *l.* :: 1 *l.* : 0,325 Decimal Parts.

Consequently, 1 *l.* : 0,325 :: 654,5 : 212,7125. That is,

654,50		212,71250		*A.*
543,75		176,71875		*B.*
480,00	×0,325 =	156,00000	for	*C.*
254,50		82,71250		*D.*
365,25		118,70625		*E.*
260,00		84,50000		*F.*

		l. *parts.*	*l.*	*s.*	*d.*
That is, *A*	gains	212,71250 =	212 .	14 .	03
B		176,71875 =	176 .	14 .	04½
C		156,00000 =	156 .	00 .	00
D		82,71250 =	82 .	14 .	03
E		118,70625 =	118 .	14 .	01½
F		84,50000 =	84 .	10 .	00
Proof.	Sum	831,35 =	831 .	07 .	00

I have omitted resolving this Question according to the usual Method (as before directed) of finding every Man's particular Part of the Gain by the Golden Rule, as in the first Work of *Example* 1. leaving that for the Learner's Practice.

Sect. 2. *The* Double Rule of Fellowship; *or that with* Time.

THIS is usually called the *Double Rule of Fellowship*, because every particular Man's Money is to be considered with Relation to the Time of it's Continuance in the Joint-Stock.

Question 1. *A* and *B* join in Partnership upon these Terms, *viz.* *A* agrees to lay down 100 *l.* and to employ it in Trade 3 Months: Then *B* is to lay down his 100 *l.* and with the whole Stock of 200 *l.* they are to trade 3 Months more. Now at the End of that Time, they find their whole Gain to be 21 *l.* It is required to know what each Man's Part of the Gain ought to be, according to his stock, and the Time of employing it.

Here

Here it is but reaſonable to conclude, that *A* ought to gain more than *B*, notwithſtanding their Stocks of Money are equal; becauſe *A* employed his Money a longer Time than *B*.

Now for ſolving of this Queſtion, let us ſuppoſe *A*'s 100 *l.* employed the firſt 3 Months to gain Z = a Sum as yet unknown; then it muſt gain 2 Z in 6 Months; and to find what *B* muſt gain, it will be,

l. *Months.*
100 . 6 . 2 Z = *A*'s Gain
100 . 3 . to *B*'s Gain } *per Rule* 1. *Page* 97.

Ergo $\frac{100 \times 3 \times 2Z}{100 \times 6} = B$'s Gain.

But *A*'s Gain added to *B*'s Gain muſt = 21 *l.* the whole Gain by the Queſtion.

Therefore $2Z + \frac{100 \times 3 \times 2Z}{100 \times 6} = 21\ l.$

That is, $100 \times 6 \times 2Z + 100 \times 3 \times 2Z = 21 \times 100 \times 6$.
Which contracted is, $900 \times 2Z = 21 \times 600$.

Conſequently, $2Z = \frac{21 \times 600}{900}$, which gives the following Analogy.

Viz. 900 : 21 :: 600 : 2 Z = 14 *l.* for *A*'s Gain.
And 900 : 21 :: 100 × 3 = 300 : 7 *l.* for *B*'s Gain.

Now this way of arguing hath not only reſolved the preſent Queſtion, but it alſo affords (and demonſtrates) a general Rule for reſolving all Queſtions of this Nature, be the Partners never ſo many.

Rule. { *Multiply every particular Man's Stock, with the Time it is employed, then it will be, as the Sum of all thoſe Products; is to the whole Gain (or Loſs). So is every one of thoſe Products: to it's proportional Part of that whole Gain (or Loſs).*

Queſtion 2. Three Merchants, *A*, *B*, and *C*, enter into Partnerſhip, thus; *A* puts into the Stock 65 *l.* for 8 Months; *B* puts in 78 *l.* for 12 Months; and *C* puts in 84 *l.* for 6 Months. With theſe they traffick, and gain 166 *l.* 12 *s.* It is required to find each Man's Share of the Gain, proportionable to the Stock and Time of employing it.

1. *A*'s

1. *A*'s
2. *B*'s
3. *C*'s

} Stock { 65 *l.* × 8
78 *l.* × 12
84 *l.* × 6 } *Months, the Time it was employed* = { 520
936
504

The Sum of thoſe Products is, 1960

Then, according to the Rule, the ſeveral Proportions will ſtand thus,

1960 : 166,6 :: { 520 : 44,20 = 44 *l.* 4 *s.* 0 *d.*
936 : 79,56 = 79 *l.* 11 *s.* 2 $\frac{1}{2}$ *d.*
504 : 42,84 = 42 *l.* 16 *s.* 9 $\frac{1}{2}$ *d.* } for { *A.*
B.
C.

The whole Gain = 166 *l.* 12 *s.* 0 *d.*

Or you may work as in ſome of the former *Examples*, viz. by finding the proportional Part of the Gain due to one Pound, *&c.*

Thus 1960 : 166,6 :: 1 : 0,085 the common Multiplier.

Then { 520
936
504 } × 0,085 = { 44,2
79,56
42,84 } for { *A.*
B.
C. } *&c.* As before.

Queſtion 3. Six Merchants, *viz. A*, *B*, *C*, *D*, *E*, and *F*, enter into Partnerſhip, and compoſe a Joint-Stock in this manner;

		l.	*s.*	
Viz.	*A*	64	10	$4\frac{1}{2}$
	B	78	15	6
	C	100	00	$8\frac{1}{4}$
	D	80	10	12
	E	74	12	$9\frac{1}{2}$
	F	125	15	7
puts in				for Months.

They traffick, and gain 258 *l.* 18 *s.* 4 $\frac{1}{2}$ *d.* It is required to find every Man's Share of the Gain, according to the Stock and Time it was employed.

The ſeveral Stocks of Money, and their reſpective Times being firſt brought into Decimals, and then multiplied together, will produce theſe following Products.

A's
B's
C's
D's
E's
F's

} Stock { *l.* *Months.*
64,50 × 4,50
78,75 × 6,00
100,00 × 8,25
80,50 × 12,00
74,60 × 9,50
125,15 × 7,00 } *the Time it was employed* = { 290,25
472,50
825,00
966,00
708,70
880,25

The Sum of thoſe Products = 4142,70

Then

Then if you work by the common Way; it will be 4142,7 : 258,91875 :: 290,25 : 18,140625 = 18 *l.* 2 *s.* 9 $\frac{3}{4}$ *d.* for *A*'s part of the Gain; and ſo on for the reſt.

But if you work by the eaſieſt Way, *viz.* by finding the proportional Part of the Gain due to one Pound.

Thus 4142,7 : 258,91875 :: 1 : 0,0625.

Then		*l.* *s.* *d.*	
290,25		18,140625 = 18 . 02 . 09 $\frac{3}{4}$	*A*
472,50		29,531250 = 29 . 10 . 07 $\frac{1}{2}$	*B*
825,00	× 0,0625 =	51,562500 = 51 . 11 . 03	for *C*
966,00		60,375000 = 60 . 07 . 06	*D*
708,70		44,293750 = 44 . 05 . 10 $\frac{1}{2}$	*E*
880,25		55,015625 = 55 . 00 . 03 $\frac{3}{4}$	*F*
		The whole Gain = 258 . 18 . 04 $\frac{1}{2}$	

Theſe few *Examples* being well underſtood, are ſufficient to ſhew the whole Buſineſs of Fellowſhip, *&c.*

Sect. 3. *Of* Bartering.

WHEN Merchants or Tradeſmen exchange one Commodity for another, it is called *Bartering*; and the only Difficulty in this way of dealing, lies in duly proportioning the Commodities to be exchanged ſo, as that neither Party may ſuſtain Loſs.

Queſtion 1. Two Merchants, *A* and *B*, *Barter*; *A* would exchange 5 *C.* 3 *qrs.* 14 lb of *Pepper*, which is worth 3 *l.* 10 *s.* *per C.* with *B* for *Cotton*, worth 10 *d.* *per pound* weight; how much *Cotton* muſt *B* give to *A* for his *Pepper?*

Note, *In order to the reſolving of this Queſtion (and all other Queſtions of this Nature) you muſt firſt find, by the Rule of Three (or otherwiſe) the true Value of that Commodity whoſe Quantity is given (which in this Queſtion is Pepper). And then find how much of the other Commodity will amount to that Sum, at the Rate propoſed.*

First 5 *C.* 3 *qrs.* 14 lb. = 5,875
And 3 *l.* 10 *s.* 0 *d.* = 3,500 } in Decimals.

Then 1 : 3,5 :: 5,875 : 20,5625 = 20 *l.* 11 *s.* 3 *d.* the true Value of the *Pepper*.

Next, It is eaſy to conceive, that *A* ought to have as much *Cotton* at 10 *d.* *per Pound*, as will amount to 20 *l.* 11 *s.* 3 *d.* which may be thus found;

10 *d.* : 1 lb. :: 20 *l.* 11 *s.* 3 *d.* = 4235 *d.* : 493,5 lb.

That

That is, 4 *C.* 1 *qr.* 17 ½ *pound* of *Cotton.* And ſo much *B* muſt give to *A* in exchange for his 5 *C.* 3 *qrs.* 14 *pound* of *Pepper.*

Queſtion 2. Two Merchants *A* and *B* barter thus; *A* hath 86 Yards of Broad Cloth worth 9 *s.* 2 *d. per Yard* ready Money: but in Barter he will have 11 *s. per Yard.* *B* hath Shalloon worth 2 *s.* 1 *d. per Yard* ready Money; it is required to find how many Yards of the Shalloon *B* muſt give to *A* for his Cloth, to make his Gain in the Barter equal to that of *A*'s.

The Method of reſolving this, and the like Queſtions, differs a little from the laſt Caſe; for in this you muſt firſt find what Advance *B* ought to make *per Yard* upon his Shalloon, in proportion to what *A* hath done upon a Yard of his Cloth.

Thus
s. d. d. s. d. s. d. d. s. d. d.
9. 2 = 110 : 11 = 132 :: 2. 1 = 25 : 2. 6 = 30

the advanced Price for a Yard of *B*'s Shalloon. Then proceed as before in the laſt *Example.*

Thus 1 Yard : 11 *s.* :: 86 Yards : 946 *s.* = 47 *l.* 6 *s.* the advanced Value of all the Cloth.

Next, If 2 *s.* 6 *d.* will buy one Yard of Shalloon, at it's advanced Price, how many Yards will 47 *l.* 6 *s.* buy.

Thus 2, 5 : 1 :: 946 : 378,4 Yards.

That is, *B* muſt give 378 ⅖ Yards of his Shalloon to *A*, for his 86 Yards of Broad Cloth.

Theſe two Examples are ſufficient to ſhew the Learner, that the Method of bartering, or exchanging Commodities for Commodities, wholly depends upon a clear underſtanding of the Golden Rule; which indeed is ſo called, becauſe of it's Univerſal Uſe.

Sect. 4. *Of* Exchanging Coins.

EXchanging the Coins of one Country for thoſe of another, is like the Buſineſs of bartering Commodities. That is, it conſiſts in finding what Sum of one Country Coin will be equal in Value to any propoſed Sum of another Country Coin. And, in order to perform that, it will be very neceſſary to have a true Account at all Times of the juſt Value of thoſe Foreign Coins which are to be exchanged, as they are compared in Value with our *Engliſh* Coin. I ſay at all Times, becauſe the Par of Exchange (as the Merchants call it) differs almoſt every Day from *London* to other Countries. That is, it riſes and falls, according as Money is plenty or ſcarce; or according to the Time allowed for Payment of the Money in Exchange, *&c.*

Thoſe that deſire to be fully ſatisfied in the common Values of Foreign Coins, Weights, Meaſures, &c. may find them in a Book called the *Merchants Map of Commerce*, which for Brevity ſake I have omitted tranſcribing, and only collected theſe few of Coin.

	Foreign Coins.	*Engliſh Coin.* l.	s.	d.
French Coin.	A *Denier* =	0	0	$0\frac{3}{10}$
	12 *Deniers* = 1 *Soulz* =	0	0	$0\frac{9}{10}$
	12 *Soulz* = 1 *Livre* =	0	1	6
	3 *Livres* = 1 *Crown* =	0	4	6
Low-Country Coin.	A *Stiver* =	0	0	$1\frac{1}{5}$
	6 *Stivers* = 1 *Flemiſh Shilling* =	0	0	$7\frac{1}{5}$
	20 *Stivers* = 1 *Gilder* =	0	2	0
	10 *Gilders* = $33\frac{1}{3}$ *Shillings* or a *Flemiſh Pound* =	1	0	0
	A *Embden Dollar* =	0	2	$3\frac{3}{5}$
	A *Campen Dollar* =	0	2	$7\frac{1}{5}$
	A *Zealand Dollar* =	0	3	0
	A *Lyons Dollar* =	0	4	0
	A *Specie Dollar* =	0	5	0
	A *Duccatoon* =	0	6	$3\frac{3}{5}$
Germany	A *Rixdolar* of the *Empire* =	0	4	$5\frac{3}{4}$
	A *Gilder* of *Nuremberg* =	0	7	1
In Italy *and* Spain,	The *Livre at Leghorn* =	0	0	9
	Florence Crown Current =	0	5	3
	Venice Ducat de Banco =	0	4	4
	The *Current Ducat* =	0	3	4
	The *Naples Ducat* =	0	5	0
	The *Cadiz Ducat* =	0	5	$6\frac{1}{4}$
	The *Barcelona Ducat* =	0	6	0
	The *Valencia Ducat* =	0	5	3
	The *Bergonia Ducat* =	0	4	4
	The *Portugal Teſtoon* =	0	1	3
	The Piece of Eight =	0	4	6

Note, The *Engliſh* generally reckon their Exchange with other Countries by Pence, *viz.* other Countries value their Crowns, Dollars, or Ducats, &c. by *Engliſh* Pence. Except with ſome Parts of the *Low-Countries*, with whom the Exchange is in Pounds Sterling.

Queſt. 1. How many Dollars at 4 *s.* 6 *d. per Dollar*, may one have for 162 *l.* 18 *s.* Anſwer 724 Dollars.

Thus

Thus 162 *l.* 18 *s.* = 3258 *s.* and 4 *s.* 6 *d.* = 54 *d.*
Then 54 : 1 :: 3258 : 724 the Anſwer.

Queſt. 2. How many *Saragoſſa* Ducats, of 5 *s.* 6 *d.* the Ducat, may be had for 275 *Bergonia* Ducats, at 4 *s.* 4 *d.* the Piece?
Anſwer 216 and 3 *s.* 8 *d.* over.

Thus 5 *s.* 6 *d.* = 66 *d.* and 4 *s.* 4 *d.* = 52 *d.*
Then 275 × 52 = 14300 *d.* = 275 Ducats.
Conſequently 66) 14300 (216 $\frac{2}{3}$ the Anſwer required.

Queſt. 3. A Traveller would change 233 *l.* 16 *s.* 8 *d.* Sterling Money; for *Venice* Ducats at 4 *s.* 9 $\frac{1}{2}$ *d. per Ducat*; How many Ducats muſt he have? Anſwer 976 Ducats.

Thus 4 *s.* 9 $\frac{1}{2}$ *d.* = 57,5 *d.* and 233 *l.* 16 *s.* 8 *d.* = 56120 *d.*
Then 57,5 *d.*) 56120 *d.* (976 the Anſwer required.

Queſt. 4. A Caſhier hath received 759 Ducats, at 7 *s.* 6 *d. per Ducat*; And 579 Dollars at 4 *s.* 8 *d. per Dollar*: Which he would exchange for *Flemiſh* Marks at 14 *s.* 3 *d. per* Piece: How many ought he to have?
Anſwer 589 Marks, and 15 *d.* over.

For 7 *s.* 6 *d.* = 90 *d.* and 4 *s.* 8 *d.* = 56 *d.*
Then { 759 × 90 = 68310 *d.* the Value of the Ducats.
{ 579 × 56 = 32424 *d.* the Value of the Dollars.

their Sum = 100734 *d.*
And 14 *s.* 3 *d.* = 171 *d.* the *Flemiſh* Mark in Pence.
Conſequently 171) 100734 (589 *&c.* the Anſwer required.

Queſt. 5. A Bill of Exchange was accepted at *London* for the Payment of 400 *l.* Sterling, for the like Value delivered in *Amſterdam*, at 1 *l.* 13 *s.* 6 *d.* for 1 *l.* Sterling; How much Money was delivered at *Amſterdrm?* Anſwer 670 *l. Flemiſh.*

For 1 *l.* = 240 *d.* and 1 *l.* 13 *s.* 6 *d.* = 402 *d.*
Then 240 : 402 :: 400 : 670 the Anſwer required.

Queſt. 6. When the Exchange from *Antwerp* to *London* is at 1 *l.* 4 *s.* 7 *d. Flemiſh*, for 1 *l.* Sterling; How many Pounds Sterling muſt be paid at *London*; to ballance 236 *l. Flemiſh* at *Antwerp?* Anſwer 192 *l.* Sterling.

Thus 1 *l.* 4 *s.* 7 *d.* = 295 *d.* and 1 *l.* = 240 *d.*
Then 295 : 240 :: 236 : 192 the Anſwer.

Queſt. 7. A Merchant delivered at *London* 120 *l.* Sterling to receive 147 *l. Flemiſh* in *Amſterdam*; How much was 1 *l.* Sterling valued at, in *Flemiſh* Money? Anſwer 1 *l.* 4 *s.* 6 *d.*

Thus 120 : 147 :: 240 *d.* : 294 *d.* = 1 *l.* 4 *s.* 6 *d.* *&c.*

Queſt. 8. A Factor hath ſold Goods at *Cadiz* for 1468 Pieces of Eight, valued at 4 *s.* 6 ½ *d.* Sterling per Piece; How much Sterling Money do thoſe Pieces of Eight amount to?

Anſwer 333 *l.* 7 *s.* 2 *d.*

Thus, if 1 = 54,5 *d.* then 1468 × 54,5 = 83006 *d.* *&c.*

Queſt. 9. A Traveller would have an equal Number of Crowns at 5 *s.* 6 *d. per* Crown; and Dollars at 4 *s.* 5 *d. per* Piece; How many of each ſort may he have for 309 *l.* 8 *s.*?

Anſwer 624 of each.

Thus 309 *l.* 8 *s.* = 74256 *d.*

And 5 *s.* 6 *d.* + 4 *s.* 5 *d.* = 119 *d.*

Then 119) 74256 (624 the Anſwer required.

Queſt. 10. Suppoſe I would exchange 527 *l.* 17 *s.* 6 *d.* for Dollars at 4 *s.* 6 *d.* a Piece, Ducats at 5 *s.* 8 *d.* a Piece, and Crowns at 6 *s.* 1 *d.* a Piece; and would have 2 Dollars for 1 Ducat, and 3 Dollars for 2 Crowns. How many of each ſort muſt I have?

Anſwer 927 Dollars, 463 ½ Ducats, and 618 Crowns.

For { 54 *d.* = 1 Dollar / 68 *d.* = 1 Ducat / 73 *d.* = 1 Crown } *per Queſtion.*

And 126690 *d.* = 527 *l.* 17 *s.* 6 *d.*

Now if the Crowns, Dollars, and Ducats, were to be equal in Number; then 73 + 54 + 68 muſt have been the Diviſor, by which 126690 muſt have been divided, and the Quotient would have been the Anſwer to the Queſtion. As in the laſt *Example.*

But here inſtead of their Sum, ſuch Parts of them muſt be taken as are aſſigned or limited by the Queſtion; that ſo the Number of ſome one of them may be found.

And becauſe there muſt be { 2 Dollars for 1 Ducat, and / 3 Dollars for 2 Crowns,

Therefore it will be ½ of a Ducat for one Dollar, and ⅔ of a Crown for one Dollar.

Conſequently,

Conſequently, $54 + \frac{68}{2} : + \frac{2}{3}$ of $73 = 136\frac{2}{3}$, or $\frac{410}{3}$ will be the Diviſor to find the Number of Dollars.

Thus $\frac{410}{3}$) 126690 (927 the Number of Dollars.

Then $\frac{1}{2}$ of $927 = 463\frac{1}{2}$ is the Number of Ducats.

And $\frac{2}{3}$ of $927 = 618$ is the Number of Crowns.

Or if you pleaſe you may form Diviſors to find either the Ducats or Crowns firſt: For if it be 2 Dollars for 1 Ducat, and 3 Dollars for 2 Crowns, as before;

Then will 6 Dollars be for 3 Ducats, and 6 Dollars for 4 Crowns.

Therefore, $\left\{\begin{array}{l}\frac{3}{2} \text{ of a Dollar} \\ \frac{3}{4} \text{ of a Ducat}\end{array}\right\}$ will be for 1 Crown.

Conſequently, $\frac{3}{2}$ of $54 : + \frac{3}{4}$ of $68 : + 73 = 205$ will be the Diviſor to find the Crowns firſt, *&c.*

Queſt. 11. A Caſhier is to receive 500 *l.* He is offered Crowns at 6 *s.* 1 $\frac{1}{2}$ *d. per* Crown, which are worth but 6 *s.* Or he may have Dollars at 4 *s.* 5 *d.* the Piece, which are worth but 4 *s.* 4 *d.* Which of theſe ſhall he receive to have the leaſt Loſs? And how much will he loſe in the Payment?

1 $\left\{\begin{array}{l}1 \text{ Crown} = 72\ d. \\ 1 \text{ Dollar} = 52\ d.\end{array}\right\}$ according to the true Values.

2 $\left\{\begin{array}{l}1 \text{ Crown} = 73{,}5\ d. \\ 1 \text{ Dollar} = 53{,}0\ d.\end{array}\right\}$ the advanced Values.

Now to find which will be the leaſt Loſs; find what the advanced Value of a Dollar ought to be in Proportion to that of 1 Crown.

Thus $72 : 73{,}5 :: 52 : 53{,}083$ *&c.* But he may have Dollars at 53 *d. per* Piece, therefore the Payment in Dollars will be the leaſt Loſs; *viz.* 53 is leſs than 53,083 *&c.*

Next, to find what the whole Loſs will be by receiving Dollars. Becauſe the 500 *l.* = 120000 *d.* is advanced as much above the true Value, as 53 *d.* is above 52 *d.*: therefore ſay, If 53 *d.* advance 1 *d.* = 53 *d.* — 52 *d.*; what will 120000 *d.* advance? *i. e.*

$53\ d. : 1\ d. :: 120000\ d. : 2264\frac{8}{53}\ d. = 9\ l.\ 8\ s.\ 4\frac{8}{53}\ d.$ = the Loſs.

Queſt. 12. Suppoſe I exchange 4 *l.* 10 *s.* 10 *d.* for 11 Crowns and 7 Dollars; and at another Time I have 4 Crowns and 3 Dollars for 1 *l.* 15 *s.* each being of the ſame Value with the firſt. What is the Value of a Crown, and of a Dollar?

Firſt

First 11 Crowns + 7 Dollars = 1090 *d.*
Second 2 Crowns + 3 Dollars = 420 *d.* } by the Queſtion.

Then in order to find the Value of 1 Crown, you muſt caſt off the Dollars by making them of the ſame Number; Thus,

33 Crowns + 21 Dollars = 3270 *d.* the firſt multipl. with 3.
28 Crowns + 21 Dollars = 2940 *d.* the 2d multipl. with 7.

Then 5 Crowns = 330 *d.* being the Difference.

Conſequently 5) 330 (66 = 5 *s.* 6 *d.* is the Value of 1 Crown.
And 4 Crowns = 264 *d.*
Then will 3 Dollars = 420 *d.* — 264 *d.* = 156 *d.*
Conſequently 3) 156 (52 *d.* = 4 *s.* 4 *d.* the Value of 1 Dollar.

CHAP. IX.

Of Alligation.

WHEN it is required to mix ſeveral Sorts of Ingredients together; as different ſorts of Corn, Wines, Wool, Spices, or Metals; or to compoſe Medicines, *&c.* the Method of proportioning ſuch Mixtures is called the *Rule of Alligation*; and is divided into two Parts or Branches, called *Medial* and *Alternate*.

Sect. 1. *Of* Alligation Medial.

Alligation Medial, is that by which the Mean Rate or Price of any Mixture is found, when the particular Quantities of the Mixtures and Rates are given; and is thus performed.

Firſt find the Sum of all the Quantities propoſed to be mixed? And alſo the Sum of all their particular Rates.

Then the Proportion will be,

Rule { *As the Sum of all the Quantities : Is to the Sum of all their Rates : : So is any Part of the Mixture : To the Mean Rate or Price of that Part.*

Queſt. 1. Suppoſe 15 Buſhels of Wheat at 5 *s.* the Buſhel, and 12 Buſh. of Rye at 3 *s.* 6 *d.* the Buſh. were mixed together; What

What is the Mean Rate or Price, it may be ſold for a Buſhel, without Loſs or Gain?

This Queſtion prepared as directed above will ſtand

Thus { 15 Buſhels of Wheat at 5 *s.* *per* Buſhel come to 900*d.*
12 Buſhels of Rye at 3 *s.* 6 *d.* each come to 504*d.*

27 = their Sum. And their total Value = 1404*d.*

Then 27 Buſhels : 1404 *d.* :: 1 Buſhel : 52 *d.* = 4 *s.* 4 *d.* the Anſwer required.

Queſt. 2. A Grocer mixeth 36 Pounds of Tobacco, worth 1 *s.* 6 *d.* a Pound, with 12 Pounds of another ſort at 2 *s.* a Pound, and 12 Pounds of a third ſort at 1 *s.* 10 *d.* the Pound. How may he ſell the Mixture *per* Pound?

	lb.		*s.*	*d.*		*d.*
First	36	at	1	6	*per* Pound amount to	648
	12	at	2	0		288
	12	at	1	10		264

60 = the Number of Pounds. Their Value = 1200

Then 60 lb : 1200 *d.* :: 1 lb : 20 *d.* = 1 *s.* 8 *d.* the Anſwer required.

Queſt. 3. A Vintner mixeth 31 Gallons and a half of *Malaga* Sack worth 7 *s.* 6 *d.* the Gallon; with 18 Gallons of Canary at 6 *s.* 9 *d.* the Gallon; 13 Gallons and a half of Sherry at 5 *s.* the Gallon; and 27 Gallons of White Wine at 4 *s.* 3 *d.* the Gallon. It is required to find what one Gallon of this Mixture is worth.

	Gal.		*s.*	*d.*		*Pence.*
First	31½	at	7	6	*per* Gallon come to	2835
	18	at	6	9		1458
	13½	at	5	0		810
	27	at	4	3		1377

90 = the Number of *Gal.* Their Value = 6480

Then 90 : 6480 :: 1 : 72 *d.* = 6 *s.* the Rate or Price of one Gallon, as was required.

The Proof of all Operations in theſe ſort of Mixtures, is done by comparing the Value of all the Mixture (being ſold at the Mean Rate) with the total Value of all the particular Quantities, ſuppoſing they had been ſold at their reſpective Rates unmixed; if thoſe Sums are equal, the Work is true.

Sect.

Sect. 2. *Of* Alligation Alternate.

Alligation Alternate, is that by which the particular Quantities of every Ingredient concerned in any Mixture are found; when the particular Rates of every one of thoſe Ingredients, and the mean Rate are given; being (as it were) the Converſe to Alligation Medial; as will appear by the following Operations, which admit of three Caſes.

Caſe I. The Particular Rates of any Ingredients propoſed to be mixed, and the Mean Rate of the whole Mixture being given. To find how much of each Ingredient is requiſite to compoſe the Mixture; when the whole Quantity, or any Part thereof, is not limitted.

Queſt. 1. How much Wheat at 5 *s.* the Buſhel, and Rye at 3 *s.* 6 *d.* the Buſhel, will compoſe a Mixture that may be ſold for 4 *s.* 4 *d.* the Buſhel?

Note, *In all Queſtions of this Nature, it will be convenient to place the Mean Rate ſo, as that it may be eaſily compared with the Particular Rates, in order to find every one of their Differences from the Mean Rate, by Inſpection only.*

Thus, the Mean Rate = 52 *d.* { Wheat 60 *d.* / Rye 42 *d.*

Then take the ſeveral Differences between the Mean Rate and the Particular Rates; ſetting down thoſe Differences Alternately, and they will be the Quantities required.

Thus 52 { 60 / 42 } { 10 = 52 − 42 / 8 = 60 − 52

That is 52 − 42 = 10 for the Quantity of Wheat.

And 60 − 52 = 8 for the Quantity of Rye that will compoſe the Mixture required.

The Proof by *Alligation Medial.*

Add { 10 Buſhels of Wheat at 60 *d.* *per* Buſhel = 600 *d.*
{ 8 Buſhels of Rye at 42 *d.* *per* Buſhel = 336 *d.*

18 = the Number of Buſhels. = 936 *d.*

Then 18 : 936 :: 1 : 52 *d.* = 4 *s.* 4 *d.* the Mean Rate.

Note, Although 10 and 8 do anſwer the Queſtion, as plainly appears by the Proof; yet they are not the only two Numbers; for this Queſtion, and all others of this kind, will admit of various Anſwers, and all whole Numbers; for any two Numbers that are in the ſame Proportion to one another, as 10 is to 8, will as truly anſwer the Queſtion.

Viz.

Viz. 10 : 8 :: $\left\{\begin{matrix} 5 : 4 \\ 15 : 12 \\ 20 : 16 \\ 25 : 20 \end{matrix}\right\}$ *&c. ad infinitum.*

Queſt. 2. A Grocer would mix 3 ſorts of Tobacco together, *viz.* One Sort of 18 *d. per* lb, another Sort of 22 *d. per* lb, and a third Sort of 2 *s.* the lb. How much of each Sort muſt he take, that the whole Mixture may be ſold for 20 *d.* the Pound?

Having ſet down the given Rates, as before, then find each of their Differences from the propoſed Mean Rate, and place thoſe Differences alternately. Thus,

Mean Rate 20 $\left\{\begin{matrix} 18 \\ 22 \\ 24 \end{matrix}\right\}$ $\left\{\begin{matrix} 4 + 2 = 24 - 20 \text{ and } 22 - 20 \\ 2 = 20 - 18 \\ 2 = 20 - 18 \end{matrix}\right.$

Theſe Differences, *viz.* 6 . 2 . 2 are the Quantities required.

Proof $\left\{\begin{matrix} 6\text{lb of Tobacco at } 18d. \\ 2\text{lb at } 22\ d. \\ 2\text{lb at } 24\ d. \end{matrix}\right\}$ the Pound come to $\left\{\begin{matrix} 108 \\ 44 \\ 48 \end{matrix}\right\}$ *d.*

10 = the Number of Pounds. Their Value = 200 *d.*

Then 10) 200 (20 the Mean Rate.

Or indeed any three Numbers that have the ſame Ratio to one another as 6 and 2 have, will anſwer the Queſtion.

That is, 6 : 2 :: $\left\{\begin{matrix} 9 : 3 \\ 12 : 4 \\ 15 : 5 \end{matrix}\right\}$ *&c.*

But if only one of the three given Rates had been greater than the Mean Rate; as ſuppoſe 14 *d. per* Pound, 18 *d. per* Pound, and 24 *d. per* Pound, and the Mean Rate 20 *d.* as before. Then their Differences muſt have been placed

Thus, 20 $\left\{\begin{matrix} 14 \\ 18 \\ 24 \end{matrix}\right\}$ $\left\{\begin{matrix} 4 \\ 4 \\ 6 + 2 \end{matrix}\right\}$ *&c.* as before.

Queſt. 3. A Vintner would make a Mixture of Malaga, worth 7 *s.* 6 *d. per* Gallon, with Canary at 6 *s.* 9 *d. per* Gallon, Sherry at 5 *s. per* Gallon, and White Wine at 4 *s.* 3 *d. per* Gallon; What Quantity of each Sort muſt he take, that the Mixture may be ſold for 6 *s. per* Gallon?

In all Queſtions of this Kind, wherein it is required to mix four Things together, two of them having their Prices greater, and two leſſer than the mean Rate: you muſt always alligate or

compare a greater and leſſer Price with the mean Price, ſetting down their Differences alternately, as in the firſt *Example* of this *Section*.

Thus, Mean Rate = 72 *d.*

Malaga	90 *d.*	21	= 72 — 51
White	51 *d.*	18	= 90 — 72
Sherry	60 *d.*	9	= 81 — 72
Canary	81 *d.*	12	= 72 — 60

Hence 21 Gallons of Malaga, 12 of Canary, 9 of Sherry, and 18 of White, will compoſe the Mixture required.

Or thus, 72

Malaga	90 *d.*	12	Malaga
Sherry	60 *d.*	18	Sherry
Canary	81 *d.*	21	Canary
White	51 *d.*	9	White

will, *&c.*

Either of theſe Mixtures equally anſwer the Queſtion, which may be eaſily tried as before in the laſt, *&c.*

Caſe II. The particular Rates of all the Ingredients propoſed to be mixed, the Mean Rate of the whole Mixture, and any one of the Quantities to be mixed being given: Thence to find how much of every one of the other Ingredients is requiſite to compoſe the Mixture.

Note, This is uſually called *Alligation Partial.*

Queſt. 4. How much Wheat at 5 *s.* the Buſhel, muſt be mixed with 12 Buſhels of Rye at 3 *s.* 6 *d.* a Buſhel; that the whole Mixture may be ſold for 4 *s.* 4 *d.* the Buſhel?

In this Caſe you muſt ſet down all the particular Rates, with the Mean Rate, and find their Differences juſt as before; without any regard had to the Quantity given.

Thus, Mean Rate 52 *d.*

Wheat	60 *d.*	10
Rye	42 *d.*	8

Then *As the Quantity found by the Differences of the ſame Name with the Quantity given : Is to the Quantity given :: So is any of the other Quantities found by the Differences : To the Quantity of it's Name.*

Thus 8 : 82 :: 10 : 15, the Quantity or Number of Buſhels of Wheat required.

Queſt. 5. How much Malaga at 7 *s.* 6 *d.* the Gallon, Sherry at 5 *s.* the Gallon, and White Wine at 4 *s.* 3 *d.* the Gallon, muſt be mixed with 18 Gallons of Canary at 6 *s.* 9 *d.* the Gallon; that the whole Mixture may be ſold for 6 *s.* the Gallon?

The

The Terms being ſet down, *&c.* as before, will ſtand

Thus, Mean Rate 72 *d.* { Malaga 90 *d.* } { 21 / White 51 *d.* } { 18 / Sherry 60 *d.* } { 9 / Canary 81 *d.* } { 12

Then, as 12 : 18 :: { 21 : 31 ½ Gallons of Malaga. / 18 : 27 Gallons of White. / 9 : 13 ½ Gallons of Sherry.

That is, 31 ½ Gallons of Malaga, 27 of White Wine, and 13 ½ of Sherry, being mixed with 18 Gallons of Canary, will make the Mixture required.

Or thus, 72 { Malaga 90 } { 12 / Sherry 60 } { 18 / Canary 81 } { 21 / White 51 } { 9

Then, as 21 : 8 :: { 12 : $10\frac{6}{21}$ the Malaga. / 18 : $15\frac{9}{21}$ the Sherry. / 9 : $7\frac{15}{21}$ the White. } *&c.*

	Gallons.		*Pence.*
Proof.	$10\frac{6}{21}$ at 90 *d.*	each	$925\frac{15}{21}$
	$15\frac{9}{21}$ at 90 *d.*		$925\frac{1}{21}$
	$7\frac{15}{21}$ at 51 *d.*		$393\frac{9}{21}$
	18 at 81 *d.*		1458
	Sum $51\frac{9}{21}$		Value = $3702\frac{18}{21}$

Then $51\frac{9}{21}$) $3702\frac{18}{21}$ (72 *d.* = 6 *s.* the Mean Rate.

Therefore the Quantities are as truly aſſigned here, as in the laſt Work.

Caſe III. The particular Rates of all the Ingredients propoſed to be mixed; and the Sum of all their Quantities with the Mean Rate of that Sum being given; to find the particular Quantities of the Mixture.

This is called *Alligation Total*, and is thus performed.

Set down all the particular Rates, with the Mean Rate, and find their Differences as before: add together all the Differences into one Sum;

Then { *As the Sum of all the Differences : Is to the Sum of all the Quantities given :: So is every particular Difference : To it's particular Quantity.*

Queſt. 6. Let it be required to mix Wheat at 5 *s.* the Buſhel, with Rye at 3 *s.* 6 *d.* the Buſhel; ſo that the whole Quantity may be 27 Buſhels, to be ſold for 4 *s.* 4 *d.* a Buſhel; what Quantity of each muſt be taken to make up the Mixture?

Mean Rate 52 { Wheat 60 *d.* } { 10
Rye 42 *d.* } { 8

18 = their Sum.

Then 18 : 27 :: { 10 : 15 } the Quantities required.
{ 8 : 12 }

Queſtion 7. Suppoſe it were required to mix Malaga at 7 *s.* 6 *d.* the Gallon, with *Canary* at 6 *s.* 9 *d.* the Gallon; Sherry at 5 *s.* the Gallon, and White Wine at 4 *s.* 3 *d.* the Gallon; ſo that the whole Mixture may be 90 Gallons; to be ſold for 6 *s.* the Gallon: How much of each ſort will compoſe that Mixture?

Mean Rate = 72 *d.* { Malaga 90 } { 21
White 51 } { 18
Canary 81 } { 9
Sherry 60 } { 12

60 = their Sum.

Then 60 : 90 :: { 21 : 31½ / 18 : 27 / 9 : 13½ / 12 : 18 } the Gallons of { Malaga. / White Wine / Sherry. / Canary.

Or thus, 72 { Malaga 90 } { 12
Sherry 60 } { 18
Canary 81 } { 21
White 51 } { 9

60 their Sum.

Then 60 : 90 :: { 12 : 18 / 28 : 27 / 21 : 31½ / 9 : 13½ } Gallons of { Malaga. / Sherry. / Canary. / White Wine.

Either of theſe Ways do equally anſwer the Queſtion, as may be eaſily tried by *Alligation Medial*. As before, *&c.*

Note, *The Work of theſe Proportions may be much ſhortened (eſpecially when there are many Ingredients to be mixed) if you obſerve the ſame Method as was propoſed in the* Rule of Fellowſhip, *page* 99, *&c.*

I have made Uſe of the very ſame *Examples* both in *Alligation Medial*, and *Alternate*, throughout the three *Caſes*; being, as I preſume, much better than if they had been different ones; becauſe the Learner may (if he conſider them a little) eaſily perceive, not only the Difference between the two Rules, but alſo wherein the

the chief Differeuce of each Caſe in the *Alternate Rule* depends, *&c.* Not but that I could have inſerted many various *Examples*, as alſo the Manner of compoſing Medicines, *&c.* which, for Brevity ſake, I have omitted, and refer thoſe that deſire to ſee into that Buſineſs to Sir *Jonas More*'s *Arithmetick*, wherein he will find it largely handled. And ſo I ſhall conclude with *Alligation Alternate*, which altho' it gives true Anſwers to Queſtions of that Kind, wirh ſome little Variety, according as the Ingredients are more or leſs in Number; as appears by the foregoing *Examples*; yet it will not give all the Anſwers ſuch Queſtions are capable of, nor perhaps thoſe which ſuit beſt with the preſent Occaſion: Nor can this Imperfection be remedied by common *Arithmetick*; but by an *Algebraick* way of arguing it may; whereby all the poſſible Anſwers to any Queſtion may be clearly and eaſily diſcovered; as ſhall be ſhewed further on in the Second Part.

CHAP. X.

Of Metals *and their* Specifick Gravities, &c.

Sect. 1. *Of* Gold *and* Silver.

PURE Gold, free from Mixture with other Metals, uſually called Fine Gold, is of ſuch a Nature and Purity that it will endure the Fire without waſting, although it be kept continually melted: And therefore ſome of the ancient Philoſophers have ſuppoſed the Sun to be a Globe of liquid or melted Gold.

Silver having not the Purity of Gold, will not endure the Fire like it: Yet Fine Silver will waſte but a very little by being in the Fire any reaſonable time; whereas Copper, Tin, Lead, *&c.* will not only waſte, but may be calcined or burnt to a Powder.

Both Gold and Silver in their Purity, are ſo very flexible or ſoft (like new Lead, *&c.*) that they are not ſo uſeful either in Coin, or otherwiſe (except to beat in Leaf-Gold or Silver) as when they are allay'd, or mixed and hardened with Copper or Braſs. And altho' moſt Places differ more or leſs in the Quantity of ſuch Allay, yet in *England* it is generally agreed on, that,

Standard

Standard for Gold.

22 Carracts of Fine Gold, and 2 Carracts of Copper, being melted together ſhall be eſteemed the true Standard for Gold Coin, *&c. (The French and Spaniſh Gold being very near of the ſame Standard).* That is, if any Quantity or Weight of Fine Gold, be divided into Twenty-four equal Parts, and 22 of thoſe Parts be mixed with 2 of the like Parts of Copper; that Mixture is called Standard Gold.

Whence you may obſerve, that a Carract is not any certain Quantity or Weight, but $\frac{1}{24}$ part of any Quantity or Weight; and the *Minters* and *Goldſmiths* divide it into 4 equal Parts, which they call Grains of a Carract; alſo they ſubdivide one of thoſe Grains, into Halves, Quarters, *&c.*

Standard for Silver.

Eleven Ounces and Two Penny-weight of Fine Silver, and 18 Penny-weight of Copper being melted together, is eſteemed the true Standard for Silver Coin, called Sterling Silver. And ſo in Proportion for a greater or leſſer Quantity; which is a leſs Proportion of Allay for Silver, than the other is for Gold.

Note, When either Silver or Gold is finer than Standard, it is called Better; if coarſer, it is called Worſe; and that Betterneſs or Worſeneſs, is reckoned by Carracts and Grains of a Carract in Gold, and by Penny-weights in Silver; and is thus diſcovered: The *Goldſmiths* or *Refiners*, &c. take a ſmall Quantity of ſuch Gold as they intend to try (which they call making an *Aſſay)* and weigh it very exactly, then they put it into a Crucible, and melt it in a ſtrong Fire, ſo long, that if there be any Copper, or other Allay mixt with it, that Allay may be conſumed or burnt away: When it is cold they weigh it very exactly again, and if it has loſt nothing of it's firſt Weight, they conclude it is Fine Gold, but if the Loſs be $\frac{1}{24}$ Part, they call it 23 Carracts Fine, or one Carract better than Standard: If it has loſt $\frac{2}{24}$ Parts it is 22 Carracts Fine, or Standard: If $\frac{3}{24}$ Parts it is ſaid to be 21 Carracts Fine, or rather one Carract worſe than Standard, and ſo in Proportion as it happens to be better or worſe.

In the ſame Manner they make their Aſſay on Silver, only they compute it's Loſs by Penny-weights, *&c.*

The Author of the *Preſent State of England*, mentioned before (*page* 32.) ſays,

'That

'That the *Engliſh* Coin may want neither the Purity nor 'Weight required, it is moſt wiſely and carefully provided, that 'once every Year the chief Officers of the *Mint* appear before 'the Lords of the Council in the *Star-Chamber* at *Weſtminſter*, 'with ſome Pieces of all ſorts of Monies coined the foregoing 'Year, taken at adventure out of the *Mint*, and kept under 'ſeveral Locks, by ſeveral Perſons, till that Appearance, and 'then by a Jury of 24 able *Goldſmiths*, in the Preſence of the 'ſaid Lords, every Piece is moſt exactly weighed and Aſſay'd.'

This if it were conſtantly practiſed would keep our Coin to it's true Standard, *&c.*

Many pretty Queſtions may be ſtarted concerning the Fineneſs of Gold and Silver, *&c.*

EXAMPLE 1.

If an Ingot of Silver weighing 787 *Oz.* 14 *Pwt.* 6 *Grains*, be 11 *Oz.* 6 *Pwt.* fine; How much fine Silver is there in it, and what amounts it to, at 5 *s.* 1 $\frac{1}{2}$ *d.* the Ounce?

This Ingot is better than Standard by 4 *Pwt.* For 11 *Oz.* 2 *Pwt.* = 222 *Pwt.* the fine Silver in 12 *Oz.* of Standard. But 11 *Oz.* 6 *Pwt.* = 226 *Pwt.* the fine Silver in 12 *Oz.* according to the Queſtion.

Firſt 787 *Oz.* 14 *Pwt.* 6 *Grains* = 378102 *Grains.*
And 12 *Oz.* = 240 *Pwt.*
Then, As 240 : 226 :: 378102 : 356046 $\frac{1}{20}$ = 741 *Oz.* 15 *Pwt.* 6 $\frac{1}{20}$ *Grains* the fine Silver in that Ingot.
Which at 5 *s.* 1 $\frac{1}{2}$ *d.* the Ounce, amounts to 190 *l.* 1 *s.* 6 *d.* and near a Half-penny.

EXAMPLE 2.

If an Ingot of Gold weighing 115 *Oz.* 13 *Pwt.* 18 *Grains*; be $\frac{1}{4}$ of a Grain worſe than Standard: How much Standard Gold is there in it, and what comes it to at 3 *l.* 11 *s.* an Ounce?

Firſt 115 *Oz.* 13 *Pwt.* 18 *Grains* = 55530 Grains *Troy.*
Then 24) 55530 (2313,75 = a Carract of that Quantity.
And 4) 2313,75 (578,4375 = a Grain of that Carract.
Conſequently 4) 578,4375 (144,609375 = $\frac{1}{4}$ of a Grain.

Again, 2313,75 × 22 = 50902,5 ought to be the fine Gold in that Ingot, if it had been Standard:

Bu

But 50902,5 — 144,609375 = 50757,890625 is the Quantity of fine Gold according to the Queſtion. Therefore 509025 : 50757,890625 :: 55530 : 55372,244 &c. Grains = 115 *Oz.* 7 *Pwt.* 4,244 &c. Grains *Troy*, being the Quantity of Standard Gold in that Ingot, as was required.

Next for the Value of it at 3 *l.* 11 *s. per* Ounce; 1 *Oz.* = 480 Grains; and 3 *l.* 11 *s.* = 71 *s.* Conſequently 480 : 71 :: 55372,244 &c. : 8190,4777 &c. = 409 *l.* 10 *s.* 5 $\frac{3}{4}$ *d.* very near; being the Value of that Ingot, as was required.

Or the laſt Queſtion may be otherwiſe wrought thus; 115 *Oz.* 13 *Pwt.* 18 *Grains* = 115,6875. And $\frac{1}{4}$ of a Grain of a Carract is $\frac{1}{16}$ (*viz.* the $\frac{1}{4}$ of $\frac{1}{4}$) Then 22 — $\frac{1}{16}$ = 21 $\frac{15}{16}$ = 21,9375. Conſequently 22 : 21,9375 :: 115,6875 : 115,358842 &c. = 115 *Oz.* 7 *Pwt.* 4,244 *Grains*, &c. as before.

Next for the Value; as 1 : 3,55 :: 115,358842 : 409,523889 = 409 *l.* 10 *s.* 5 $\frac{3}{4}$ *d.* very near: as before.

Sect. 2. *The* Specifick Gravity *of* Metals, &c.

I Take an Enquiry made about the different Gravities, or Weights of Metals, and other Bodies, to be (not only a Work of Curioſity, but alſo) of very good Uſe upon many Occaſions. Therefore ſeveral Authors have given us ſuch Proportions, or Difference of their Weights, as they are ſaid to have one to another; ſuppoſing every one of them to be of the ſame Magnitude or Bigneſs. Some of which I ſhall here inſert.

1. *Henry Van Etten*, in his *Mathematical Recreations*, printed *Anno* 1633, ſets down the Proportion of their Weights thus; Gold 1875. Lead 1165. Silver 1040. Copper 910. Iron 810. Tin 750. Water 100.

2. One *Alſted*, in his *Encyclopædia*, printed 1649, hath them thus: Gold 1875. Quickſilver 1500. Lead 1165. Silver 1040. Copper 910. Iron 806. Tin 750. Honey 150. Water 100. Oil 90. Theſe ſeem to be taken from thoſe of *Van Etten*'s, with ſome Additions only.

3. The ingenious Mr *Oughtred*, in his *Circles of Proportions*, printed *Anno* 1660, hath their Proportions (according to the Experiments of one *Marinus Ghetaldi*, in his Tract called *Archimedes Promotus*) thus: Gold 3990. Quickſilver 2850. Lead 2415. Silver 2170. Braſs 1890. Iron 1680. Tin 1554.

4. In

4. In the Philosophical Transactions, (*Number* 169 and 199) there is an Account of a great many Experiments of this Kind; from whence I collected these following, *viz.* Gold 18888. Mercury 14019. Lead 11343. Silver 11087. Copper 8843. Hammered Brass 8349. Cast Brass 8100. Steel 7852. Iron 7643. Tin 7321. Pump-water 1000.

These last Proportions being approved of and published by Order of the *Royal Society* seem to be unquestionably true: Nevertheless, because they differ so much from the beforementioned (*and those from one another*) I have for my own Satisfaction made several Experiments of that Kind: And have (*I presume*) obtained the Proportions of Weight that one Body bears to another of the same Bulk or Magnitude, as nicely as the Nature of such Matter, which may be contracted or brought into a lesser Body (*viz.* either by Drying, or Hammering, or otherwise) will admit of; which are as followeth:

		Oz. Troy	||	*Oz. Avoir.*
A *Cubick Inch of*	*Fine Gold*, is	10,359273	=	11,365602
	Standard Gold,	9,962625	=	10,930422
	Quicksilver,	7,384411	=	8,101753
	Lead,	5,984010	=	6,553885
	Fine Silver,	5,850035	=	6,418324
	Standard Silver,	5,556769	=	6,096569
	Rose Copper,	4,747121	=	5,208369
	Plate Brass,	4,404273	=	4,832116
	Cast Brass,	4,272409	=	4,630300
	Steel,	4,142127	=	4,544505
	Common Iron,	4,031361	=	4,422979
	Block Tin,	3,801519	=	4,236638
	Fine Marble,	1,429411	=	1,568859
	Common Glass,	1,360841	=	1,493037
	Alabaster,	0,988456	=	1,084477
	Dry Ivory,	0,962083	=	1,055542
	Dry Box-wood,	0,543282	=	0,596057
	Sea Water,	0,542742	=	0,594894
	Common Clear Water,	0,527458	=	0,578697
	Red Wine,	0,523766	=	0,574646
	Proof Spirits of Brandy,	0,489268	=	0,536796
	Sound Dry Oak,	0,489008	=	0,536569
	Linseed Oil,	0,491591	=	0,539345
	Oil Olive,	0,481569	=	0,528350

In this Table you have the Specifick Gravity or Weight of a Cubick Inch of various ſorts of Bodies, both in *Troy* Ounces and *Avoirdupois* Ounces, and Decimal Parts of an Ounce, which I can aſſure you required more Charge, Care, and Trouble, to find out nicely, than I was at firſt aware of.

Now from hence it will be eaſy to determine the Weight of any propoſed Quantity, of the ſame Matter and Kind with thoſe in the Table; it's Solid Content being given in Cubick Inches. For it is plain, that if the Number of Cubick Inches contained in any given Quantity, be multiplied with the tabular Weight of one Inch, *(of the ſame kind of Matter)* the Product will be the Weight of that Quantity in Ounces, *&c.*

EXAMPLE.

Suppoſe it were required to find the Weight of a Piece of Marble, containing three Solid Feet, and 40 Cubick Inches.

Firſt $1728 \times 3 = 5184$ the Cubick Inches in 3 Solid Feet.

And $5184 + 40 = 5224$ the Number of Cubick Inches in the Piece of Marble.

Then $5224 \times 1{,}429411 = 7410{,}066624$ Ounces *Troy*.

Or $5224 \times 1{,}568859 = 8195{,}719416$ Ounces *Avoirdupois*.

The Weight of that Piece of Marble in Ounces, *&c.* which is eaſily brought into Pounds, *&c.* The like for any of the reſt.

The Converſe of this Work is as eaſy; *viz.* If the Weight of any propoſed Quantity be given, thence to find the Solid Content of that Quantity in Cubick Inches, *&c.*

Thus, divide the given Weight of the propoſed Quantity *(it being firſt reduced into Ounces,* &c.) by the tabular Weight of one Inch *(of the ſame kind of Matter)* and the Quotient will be the Number of Cubick Inches contained in that Quantity.

Note, If you would find what Weight any Quantity of thoſe Bodies mentioned in the Table will have, when it is immerſed or put into Water, you muſt ſubſtract the Weight of an equal Quantity of Water (with that of the Body) from the Weight of the propoſed Body (if it be heavier than Water) and there will remain the Weight required. As for Inſtance,

A Cubick Inch of Lead $= 5{,}984010$
A Cubick Inch of Water $= 0{,}542742$ } Ounces *Troy*, &c.

their Difference is $= 5{,}441268$ the Weight of a Cubick Inch of Lead in the Water, *&c.*

CHAP.

CHAP. XI.

Evolution, *or Extracting the* Roots *out of all* Single Powers; *by one Geometrical Method.*

SECT. 1.

Evolution is the Unravelling, or as it were the Unfolding and Resolving any proposed Power or Number, into the same Parts of which it was composed, or supposed to be made up. Now in order to perform that, it will be convenient to consider how those Powers are composed, *&c.*

A Square Number is that which is equally equal; or which is contained under two equal Numbers. *Euclid.* 7. *Def.* 18. Thus the Square Number 4 is composed of the two equal Numbers 2 and 2. *viz.* $2 \times 2 = 4$. Or the Square Number 9 is composed of the two equal Numbers 3 and 3. *viz.* $3 \times 3 = 9$: according to *Euclid.* That is, if any Number be multiplied into itself, that Product is called a Square Number.

A Cube is that Number which is equally equally equal, or which is contained under three equal Numbers. *Euc.* 7. *Def.* 19. Thus the Cube Number 8 is composed of the three equal Numbers 2 and 2 and 2. *viz.*, $2 \times 2 \times 2 = 8$, *&c.* That is, if any Number be multiplied into itself, and that Product be multiplied with the same Number; the second Product is called a Cube Number.

These two, *viz.* the Square and Cube Numbers, borrow their Names from *Geometrical* Extensions or Figures; as from the three Signal Quantities mentioned in *page* 2. That is, a *Root* is represented by a Line or Side, having but one Dimension, *viz.* that of Length only. The Square is a Plane or Figure of two Dimensions, having equal Length and Breadth. The Cube is a Solid Body of three Dimensions, having equal Length, Breadth, and Thickness: But beyond these three, Nature proceeds not, as to *Local* Extension. That is, the Nature of Place or Space admits no Room for other Ways of Extension, than Length, Breadth, and Thickness. Neither is it possible to form or compose any Figure or Body beyond that of a Solid.

And therefore all the superior Powers above the Cube or third Power; as the *Biquadrat* or fourth Power, the *Surfolid* or fifth Power, *&c.* are best explained and understood by a Rank or Series of Numbers in *Geometrical Proportion.* For instance: Suppose any Rank of *Geometrical Proportionals*, whose first Term and Ratio are the same; and to them let there be assigned a Series

of Numbers in *Arithmetical Progreſſion*, beginning with an Unit or 1, whoſe common Difference is alſo 1, as in *page* 79.

Thus, { 1 . 2 . 3 . 4 . 5 . 6 . 7 Indices.
2 . 4 . 8 . 16 . 32 . 64 . 128 &c. in ÷

Then are thoſe Numbers in ÷ produced by a contiued Multiplication of the firſt Term or Root into it ſelf; and thoſe in *Arithmetical Progreſſion* or **Indices**, do ſhew what Degree or Power each Term in the *Geometrical Proportion* is of. For Example; In this Series of ÷ 2 is both the firſt Term or Root, and common Ratio of the Series. Then $2 \times 2 = 4$ the ſecond Term or Square; and $2 \times 2 \times 2 = 8$, or $4 \times 2 = 8$, the Cube or third Term; $2 \times 2 \times 2 \times 2 = 16$, or $8 \times 2 = 16$ the fourth Term or Biquadrat. And ſo on for the reſt.

Note, *This is called* **Involution**, viz. *When any Number is drawn into it ſelf, and afterwards into that Product*, &c. *it is ſaid to be ſo often involved into itſelf; and the Indices are the Exponents of their reſpective Powers ſo involved.*

And according to theſe Involutions, is formed the following Table of Powers; wherein the Root is only one ſingle Figure.

Root, or ſingle ſide.	*Square, or ſecond Power.*	*Cube, or the third Power.*	*Biquadrat, or Square ſquared; being the fourth Power.*	*Surſolid, or the fifth Power.*	*Square cubed, or Cube ſquared; the ſixth Power.*	*The ſecond Surſolid, or ſeventh Power.*	*The Biquadrat ſquared, or the eighth Power.*	*The Cube cubed, or the ninth Power, &c.*
	Index (2)	*Index* (3)	*Index* (4)	*Index* (5)	*Index* (6)	*Index* (7)	*Index* (8)	*Index* (9)
1	1	1	1	1	1	1	1	1
2	4	8	16	32	64	128	256	512
3	9	27	81	243	729	2187	6561	19683
4	16	64	256	1024	4096	16384	65536	262144
5	25	125	625	3125	15625	78125	390625	1953125
6	36	216	1296	7776	46656	279936	1679616	10077696
7	49	343	2401	16807	117649	823543	5764801	40353607
8	64	512	4096	32768	262144	2097152	16777216	134217728
9	81	729	6561	59049	531441	4782969	43046721	387420489

This Table plainly ſhews (by Inſpection) any Power (under the Tenth) of all the nine Figures; and from thence may be taken the neareſt Root of any Square, Cube, Biquadrat, &c. of any Number whoſe Root or Side is a ſingle Figure.

But

But if the Root consists of two, three, or more Places of Figures, then it must be found by piece-meal, or Figure after Figure, at several Operations.

The Extraction of all Roots, above the Square (*viz.* of the Cube, Biquadrat, Surſolid, *&c.*) hath heretofore been a very tedious and troublesome Piece of Work: All which is now very much shortened, and rendered easy, as will appear further on.

When any Number is proposed to have it's Root extracted, the first Work is to prepare it, by Points set over (or under) their proper Figures; according as the given Power whose Root is sought doth require; and that is done by considering the Index of the given Power, which for the Square is 2, for the Cube 3, for the Biquadrat is 4, *&c.* (as in the precedent Table) Then allow so many Places of Figures in the given Power, for each single Figure of the Root, as it's Index denotes; always beginning those Points over the Place of Unity, and ascend towards the Left-Hand if the given Number be Integers, and descend towards the Right-Hand in Decimal Parts. As in these following.

Suppose any given Number; as 75640387246 which I shall all along hereafter call the Resolvend.

Then if it be required to extract any of the following Roots, it must be pointed (according to the forementioned Consideration) in this Manner:

	Square Root Thus	75640387246
Viz. For the	*Cube Root*	75640387246
	Biquadrat Root	75640387246
	Surſolid Root	75640387246
Or suppose the Number to be		0,674035982
	Square Root Thus	0,6740359820
Then for the	*Cube Root*	0,674035982
	Biquadrat Root	0,674035982000

Now the Reason of pointing the given Resolvend in this manner; *viz.* the allowing two Figures in the Square; three Figures in the Cube, and four Figures in the Biquadrat, *&c.* for one Figure in the Root, may be made evident several ways; but I think it is easily conceived from the Table of single Powers, wherein you may observe that all the Powers of the Figure 9 (which

(which is but a ſingle Figure) have the ſame Number of Places of Figures, as the Index of thoſe Powers denotes: Therefore ſo many Places of Figures muſt be taken or aſſigned for every ſingle Figure in the Root. Conſequently by theſe Points is known how many Places of Figures there will be in the Root, *viz.* So many Points as there are, ſo many Figures there muſt be in the Root, and whether they muſt be Integers or Decimal Parts, is eaſily determined by the reſpective Places of the Points.

Sect. 2. *To* Extract *the* Square Root.

AND firſt how to extract the Square Root, according to the common Method.

Having pointed the given Reſolvend into Periods of two Figures as before directed; then by the Table of Powers (or otherwiſe) find the greateſt Square that is contained in the firſt Period towards the Left-Hand (ſetting down it's Root, like a Quotient Figure in *Diviſion*) and ſubſtract that Square out of the ſaid Period of the Reſolvend: To the Remainder bring down the next Period of Figures, for a Dividend, and double the Root of the firſt Square for a Diviſor; enquiring how often it may be had in that Dividend, ſo as when the Quotient Figure is annexed to the Diviſor, and that increaſed Diviſor multiplied with the ſame Quotient Figure, the Product may be the greateſt Number that can be taken out of that Dividend; which ſubſtract from the ſaid Dividend, and to the Remainder bring down the next Period of Figures, for another new Dividend: Then ſee how often the laſt increaſed Diviſor can be had in the new Dividend (*with the ſame Caution as before*, viz.) ſo as that the Quotient Figure being annexed to the Diviſor, and that increaſed Diviſor multiplied with the ſame Quotient Figure, their Product may be the greateſt Number that can be ſubſtracted from the new Dividend. (As before) And ſo proceed on from Period to Period (*viz.* from Point to Point) in the very ſame Manner, until all be finiſhed.

An Example or two being well obſerved will render the Work of forming the new Diviſors, *&c.* more plain and eaſy than can be expreſſed in a Multitude of Words.

Example 1. Let it be required to extract the Square Root out of 572199960721. This Reſolvend being prepared or pointed as before directed, will ſtand

Thus,

Thus, 57219960721 (756439 the Root.
49 = the greateſt Square in 57.

1. Diviſor	145)	821
	5	725 = 145 × 5
2. Diviſor	1506)	9699
	6	9036 = 1506 × 6
3. Diviſor	15124)	66396
	4	60496 = 15124 × 4
4. Diviſor	151283)	590007
	3	453849 = 151283 × 3
5. Diviſor	1512869)	13615821
	9	13615821 = 1512869 × 9

Proof 756439 × 756439 = 57219960721 the Reſolvend.

Example 2. What is the Square Root of 1850701,764025?

Operation 1850701,764025 (1360,405

1	
23)	85
3	69
266)	1607
6	1596
17204)	1101,76
4	1088 16
1720805)	13 604025
5	13 604025
	(0)

{ Hence 1360,405 is the Root required.

Ex. 3. What is the Square Root of 0,06076225 Dec. Parts?

Operation 0,06076225 (0,2465 the Root required.

,04 = ,2 × ,2	
,44)	207
4	176
,486	3162
6	2916
,4925)	24625
5	24625
	(0)

Proof { 0,2465 × 0,2465 = 0,06076225 the Reſolvend.

What

What is here done in whole Numbers, mixed Numbers, and Decimals, may alſo be done in Vulgar Fractions; if you firſt change the given Fraction into Decimals. (As in *Sect.* 5. *p.* 68.)

Example 4. Let it be required to extract the Square Root of $\frac{16}{25}$. Firſt $\frac{16}{25} = 0{,}64$

Then 0,64 (,8 *the Root required.*
,64
(0)

In theſe 4 Examples the Reſolvend hath been a perfect Square; and therefore the Root hath been extracted without leaving any Remainder: But it very often happens that the Reſolvend is not a true Figurate Number, according to the propoſed Power. That is, it is not a perfect Square, Cube, Biquadrat, *&c.* and then ſomething will remain after the Extraction hath been made throughout all the Points. Such Numbers are called **Surd** Numbers, and their Roots can never be truly found, but will become a continued Series, *ad infinitum:* If to the Remainder there be ſtill annexed Cyphers according as the propoſed Power requires, *viz.* by two's in the Square; three's in the Cube; four's in the Biquadrat, *&c.* and the Operations continued on as before.

Example 5. Suppoſe it were required to extract the Square Root of 6968.

Operation 6968 (83,4745, *&c.*
64
163) 568
3 489
1664) 79,00
4 66 56
16687) 12 4400
7 11 6809
166944) 759100
4 667776
1669485) 9132400
5 8347425
1669490) 784975 *&c.*

Then the Root of any Surd Number may be continued on to what Exactneſs you pleaſe, but cannot be truly found.

In my *Compendium of Algebra*, Chap. 9. I have propoſed another Way of extracting the Square Root, and there given Examples of the Work; which to avoid Prolixity is thus:

Having

Having pointed the given Reſolvend, and taken the greateſt Square to the firſt Point from it, as before. Then divide the Remainder of the whole Reſolvend by 2 (that is, half it) and point it a-new. (This I call a new Dividend) Then make the Root of the firſt Square a Diviſor, inquiring how often it may be found in the new Dividend to the next Figure forward, reſerving that Figure under the next Point for the half Square of the Quotient Figure. Which being found, multiply the Diviſor with it, adding to that Product the Tens of the half Square if there be any, as in plain Diviſion. Then annex the Quotient Figure to the laſt Diviſor for a new Diviſor, with which proceed in all Reſpects as with the laſt Diviſor; and ſo on until all be finiſhed.

Example 6. What is the Square Root of 2990667969?

Operation 2990667969
−25 (5 The firſt ſingle Root
2) 490667969 The Remainder to be divided by 2.

First Root 5) 24533398 4,5 (54687
+4 208 = 56 × 4 : + ½ the Square of 4, *viz.* $\frac{16}{2}$ = 8.
Diviſor 54) 3733
+6 3258 = 54 × 6 : + ½ the Square of 6.
Diviſor 546) 47539
+8 43712 = 546 × 8 : + ½ the Square of 8.
Diviſor 5468) 382784,5
+7 382784,5 = 5468 × 7 : + ½ the Square of 7.
(0)

Hence the Root is found to be 54687, as was required.

All the Difficulty in this Method is only the true placing of the half Square of the Quotienr Figure, when it happens to be an odd Number: In that Caſe you muſt bring down one Figure more of the Dividend; *viz.* of the next Period; under which, place the odd 5 that will always ariſe from the half Square of an odd Number: As 7 whoſe Square is 49; the Half of which is 24,5 to be placed as in the laſt Operation of this Example.

N. B. *When the Number of Figures in the Root of any Surd Number are limited; you need not proceed in extracting the whole Root as before; but only to one Figure more than half the deſigned Number of Figures; for the Reſt may be obtained by plain Diviſion only.*

Example 7. Suppoſe it were required to extract the Square Root of 7 (a Surd Number) to have 12 places of Figures in it.

	7 (2,645751	Firſt Part of the Root.
	4	
Remainder	3 .	
2)	1,50 = Half the Remainder.	
+ ,6	1,38 = 2×,6 : + ½ the Square of 0,6=0,18	
2,6)	1200	
+ ,04	1048	
2,64)	152000	
+ ,005	132125	
2,645)	1987500	
+ ,0007	1851745	
2,6457)	13575500	
+ ,00005	13228625	
2,64575)	34687500	
+ ,000001	26457505	
2,645751	8229995	

Having thus got 7 of the 12 Figures required in the Root; the reſt may be eaſily found by the contract Way of Diviſion propoſed in *page* 68.

Thus 2,645751)	8229995	2,64575131106
....	7937253	
	292742	
	264575	
	28167	
	26457	
	1710	
	1097	
	(13)	

Hence I find the Root of 7 to be 2,64575131106, *as was required.*

Thus you have two ways of extracting the Square Root, either of which may be practiſed as every one likes beſt.

Sect.

Sect. 3. *To* extract *the* Cube Root.

THE Method I ſhall here propoſe for extracting the Cube Root admits of two Caſes; both which are to be very well obſerved.

Having pointed the given Reſolvend, (as before directed) *viz.* into Periods of 3 Figures; then ſeek a Cube Number by the Table of Powers (or otherwiſe) that comes neareſt to the firſt Period of the Reſolvend, whether it be greater or leſs than that Period.

Caſe 1. If the Cube Number ſo taken, be leſs than the firſt Period of the Reſolvend, call it's Root Leſs than Juſt: And ſubſtract that Cube from the firſt Period of the Reſolvend.

Caſe 2. But if that Cube be greater than the firſt Period of the Reſolvend, call it's Root More than Juſt: And ſubſtract the Reſolvend from that Cube, annexing Cyphers to it, that ſo Subſtraction may be made.

To the firſt Root, whether it be leſs or more than Juſt, annex ſo many Cyphers as there are remaining Points over the whole Numbers of the Reſolvend, and multiply it with 3: Then making that Product a Diviſor, by which you muſt divide the Difference between the Reſolvend and the foreſaid Cube; that Quotient will be the Reſolvend depreſſed to a Square, and therefore muſt be pointed as ſuch, *viz.* into Periods of two Figures each. That being done, make the firſt Root (without thoſe Cyphers that were annexed to it) a Diviſor, enquiring how often it may be found in the firſt Period of the new Reſolvend (as before in extracting the Square Root) with this Conſideration, that if the Root (now a Diviſor) be leſs than Juſt, as in *Caſe* 1. you muſt annex the Quotient Figure to it, and then multiply the Root ſo increaſed into the ſaid Quotient Figure, ſetting down the Unit's Place of their Product under the pointed Figure of that Period, ſubſtracting it, as in Diviſion. And ſo on from one Period to another, as before.

But if the ſaid Root (now a Diviſor) be more than Juſt, as in *Caſe* 2. Then you muſt ſubſtract the Quotient Figure from a Cypher annexed, or ſuppoſed to be annexed, to the ſaid Diviſor; multiplying the Root ſo decreaſed into the Quotient Figure; ſetting down their Product as before, *&c.* An *Example* or two in each *Caſe* will render the Work plain and eaſy.

Note, *Each Quotient Figure ought always to be twice added to the Diviſor, if the Tabular Cube was taken leſs than juſt, or twice ſubſtracted from it, if greater*; viz. *once before you multiply by it, and once with the next Quotient Figure: as will be ſhewn in the*

following Examples; which are therefore more exact and conciſe than as done by the Author in all the former Editions of his Work.

Ex. 1. What is the Cube Root of 146363183 the given Reſolvend, to be pointed thus 146̇363̇183̇ (the firſt Root, leſs than Juſt.

125 = the neareſt Cube to 146

300 × 3 = 1500) 21363183 (14242,12 new Reſolvend

First Root 5

+ 2

1 Diviſor 52) 14242,12 (527 the Root required.

+ 27 104

Diviſor 547) 3842

3829

13 the Remainder to be rejected.

Here the Root 527 is the true Root at the firſt Operation, as may be eaſily tried by involving it.

That is 527 × 527 × 527 = 146363183 the given Reſolvend. But if it had not been the true Root, then every thing that hath been here done muſt have been repeated; only inſtead of the firſt ſingle Root (*viz.* 5.) you muſt have taken the increaſed Root (*viz.* 527) and this I call a ſecond Operation; which would increaſe the laſt Root to nine Places of Figures; *viz.* every Operation triples the Number of Places in the laſt Root; as will appear further on.

N. B. *It often happens that four, five, and ſometimes more Places of Figures may be taken into the Root: eſpecially when the ſecond Place proves to be a Cypher. That is, when the firſt Cube comes very near to the firſt Period of the Reſolvend.*

EXAMPLE 2.

What is the Cube Root of 67̇507̇824̇239̇ (4000 Root leſs than Juſt

First neareſt Cube = 64

Root 4000 × 3 = 12000) 3507824239 (292318,68

First Root 4

+ 07

1 Diviſor 407) 292̇318,68̇ (4071,79

+ 071 2849

2 Diviſor 4141) 7418

+ 1,7 4141

3 Diviſor 4142,7) 3277,68

+ ,79 2899,87

4 Diviſor 4143,49) 377,79 &c.

In

In this *Example* I have taken ſix Figures into the Roots, becauſe the ſecond Place proved to be a Cypher. And in theſe ſix the Exceſs is not an Unit in the laſt Place; for if there were made a ſecond Operation, the Root would be 4071,78 *&c.* as may be eaſily tried.

EXAMPLE 3.

Let it be required to extract the Cube Root out of this Number; . . . . . .

Viz. 976379602989073960279630298890

The neareſt Cube to 976 is 1000 whoſe Root is 10 more than juſt, it's Cube 1000000000000000000000000000000

— 976379602989073960279630298890 the Reſolvend

Remains 23620397010926039720309701110

The firſt Root 10000000000 × 3 = 30000000000 the Diviſor

Then 30000000000) 23620397010926039720309701110 (78734-6567030867990 for a new Reſolvend.

1ſt Root 10

— 007

——

1 Div. 993) 787346567030867990 (10000000000 = 1ſt Root.
— 79 6951 0079364 &c. ſubſtract.

2 Div. 9851) 92246 Remains 9920636000 the Root true to the ſixth Figure, and only too little by an Unit at the ſeventh, at the firſt Operation.
— 93 88659

3 Div. 98417) 358756
— 36 295251

4 Div. 984134) 6350570
— 64 5904804

5 Div. 9841276) 44576630
&c. &c.

For a ſecond Operation (if you require no more than ten Places of Figures true in the Root) you need only aſſume 9920000000; which being leſs than juſt, proceed with as follows:

From the given Reſolvend = 976379602989073960279630298890

Sub. the Cube of 9920000000 = 976191488000000000000000000000

Remainder 1881149890739 6 &c.

Then 3×992 &c. =2976 &c.) 1881149890739 6 &c. (632106818 1 &c. for a new Reſolvend.

99200

99200		
+ 06		
99206)	6321068181	(9920000000 the Root assumed
+ 63	595236	637163,5 add
992123)	3687081	9920637163,5 the Root true to
+ 37	2976369	the tenth Figure, and only too
9921267)	71071281	much by an Unit in the ele-
+ 7 &c.	69448869	venth.
992127*)	1622412	* Here the Additions of the Quo-
....	992127	tient Figure being of no Con-
	630285	ſequence, therefore the Divi-
	595276	ſion is carried on from hence,
	35009	as in *page* 68.
	29763	
	5246	
	4960	
	&c.	

In the ſame manner the Cube Roots of Decimal Parts; or of vulgar Fractions, being firſt changed into Decimals, may be extracted.

Sect. 4. *To* Extract *the* Biquadrat Root.

IN extracting the Biquadrat Root, or that of the Fourth Power; (and indeed the Roots of all even Powers) there are ſome ſmall Difficulties, not ſo eaſily expreſſed and explained in a few Words, as they are by an *Algebraick Theorem* (ſuch as ſhall be ſhewed further on) I have therefore in this Place made choice of extracting ſuch Roots by two ſeveral Extractions; and the rather, becauſe I preſume the Reader by this Time thoroughly acquainted with the Buſineſs of extracting the Square Root, by which this may eaſily be performed. Thus:

Firſt, Extract the Square Root of the propoſed Reſolvend, then the Square Root of that firſt Root will be the Biquadrat Root required.

Example 1. What is the Biquadrat Root of 4857532416? Firſt extract it's Square Root,

Thus

Thus 4857532416
— 36 = the greateſt Square, whoſe Root is 6.
1257532416 Remainder to be divided by 2.

First Root 6)	628766208	(69696
+ 9	5805	
69	4826	
+ 6	4158	
696	668620	
+ 9	626805	
6969	418158	
	418158	
	(0)	

Then 69696 } being the firſt Root, whoſe Square Root muſt now be extracted.
— 4
29696 Remainder to be divided by 2.

First Root 2)	14848	(264 the Biquadrat Root as was required
+ 6	138	
26)	1048	
+ 4	1048	
264	(0)	

This is ſo eaſy I need not inſert any more Examples.

Sect. 5. *To* Extract *the* Surſolid Root.

HAving pointed the given Reſolvend according as it's Index denotes; *viz.* into Periods of five Figures; ſeeking ſuch a Surſolid Number in the Table of Powers (or otherwiſe) as comes the neareſt to the firſt Period of the Reſolvend, whether greater or leſs; and call it's reſpective Root accordingly; *viz.* more than Juſt, or leſs than Juſt; annexing ſo many Cyphers to it, as there are remaining Periods of whole Numbers in the Reſolvend; as before in extracting the Cube Root: Then find the Difference between the Reſolvend, and the Surſolid Number ſo taken, by ſubſtracting the leſſer from the greater, (as before in the Cube). Next find the Cube of the aforeſaid Surſolid Root with it's annexed Cyphers, (which you may alſo do by the Table of Powers) and multiply that Cube with 5 the Index of the Surſolid, the Product muſt be a Diviſor, by which the Difference between the Reſolvend and the Surſolid Number muſt be divided;

divided; that ſo it may be depreſſed to a Square (as before in the Cube) which muſt be pointed into Periods of two Figures each, calling it the new Reſolvend (as before). Then make the firſt Root, without it's Cyphers, a Diviſor, enquiring how often it may be found in the firſt Period of the new Reſolvend, with this Conſideration, if the Root (now a Diviſor, be leſs than Juſt, you muſt annex twice the Quotient Figure to it; but if it be more than Juſt, you muſt ſubſtract twice the Quotient Figure from a Cypher either annexed or ſuppoſed to be annexed to that Diviſor or Root, multiplying it ſo increaſed or diminiſhed, with the ſaid Quotient Figure, ſetting down their Product, *&c.* as before. An *Example* in each *Caſe* will render it plain and eaſy.

Example 1. Suppoſe it be required to extract the Surſolid Root out of this Number 12309502009375.

. . . .

12309502009375 The Reſolvend pointed.

The neareſt Surſolid Number to 1230, the firſt Period of the Reſolvend, is 1024, whoſe Root is 4 being leſs than Juſt.

. . .

Therefore 12309502009375
— 1024

2069502009375 their Difference.

Next the Cube of 400 is 64000000 per Table, *&c.* And 64000000 × 5 = 320000000 the Diviſor.

Then 320000000) 2069502009375 (6497 *&c.*

First Root 400
+ 2 × 10 = + 20

1 Diviſor 420)
+ 20 + 2 × 5 = + 30
450

. . (
6467 (+ 400
42 + 15
2267 415 Root true
2250

17 the Remainder to be rejected.

That is 415 is the Surſolid Root of the given Reſolvend. As may be eaſily tried by involving it to the fifth Power. *Viz.* 415 × 415 × 415 × 415 × 415 = 12309502009375 the given Reſolvend.

Note, *Here again the double Quotient Figure ought to be twice added or ſubſtracted, in the ſame Manner as the ſingle one was directed for the Cube Root*, page 131, *and the Operation for the Surſolid Root in theſe two Examples is performed accordingly: contrary to what was heretofore done by the Author.*

Example

Example 2. What is the Surſolid Root of 2327834559873
The neareſt Surſolid Number to 232 is 243 whoſe Root is 3 being more than juſt.

Therefore 2430000000000
— 2327834559873
Remains 102165440127 For a Dividend.

The Cube of 300 is 27000000 and 27000000×5=135000000
Then 135000000) 102165440127 (756,7810 new Reſolvend.

First Root 300
—2×2=—4

1 Diviſor	296)	756,7810	(300
—4—2×0,5=—	5,0	592	—2,566
2 Diviſor	291,0)	164,78	297,434 The Root only too little by 2 in the loweſt Figure.
—1—2×0,06=—	1,12	145,50	
3 Diviſor	289,88)	19,2810	
	&c.	&c.	

Now the Reaſon why this Root comes out to ſo many Places of Figures at the firſt Operation; is becauſe the firſt Surſolid Number was ſo near the Reſolvend, *&c.* As before.

Sect. 6. *To* extract *the* Root *of the* Square cubed.

THIS may be eaſily performed by two Extractions, according as it's Name denotes. Thus, firſt extract the Square Root of the given Reſolvend; then extract the Cube Root of that Square Root, and it will be the Root required: That is, it will be the Root of the ſixth Power. Or thus, firſt extract the Cube Root of the Reſolvend; then extract the Square Root of that Cube Root, and it will be the Root required.

EXAMPLE 1.

Let it be required to extract the Square cubed Root out of this Number 1452205373535 15625 the Reſolvend.

Firſt I extract the Square Root of this Reſolvend, which I take to be the beſt and eaſieſt Way.

Thus 1452205373535156 25
— 9
Remains 55220537353515625 to be halfed.

Then 3) 2761026867675781 2,5 (381078125
+ 8 272
38) 4102
+ 10 3805
3810) 297686 7
+ 7 2667245
38107) 309622 6
+ 8 3048592
381078) 4763475 7
+ 1 3810780 5
3810781) 9526952 8
+ 2 7621562 2
38107812) 1905390612,5
+ 5 1905390612,5
381078125 (0)

Having found the Square Root of the given Reſolvend, I proceed to extract the Cube Root of that Square Root.

That is, of 381078125
— 343 = the neareſt Cube, it's Root is 700
Then 700 × 3 = 2100) 38078125 (18161

Firſt Root 7..
+ 2
1 Diviſor 72.) 18161 (700
+ 25 144 +25
2 Diviſor 745) 3761 725
3725
(36)

Hence I find 725 to be the Square cubed Root required; as may eaſily be tried by involving it to the ſixth Power. That is, 725 × 725 × 725 × 725 × 725 × 725 will be found = 145220537-353515725 the given Reſolvend.

Sect.

Sect. 7. *To* extract *the* Root *of the* ſeventh Power.

HAving pointed the given Reſolvend, as it's Index denotes, *viz.* into Periods of 7 Figures, ſeek out ſuch a Number of the 7th Power, by the Table of Powers, as comes neareſt to the firſt Period of the Reſolvend; whether it be greater or leſſer, calling it's reſpective Root more than Juſt, or leſs than Juſt, annexing it's proper Number of Cyphers, *&c.* as in the Cube and Surſolid.

Then find the Difference between the given Reſolvend, and that Number of the 7th Power (found by the Table of Powers) by ſubſtracting the leſſer from the greater.

Next find the Surſolid or fifth Power of that Root with it's annexed Cyphers (which you may alſo do by the Table of Powers) and multiply that Surſolid Number with 7, the Index of the given Reſolvend; that Product muſt be a Diviſor, by which the foreſaid Difference muſt be divided, that ſo it may be depreſſed to a Square, to be pointed, *&c.* as before in the Cube, *&c.* then make the firſt Root, without it's Cyphers, a Diviſor; working with it and the new Reſolvend (as before) only here you muſt increaſe, or diminiſh the Diviſor with thrice the Quotient Figure *.

Example. What is the ſecond Surſolid Root, or that of the ſeventh Power,

of 38298655395507 8125 the Reſolvend pointed.
— 2187 the neareſt of the ſeventh Power.
16428655395507 8125 their Difference.

The firſt Root is 300 being leſs than Juſt, and the fifth Power of 300 is 2430000000000 which being multiplied with 7 is 17010000000000 for a Diviſor, by which the aforeſaid Difference muſt be divided; which contracted may ſtand thus,

1701) 16428655 (9658,23 *&c.*

Firſt Root	300		
+3×20=+	60		
	——	. .	(300
1 Diviſor	360)	9658	(+25
60+3×05=+	75	72	325=the true Root required.
2 Diviſor	435)	2458	
		2175	(as before
		283	the Remainder to be rejected,

* That is, by twice adding or ſubſtracting the triple Quotient Figure, as was done with the double Quotient Figure for the Root of the fifth Power, *page* 136; and the ſingle Quotient Figure for the Cube Root, *page* 131.

Hence I have found 325 to be the true Root required, that is, the true Root of the ſeventh Power.

I think it needleſs to proceed farther; *viz.* to inſert *Examples* of higher Powers. For if what is already done be well underſtood, it will be eaſy to conceive how to proceed in extracting the Root of any ſingle Power how high ſoever it be (for the Method is general and alike in all Powers) due regard being had to their Indices; and to the firſt ſingle Side or Root. That is, whether it be More or Leſs than Juſt, *&c.*

Yet methinks I hear the young Learner ſay, it is poſſible to follow the Directions and Examples, as they are here laid down; but ſtill here is not the Reaſon why they are ſo, and ſo, performed; and why there ſhould be a Remainder left after the Root is found; *viz.* when the given Reſolvend hath a true Root of it's Kind.

It is true, the Reaſons of theſe are not here laid down; neither indeed can they be rendered ſo plain and intelligible by Words, as by an Algebraick Proceſs, from whence the *Theorems* or *Rules* here given had their firſt Invention; as ſhall be ſhewed in the next Part, when I come to treat of reſolving compounded or adfected Æquations; however, take this ſhort and general Account of this Method.

This, and all other of the new Methods of Converging Series (as they are called) are very different from the former (and ſtill common) Methods of extracting Roots, which require the firſt ſingle Side or Root of the firſt Period (in any Reſolvend) to be taken exactly true, and then by involving, and other tedious Ways of ordering it, there is formed a Diviſor; which helps to grope out by Trials a ſecond Figure in the Root. And ſo proceed on from Point to Point; ſtill repeating the whole Work for every ſingle Figure that comes into the Root. And if by Chance there be a Miſtake or Error committed in any one Figure (as it is poſſible there may) it ſpoils the whole Proceſs, which muſt then be wholly begun anew, or at leaſt from that Part of it where the Error firſt entered.

But the Nature and Deſign of the Method which I have here laid down is quite otherwiſe; it being ſo contrived, as to gradually leſſen the Difference betwixt any propoſed Power, and the like Power of another Number aſſumed; *viz.* it leſſens that Difference until it is either quite vanquiſhed, or becomes ſo infinitely ſmall as to be inſignificant.

Therefore when any Number is propoſed to have it's Root extracted; it is here required to take the next neareſt Root of the firſt Period in the Reſolvend; that ſo the Difference betwixt the given

given Reſolvend, and the Homogeneal Power (*viz.* the like Power) of the Root thus taken, may be leſs either in Exceſs, or Defect. Which Difference being reduced, or depreſſed lower, becomes ſo prepared, that by plain Diviſion (comparatively) there will ariſe ſuch Quotient Figures as will both correct and increaſe the firſt Root to three Places of Figures at leaſt, ſometimes to four, or five Places of Figures; according as the ſaid firſt Difference happens to be more or leſs (of which you may have obſerved Inſtances): But yet there will be a Remainder left, and perhaps an Exceſs or Defect in the Root ſo increaſed, *viz.* in the laſt Figure of it.

Now to rectify the ſaid Exceſs or Defect in the Root, and to diſcover whether the given Reſolvend be a true Figurate Number, or not: That is, whether it have a true Root of it's kind; it will be neceſſary to make a ſecond Operation; by taking the Root ſo increaſed, and proceeding with it and the given Reſolvend, in all reſpects as in the firſt Work (like to the third *Example* of extracting the Cube Root) I ſay, if the given Reſolvend have a true Root, it will appear at this ſecond Operation, and all the aforeſaid Differences, *&c.* will be vanquiſhed; provided the Root required is not to have more than three (or four) Places of Figures in it.

But if the Root be to have more than three Figures in it; or, that the given Reſolvend prove to be a Surd Number, then there will be a Difference as before; which will afford Quotient Figures to rectify and increaſe the Root laſt taken, to three times as many Places of Figures as it had at the Beginning of that ſecond Operation. As you may ſee in the aforeſaid *Example* 3. of the Cube Root; wherein that Root is increaſed to twelve Places of Figures at two Operations; which if it were to be extracted the Old (and ſtill Common) way, it would require at leaſt forty times the Number of Figures I have here uſed.

Again, if there chance to be a Miſtake committed in any Operation performed by the Method here laid down, that Miſtake will not deſtroy the precedent Work, but will be rectified in the next Operation, although it were not diſcovered before. And thus you may proceed on to a third Operation, which will afford 27 Places of Figures in the Root, *&c.* with very little Trouble, if compared with former Methods.

The brief Account, which I have here given (*by Way of Explaining the Nature of this Method of extracting Roots*) being well conſidered and compared with the ſeveral Operations of the foregoing *Examples*, muſt needs help the Learner to form ſuch an Idea of it, that he cannot (I preſume) but underſtand how to proceed

proceed in extracting the Root out of any ſingle Power, how high ſoever it be; without the Help of an *Algebraick Theorem*. Not but when that comes to be once underſtood, the Work will be much readier and eaſier performed: As will appear in the next Part.

I did intend to have here inſerted the whole Buſineſs of Intereſt and Annuities; but finding that it would require too large a Diſcourſe, to ſhew the Grounds and Reaſons of the ſeveral *Theorems* uſeful therein, I have therefore reſerved that Work for the Cloſe of the next Part. Neither indeed can the raiſing of thoſe *Theorems* be ſo well delivered in Words, as by an *Algebraick* Way of arguing; which renders them not only much ſhorter, but alſo plainer and eaſier to be underſtood.

I have alſo omitted that Rule in *Arithmetick*, uſually called the *Rule of Poſition*, or *Rule of Falſe*: Becauſe all ſuch Queſtions, as can be anſwered by that gueſſing Rule, are much better done by any one who hath but a very ſmall ſmattering of *Algebra*. I ſhall therefore conclude this Part of *Numerical Arithmetick*; and proceed to that of *Algebraick Arithmetick*, wherein I would adviſe the young Learner not to be too haſty in paſſing from one Rule to another, and then he will find it very eaſy to be attained.

AN

INTRODUCTION

TO THE

Mathematicks.

PART II.

PROEM.

HAVING formerly wrote a ſmall Tract of **Algebra**, *perhaps it may ſeem (to ſome) very improper to write again upon the ſame Subject; but only (as the uſual Cuſtom is) to have referred my Reader to that Tract. However, becauſe the following Parts of this Treatiſe are managed by an Algebraick Method of arguing; which may fall into the Hands of thoſe who have not ſeen that Tract, or any other of that Kind; I thought it convenient to accommodate the young Geometer with the firſt Elements, or principal Rules, by which all Operations in this Art are performed; that ſo he may not be at a Loſs as he proceeds farther on: Beſides, what I formerly wrote was only a Compendium of that which is here fully handled at large.*

The Principal Rules are **Addition, Subſtraction, Multiplication, Diviſion, Involution,** and **Evolution,** as in common **Arithmetick** but differently performed; and therefore ſome call it **Algebraick Arithmetick.** Others call it **Arithmetick in Specie,** becauſe all the Quantities concerned in any Queſtion remain in their ſubſtituted Letters (howſoever managed by *Addition, Subſtraction,* or *Multiplication,* &c.) without being deſtroyed or changed into others, as Figures in common *Arithmetick* are.

Mr *Harriot* called it **Logiſtica Speciosa,** or *Specious Computation.*

CHAP.

CHAP I.

Concerning the Method *of* Noting *down* Quantities; *and* Tracing *their* Steps, &c.

Sect. 1. *Of* Notation.

THE Method of noting down Letters for Quantities, is various, according to every one's Fancy; but I shall here follow the same as in my former Tract, and represent the Quantity sought (be it Line or Number, *&c.*) by the small (*a*,) and if more Quantities than one are sought, by the other small Vowels, *e. u.* or *y*.

The given Quantities are represented by the small Consonants, *b. c. d. f. g. &c.*

And for Distinction sake, mark the Points or Ends of Lines in all Schemes, with the capital or great Letters, *viz. A. B. C. D. &c.*

When any Quantity (either given or sought) is taken more than once, you must prefix it's Number to it; as $3a$ stands for a taken three times, or three times a, and $7b$ stands for seven times b, *&c.*

All Numbers thus prefixt to any Quantity, are called Coefficients or Fellow-Factors; because they multiply the Quantity; and if any Quantity be without a Coefficient, it is always supposed or understood to have an Unit prefixed to it; as a is $1a$, or b is $1b$, *&c.*

The Signs by which Quantities are chiefly managed are the same, and have the same Signification, with those in the first Part, *page* 5. which I here presume the Reader to be very well acquainted with. To them must be here added these three more;

Viz. $\left.\begin{matrix}\text{☉}\\ \text{ɯ}\\ \sqrt{\cdot}\end{matrix}\right\}$ the Sign of $\left\{\begin{matrix}\text{Involution.}\\ \text{Evolution, or extracting Roots.}\\ \text{Irrationality, or Sign of a Surd Root.}\end{matrix}\right.$

All Quantities that are expressed by Numbers only (as in *Vulgar Arithmetick*) are called *Absolute Numbers*.

Those Quantities that are represented by single Letters, as, *a. b. c. d.* &c. or by several Letters that are immediately joined together; as *ab. cd.* or *7bd.* &c. are called Simple or Single whole Quantities.

But when different Quantities represented by different or unlike Letters, are connected together by the Signs (+ or —); as $a+b$, $a-b$, or $ab-dc$, &c. they are called Compound whole Quantities.

And

And when Quantities are expreſſed or ſet down like Vulgar Fractions, Thus $\frac{a}{b}$, or $\frac{a+b}{d}$, or $\frac{ab+dc}{b-c}$, &c. they are called Fractional or broken Quantities.

The Sign wherewith Quantities are connected, always belongs to that Quantity which immediately follows it; and therefore all the Quantities concerned in any Queſtion may ſtand in any Order at Pleaſure, *viz.* the moſt convenient for the next Operation.

As $a+b-d$ may ſtand thus $b-d+a$, or thus $a-d+b$, or $-d+a+b$ &c. theſe being ſtill the ſame, tho' differently placed.

That Quantity which hath no Sign before it (as generally the leading Quantity hath not) is always underſtood to have the Sign $+$ before it. As a is $+a$, or $b-d$ is $+b-d$, &c. for the Sign $+$ is the Affirmative Sign, and therefore all leading or Poſitive Quanties are underſtood to have it, as well as thoſe that are to be added.

But the Sign $-$ being the Negative Sign, or Sign of Defect, there is a Neceſſity of prefixing it before that Quantity to which it belongs, wherever the Quantity ſtands.

Sect. 2. Of tracing the Steps uſed in bringing Quantities to an Equation.

THE Method of tracing the Steps, uſed in bringing the Quantities concerned in any Queſtion to an Equation, is beſt performed by regiſtring the ſeveral Operations with Figures and Signs placed in the Margin of the Work, according as the ſeveral Operations require; being very uſeful in long and tedious Operations.

For Inſtance: If it be required to ſet down, and regiſter the Sum of the two Quantities, a and b, the Work will ſtand

Thus	1	a	Firſt ſet down the propoſed Quantities, a and b,
	2	b	over againſt the Figures, 1, 2, in the ſmall Co-
		——	lumn, (which are here called Steps) and againſt 3
1+2	3	$a+b$	(the third Step) ſet down their Sum, *viz.* $a+b$.

Then againſt that third Step, ſet down $1+2$ in the Margin; which denotes that the Quantities againſt the firſt and 2d Steps are added together, and that thoſe in the 3d Step are their Sum.

To illuſtrate this in Numbers, ſuppoſe $a=9$ and $b=6$. Then it will be

Thus	1	a	$=9$
	2	b	$=6$
1+2	3	$a+b=9+6=15$	being the Sum of 9 and 6.

Again, If it were required to ſet down the Difference of the ſame two Quantities; then it will be

Thus	1	$a = 9$
	2	$b = 6$
1—2	3	$a - b = 9 - 6 = 3$ the Diff. between 9 and 6.

Or if it were required to ſet down their Product. Then it will be,

Thus	1	$a = 9$
	2	$b = 6$
1 × 2	3	$a \times b$ or $ab = 9 \times 6 = 54$ the Product of 9 into 6.

&c.

Note, *Letters ſet or joined immediately together (like a Word) ſignify the Rectangle or Product of thoſe Quantities they repreſent; as in the laſt* Example, *wherein* ab = 54 *is the Product of* a = 9 *and* b = 6. *&c.*

Axioms.

1. If equal Quantities be added to equal Quantities, the Sum of theſe Quantities will be equal.

2. If equal Quantities be taken from equal Quantities, the Quantities remaining will be equal.

3. If equal Quantities be multiplied with equal Quantities, their Products will be equal.

4. If equal Quantities be divided by equal Quantities, their Quotients will be equal.

5. Thoſe Quantities, that are equal to one and the ſame Thing, are equal to one another.

Note, *I adviſe the Learner to get theſe five Axioms perfectly by Heart.*

Theſe Things being premiſed, and a perfect Knowledge of the Signs and their Significations being gained, the young *Algebraiſt* may proceed to the following *Rules*. But firſt I muſt make bold to adviſe him here (as I have formerly done) that he be very ready in one Rule before he undertakes the next.

That is, He ſhould be expert in *Addition*, before he meddles with *Subſtraction*; and in *Subſtraction*, before he undertakes *Multiplication*, &c. becauſe they have a Dependency one upon another.

CHAP.

CHAP. II.

Concerning the Six Principal Rules of Algebraick Arithmetick of whole Quantities.

Sect. 1. Addition of whole Quantities.

ADDITION of whole Quantities admits of three Cases.

Case 1. If the Quantities are like, and have like Signs; add the Co-efficients or prefixt Numbers together, and to their Sum adjoin the Quantities with the same Sign.

		Exam. 1.	*Exam.* 2.	*Exam.* 3.	*Exam.* 4.
	1	a	$-a$	$5b$	$-7bc$
	2	a	$-a$	$3b$	$-8bc$
1+2	3	$2a$	$-2a$	$8b$	$-15bc$

Thus		*Exam.* 5	*Exam.* 6.	*Exam.* 7.
	1	$3a+5b$	$3a-5b$	$6ab+12$
	2	$2a+7b$	$2a-7b$	$3ab+24$
1+2	3	$5a+12b$	$5a-12b$	$9ab+36$

The Reason of these Additions is evident from the Work of Common Arithmetick. *For suppose* a *to represent one Crown, to which if I add one Crown, the Sum will be two Crowns, or* 2a, *as in Ex.* 1.

Or if we suppose —a *to represent the Want or Debt of one Crown, to which if another Want or Debt of one Crown be added, the Sum must needs be the Want or Debt of two Crowns, or* —2a; *as in Example* 2. *And so for all the rest.*

Case 2. If the Quantities are alike, and have unlike Signs; substract the Co-efficients from each other, and to their Difference join the Quantities with the Sign of the greater.

		Exam. 8.	*Exam.* 9.	*Exam.* 10.	*Exam.* 11.
	1	$+5a$	$-5a$	$7bc$	$-9abd$
	2	$-3a$	$+3a$	$-6bc$	$+7abd$
1+2	3	$+2a$	$-2a$	bc	$-2abd$

		Exam. 12.	*Exam.* 13.
	1	$7a-5b$	$-8ab-7bc+15$
	2	$-5a+7b$	$+12ab+7bc-24$
1+2	3	$2a+2b$	$4ab-9$

The Reaſon of the Operations in this Caſe may be eaſily underſtood by any one that duly conſiders the comparing of Stocks and Debts together, or the Ballancing of Accounts betwixt Debtor and Creditor. That is, the Affirmative Quantities repreſent the Stock or Creditor: The Negative Quantities repreſent the Debts; and their Sum repreſents the Ballance, &c.

Caſe 3. When the Quantities are unlike, ſet them all down, without altering their Signs; and thence will ariſe compound Quantities, which can be no otherwiſe added but by their Signs.

Thus	1	a	a	$5b+7dc$
	2	b	$-b$	$4a+20$
1 + 2	3	$a+b$	$a-b$	$5b+7dc+4a-20$

Here follow a few Examples wherein all the 3 *Caſes* are promiſcuouſly concerned.

	1	$aa+2ab+bb$	$8ab+bc-37$
	2	$-4ab$	$-7ab-bc+42-6d$
1 + 2	3	$aa-2ab+bb$	$ab+5-6d$
	1	$aa-2ab+bb$	$9bc+7ab-45$
	2	$+4ab+bb$	$4d-6bc-7ab+da$
1 + 2	3	$aa+2ab+bb$	$3bc+4d-45+da$

	1	$5a$	$1+b-ab$
	2	$-7a$	$7c-d$
	3	$+3a$	$+e+f$
1 + 2 + 3	4	a	$1+b-ab+7c-d+4e+f$

	1	$3aa+4abc-bb+30$
	2	$2bb-3aa-2abc-25$
	3	$dd+2aa-3abc-3$
1 + 2 + 3	4	$+dd+2aa+bb-abc+2$

Sect. 2. Subſtraction of whole Quantities.

SUBSTRACTION of whole Quantities is performed by one general Rule.

RULE.

Change all the Signs of the Subtrahend, (viz. *of thoſe Quantities which are to be ſubſtracted*) *or ſuppoſe them in your Mind to be changed; then add all the Quantities together, as before in* Addition, *and their Sum will be the true Remainder or Difference required.*

This

This general Rule is deduced from theſe evident Truths.

To ſubſtract an Affirmative Quantity from an Affirmative, is the ſame as to add a Negative Quantity to an Affirmative: that is $+2a$ taken from $+3a$, is the ſame with $-2a$ added to $+3a$. Conſequently, to ſubſtract a Negative Quantity from an Affirmative, will be the ſame as to add an Affirmative Quantity to an Affirmative: that is $-2a$ taken from $+3a$ will be the ſame with $+2a$ added to $+3a$.

		Exam. 1.	*Exam.* 2.	*Exam.* 3.	*Exam.* 4.
	1	$2a$	$-2a$	$8b$	$-15bc$
		a	$-a$	$3b$	$-8bc$
1—2		a	$-a$	$5b$	$-7bc$

		Exam. 5.	*Exam.* 6.	*Exam.* 7.
	1	$5a+12b$	$5a-12b$	$9ab+36$
	2	$2a+7b$	$2a-7b$	$3ab+24$
1—2	3	$3a+5b$	$3a-5b$	$6ab+12$

		Exam. 8.	*Exam.* 9.	*Exam.* 10.	*Exam.* 11.
	1	$+2a$	$-2a$	bc	$-2abd$
	2	$-3a$	$+3a$	$-6bc$	$+7abd$
1—2	3	$+5a$	$-5a$	$+7bc$	$-9abd$

		Exam. 12.	*Exam.* 13.
	1	$2a+2b$	$4ab-9$
	2	$-5a+7b$	$-8ab-7bc+15$
1—2	3	$7a-5b$	$12ab+7bc-24$

If theſe 13 *Examples* be compared with thoſe in *Addition*, the Work will appear very evident, theſe being only the Converſe or Proof of thoſe; according to the Nature of *Addition* and *Subſtraction* in common *Arithmetick*.

More *Examples* in *Subſtraction*.

	1	$a+b$	$5bc+3da$	$8a+5bd+25$
	2	$a-b$	$5bc-4da$	$7a-3bd-12$
1—2	3	$+2b$	$+7da$	$a+8bd+27$

	1	$c+13$	a	0
	2	$3a-b-2c$	b	$2a-4b$
1—2	3	$3c+13-3a+b$	$a-b$	$-2a+4b$

	2	$a+b-54$	76
	1	$d-3b-bc-75$	$a-b-5d+7c$
1—1	3	$a+4b+bc+21-d$	$76-a+b+5d-7c$

That $a-b$ taken from $a+b$ leaves $+2b$ for the Remainder, as in the firſt of theſe *Examples*, may be thus proved:

Let	1	$a+b=z$	
And	2	$a-b=x$	
$2+b$	3	$a=x+b$	*per Axiom* 1.
$1-3$	4	$b=z-x-b$	*per Axiom* 2.
$4+b$	5	$2b=z-x$ which was to be proved.	

The Truth of all Operations in *Subſtraction*, where any Doubt ariſes, may be proved, by adding the Subtrahend to the Remainder, as in common *Arithmetick*.

EXAMPLE.

From	1	$+5a$	0	$-9bc$	
Take	2	$-2a$	$+3b$	$-6da$	Subtrahend.
1—2	3	$+7a$	$-3b$	$+6da-9bc$	Remainder.
2+3	4	$+5a$	0	$9bc$	Proof.

Sect. 3. Multiplication of whole Quantities.

MULTIPLICATION of whole Quantities admits of three Caſes.

Caſe 1. When the Quantities have like Signs, and no Co-efficients, ſet or join them together, and prefix the Sign $+$ before them; and that will be their Product.

		Exam. 1.	*Exam.* 2.	*Exam.* 3.	*Exam.* 4.
Thus {	1	a	$-a$	$a+b$	$-a-b$
	2	b	$-b$	d	$-d$
1×2	3	ab	$+ab$	$ad+bd$	$+ad+bd$

Caſe 2. If there be Co-efficients; multiply them, and to their Product adjoin the Quantities ſet together as before.

Thus

		Exam. 5.	Exam. 6.	Exam. 7.	Exam. 8.
Thus	1	$5a$	$-6d$	$3a+2b$	$a+b$
	2	$3b$	$-7b$	6	$5b$
1×2	3	$15ab$	$+42db$	$18a+12b$	$5ab+5bb$

Case 3. When the Quantities have unlike Signs; join them and the Product of their Co-efficients together (as before) but prefix the Sign — before them;

		Exam. 9	Exam. 10.	Exam. 11.	Exam. 12.
Thus	1	$+a$	$-6d$	$4a-7b$	$4a-7b$
	2	$-b$	$+7b$	$3f$	$-3f$
1×2	3	$-ab$	$-42db$	$12af-21bf$	$-12af+21bf$

That is, + into +, or — into —, gives + } in the Product.
But + into —, or — into +, gives — }

That + into + will produce + in the Product is evident from *Multiplication* in Common *Arithmetick*: viz. +5 into +7 will give +35 *&c.* But that + into —, or — into + should produce the Sign —, as in the four last Examples: and that — into — should produce the Sign +, as in the second, fourth, and sixth Examples, may perhaps seem somewhat hard to be conceived; and requires a Demonstration.

First to prove that $-7b$ into $+3f=-21bf$. As in *Ex.* 11.

Suppose	1	$4a-7b=0$
Then will	2	$4a=7b$ *per Axiom* 1.
But	3	$+3f=+3f$
2×3	4	$12af=21bf$ *per Axiom* 3.
$4-21bf$	5	$12af-21bf=0$ *per Axiom* 2.

Consequently + into —, or — into + produces —, which was the Thing to be proved.

Secondly to prove that $-7b$ into $-3f$ gives $+21bf$ as in *Example* 11.

Let	1	$4a-7b=0$ } as before,
Then	2	$4a=7b$ }
But	3	$-3f=-3f$
the 2×3 is	4	$-12af=-21bf$ by what is proved above.
$4\ 21bf$	5	$-12af+21bf=0$. *per Axiom* 1.

Consequently — into — gives + which was to be proved.

Or

Or these may be otherwise proved by Numbers.

Thus, suppose $\left\{\begin{matrix} a=20 \\ b=14 \end{matrix}\right\}$ and $\left\{\begin{matrix} c=12 \\ d=8 \end{matrix}\right\}$ { or any other Numbers.

Then $\overline{a-b}=6$ $\overline{c-d}=4$ *per Axiom* 2.

Consequently, $\overline{a-5b} \times \overline{c-d}=6\times 4=24$, *per Axiom* 3. but $\overline{a-b}\times\overline{c-d}$, according to the precedent Rules, will be, $ac-cb+bd-da$, which if true must be equal to 24.

Proof $\left\{\begin{matrix} ac=20\times 12=240 \\ bd=14\times 8=112 \end{matrix}\right.$ $\begin{matrix} cb=12\times 14=168 \\ da=8\times 20=160 \end{matrix}$

Hence $ac+bd=352$ *per Axiom* 1.

And $cb+da=328$ which being subtracted,

Leaves $ac+bd-cb-da=352-328=24$, which plainly shews,

That + into − produces − } in the Product,
And − into − produces + }

Q. E. D.

Note, If the Multiplier consists of several Terms, then every one of those Terms must be multiplied into all the Terms of the Multiplicand; and the Sum of those particular Products will be the Product required, as in Common *Arithmetick*.

EXAMPLES.

	1	$a+b-d$	$7b+5d$
	2	$a-b$	$3a-5f$
1 × *a*	3	$aa+ba-da$	$21ba+15da$
1 × *b*	4	$-ba-bb+db$	$-35bf-25df$
3+4	5	$aa-da-bb+db$	$21ba+15da-35bf-25df$

	1	$aa-ba$	$2c-3d$
	2	$a+b$	$3a-4b$
1 × 2	3	$aaa-abb$	$6ca-9da-8bc+12db$

	1	$aa+2a+4$	$aa-ba+bb$
	2	$a-2$	$a+b$
		$aaa+2aa+4a$	$aaa-baa+bba$
		$-2aa-4a-8$	$+baa-bba+bbb$
1 × 2	3	$aaa-8$	$aaa+bbb$

Sect.

Sect. 4. Diviſion of whole Quantities.

Diviſion of Species is the converſe or direct contrary to that of *Multiplication*, and conſequently is performed by converſe Operations, (as in common *Arithmetick*) and admits of four *Caſes*.

Caſe 1. When the Quantities in the Dividend have like Signs to thoſe in the Diviſor, and no Co-efficients in either; caſt off or expunge all the Quantities in the Dividend, that are like thoſe in the Diviſor; and ſet down the other Quantities with the Sign + for the Quotient required.

Thus	1	ab	$-ab$	$ad+bd$	$-ad-bd$
	2	b	$-b$	d	$-d$
1 ÷ 2	3	a	$+a$	$a+b$	$a+b$

Caſe 2. When the Quantities in the Dividend have unlike Signs to thoſe in the Diviſor; then ſet down the Quotient Quantities found as before, with the Sign — before them.

Thus	1	$+ab$	$-ab-bd$	$abc+bcd+bcf$
	2	$-b$	$+b$	$-bc$
1 ÷ 2	3	$-a$	$-a-d$	$-a-d-f$

Caſe 3. If the Quantities in the Dividend and Diviſor have Co-efficients; divide the Numbers (as in common *Arithmetick*) and to their Quotients adjoin the Quotient Quantities.

Thus	1	$15ab$	$42db$	$12af-21bf$
	2	$3b$	$-7b$	$3f$
1 ÷ 2	3	$5a$	$-6d$	$4a-7b$

Note, When the Quantities and Co-efficients in the Diviſor and Dividend are all the ſame, the Quotient will be an Unit, or 1.

Thus	1	ab	$9bc$	$7ab+5bc$	$8ab+4d$
	2	ab	$-9bc$	$7ab+5bc$	$-8ab-4d$
1 ÷ 2	3	1	-1	1	-1

Caſe 4. When the Quantities in the Diviſor cannot be exactly found in the Dividend; then ſet them both down like a *Vulgar Fraction*, as in common *Arithmetick*.

Thus	1	a	$6bc$	$5b+aa$	$8adc$
	2	b	$3d$	$5d+7b$	$4abc$
		$\frac{a}{b}$	$\frac{2bc}{d}$	$\frac{5b+aa}{5d+7b}$	$\frac{2d}{b}$
$1\div 2$	3				

N. B. In Diviſion one thing muſt be very carefully obſerved; *viz.* that like Signs give + and unlike Signs give — in the Quotient; which needs no other Proof than that already laid down in the laſt Section, if duly compared with what hath been ſaid concerning *Multiplication* and *Diviſion*, in Vulgar *Arithmetick*.

Examples of *Diviſion* at large.

	1	$21ba+15da-35bf-25df$ $(+3a$
	2	$7b+5d$
$2\times 3a$	3	$21ba+15da$
$1-3$	4	$0 \quad 0 \quad -35bf-25df$ $(-5f$
$2\times -5f$	5	$-35bf-25df$
$4-5$	6	$0 \quad 0$
$1\div 2$	7	$3a-5f$ the Quotient collected from 3d and 5th Steps.

Or *Diviſion* of Quantities may ſtand as Numbers in common *Arithmetick* do; thus

$$3a-6)\ 6aaaa-96\ (2aaa+4aa+8a+16$$
$$6aaaa-12aaa$$
$$0+\ 12aaa-96$$
$$+\ 12aaa-24aa$$
$$0\ +24aa-96$$
$$+24aa-48a$$
$$0\ +\ 48a-96$$
$$+\ 48a-96$$
$$0\quad 0$$

That is, $\overline{6aaa-96}\div\overline{3a-6}$ gives $2aaa+4aa+8a+16$ for the Quotient, as may eaſily be proved by *Multiplication*, viz. $\overline{2aaa+4aa+8a+16}\times\overline{3a-6}$ will produce $6a^4-96$; and ſo for the reſt.

Sect. 5. Involution of whole Quantities.

INvolution is the raiſing or producing of Powers, from any propoſed Root, and is performed in all reſpects like *Multiplication*, ſave only in this; *Multiplication* admits of any different Factors, but Involution ſtill retains the ſame.

EXAMPLES.

EXAMPLES.

	1	a	$-a$	the Root, or ſingle Power.
1 ⊛ 2	2	aa	$+aa$	Square, or ſecond Power.
1 ⊛ 3	3	aaa	$-aaa$	Cube, or third Power.
1 ⊛ 4	4	$aaaa$	$+aaaa$	Biquadrat, or fourth Power.
1 ⊛ 5	5	$aaaaa$	$-aaaaa$	Surſolid, or fifth Power, &c.

Note, the Figures placed in the Margin, after the Sign (⊛) of Involution, ſhew to what Height the Root is involved; and are called Indices of the Power; and are uſually placed over the involved Quantities, in order to contract the Work, eſpecially when the Powers are any thing high.

Thus $\begin{cases} a = a \\ a^2 = aa \\ a^3 = aaa \\ a^4 = aaaa \end{cases}$ And $\begin{cases} a^5 = aaaaa \\ a^6 = aaaaaa \\ a^5 b^5 = aaaaabbbbb \\ a^3 b^3 d^3 = aaabbbddd \end{cases}$

If the Quantities have Co-efficients, the Co-efficients muſt be involved along with the Quantities, as in theſe.

Thus	1	$2a$	$-3a$	$5bc$
1 ⊛ 2	2	$4aa$	$+9aa$	$25bbcc$
1 ⊛ 3	3	$8aaa$	$-27aaa$	$125bbbccc$
1 ⊛ 4	4	$16aaaa$	$+81aaaa$	$625b^4c^4$
1 ⊛ 5	5	$32aaaaa$	$-243a^5$	$3125b^5c^5$ &c.

Involution of Compound Quantities is performed in the ſame manner, due regard being had to their Signs and Co-efficients, if there be any. As for inſtance, ſuppoſe $a+b$ were given to be involved to the fifth Power.

Thus	1	$a+b$ called a Binomial Root
		$a+b$
1 × a	2	$aa+ab$
1 × b	3	$+ab+bb$
1 ⊛ 2	4	$aa+2ab+bb$, the Square of $a+b$
		$a+b$
4 × a	5	$aaa+2aab+abb$
4 × b	6	$+aab+2abb+bbb$
1 ⊛ 3	7	$aaa+3aab+3abb+bbb$, the Cube of $a+b$

	7	$aaa+3aab+3abb+bbb$
		$a+b$
$7\times a$	8	$a^4+3a^3b+3aabb+abbb$
$7\times b$	9	$+a^3b+3aabb+3abbb+b^4$
$1\textcircled{2}^4$	10	$a^4+4a^3b+6aabb+4abbb+b^4$
		$a+b$
$10\times a$	11	$a^5+4a^4b+6a^3bb+4aab^3+ab^4$
$1\times b$	12	$a^4b+4a^3bb+6aab^3+ab^4+b^5$
$1\textcircled{2}^5$	13	$a^5+5a^4b+10a^3bb+10aab^3+5ab^4+b^5$
		&c.

Again, Let $a-b$, called a Reſidual Root, be given.

Then	1	$a-b$
		$a-b$
$1\times a$	2	$aa-ab$
$1\times -b$	3	$-ab+bb$
$1\textcircled{2}^2$	4	$aa-2ab+bb$ the Square of $a-b$
		$a-b$
$4\times a$	5	$aaa-2aab+abb$
$4\times -b$	6	$-aab+2abb-bbb$
$1\textcircled{2}^3$	7	$aaa-3aab+3abb-bbb$, the Cube of $a-b$
		$a-b$
$7\times a$	8	$aaaa-3aaab+3aabb-abbb$
$7\times -b$	9	$-aaab+3aabb-3abbb+bbbb$
$1\textcircled{2}^4$	10	$aaaa-4aaab+6aabb-4abbb+bbbb$
		$a-b$
$10\times a$	11	$a^5-4a^4b+6a^3bb-4aab^3+ab^4$
$10\times -b$	12	$-a^4b+4a^3bb-6aab^3+4ab^4-b^5$
$1\textcircled{2}^5$	13	$a^5-5a^4b+10a^3bb-10aab^3+5ab^4-b^5$
		&c.

By comparing theſe two Examples together, you may make the following Obſervations.

1. That the Powers raiſed from a Reſidual Root (*viz.* the Difference of two Quantities) are the ſame with their like Powers raiſed from a Binomial Root (or the Sum of two Quantities) ſave only in their Signs; *viz.* the Binomial Powers have the Sign + to every Term, but the Reſidual Powers have the Signs + and — interchangeably to every other Term.

2. The Indices of the Powers of the leading Quantity (a) continually decreaſe in Arithmetical Progreſſion; *viz.* in the Square it

it is *aa*, *a*: In the Cube *aaa*, *aa*, *a*: In the Biquadrat, *aaaa*, *aaa*, *aa*, *a*, &c.

3. The Indices of the other Quantity *b* do continually increaſe in Arithmetical Progreſſion; *viz.* In the Square it is *b*, *bb*: In the Cube, *b*, *bb*, *bbb*: In the Biquadrat *b*, *bb*, *bbb*, *bbbb*, &c.

4. The firſt and laſt Terms are always pure Powers of the ſingle Quantities, and are both of the ſame Height.

5. The Sum of the Indices of any two Letters joined together in the intermediate Terms, are always equal to the Index of the higheſt Power, *viz.* of the firſt or laſt Term.

Theſe Obſervations being duly conſidered, it will be eaſy to conceive how the Terms of any propoſed Power raiſed from a Binomial or Reſidual Root muſt ſtand, without their Unciæ or Numeral Figures.

For Inſtance, ſuppoſe it were required to raiſe the Binomial Root $a+b$ to the ſeventh Power, then the Terms of that Power will ſtand without their Unciæ in this Order.

Viz. $a^7+a^6b+a^5b^2+a^4b^3+a^3b^4+a^2b^5+ab^6+b^7$.

And becauſe the Uncia (not only of any ſingle Letter, but alſo) of every ſingle Power, how high ſoever it be, is an Unit or 1 (which neither multiplies nor divides) and all the Powers of any Binomial or Reſidual Root are naturally raiſed by multiplying of the precedent Power into it's original Root, which is done by only joining each Letter in the Root to the precedent Power, with it's Unciæ, and then removing the ſaid Power, when it is ſo joined to the ſecond Letter, one Place forward (either to the left or right Hand) it muſt needs follow,

That the Unciæ of the ſecond Terms (in any ſuch Power) will always be the Sum of ſo many Units added together more one, as there have been Multiplications of the firſt Root; which will always be determined by the Index of the firſt Term in the Power.

And becauſe the Unciæ of all the intermediate Terms are only removed along with the Letters, it alſo follows, that if they are added together, their reſpective Sums will produce the true Unciæ of the intermediate Terms in the new raiſed Power. As doth plainly appear from the following Numbers ſo removed without their Letters; which both ſhews and demonſtrates an eaſy Way of producing the Unciæ of any ordinary Power (*viz.* of one not very high) raiſed from either a Binomial or Reſidual Root.

Thus

Thus

Add { 1 . 1 . The two Unciæ of the Root.
      1 . 1

Add { 1 . 2 . 1 The Unciæ of the Square.
      1 . 2 . 1

Add { 1 . 3 . 3 . 1 The Unciæ of the Cube.
      1 . 3 . 3 . 1

Add { 1 . 4 . 6 . 4 . 1 The Unciæ of the 4th Power.
      1 . 4 . 6 . 4 . 1

Add { 1 . 5 . 10 . 10 . 5 . 1 Unciæ of the 5th Power.
      1 . 5 . 10 . 10 . 5 . 1

Add { 1 . 6 . 15 . 20 . 15 . 6 . 1 Unciæ of 6th Power.
      1 . 6 . 15 . 20 . 15 . 6 . 1

1 . 7 . 21 . 35 . 35 . 21 . 7 . 1 Unciæ of 7 Pow.

And ſo on in this manner *ad infinitum.*

Now if theſe Numbers are prefixed to the aforeſaid Letters, all the Terms will be compleated with their reſpective Unciæ, and will ſtand thus;

$$a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b.$$

But that the Buſineſs of finding theſe Unciæ may be rendered yet more eaſy for Practice, it will be convenient to conſider what Series or Progreſſion the Unciæ of each Term do make from the aforeſaid Additions.

Unciæ of the firſt Term.	Unciæ of the ſecond Term.	Unciæ of the third Term.	Unciæ of the 4th Term	Unciæ of the fifth Term.	Unciæ of the ſixth Term.	Unciæ of the 7th Term.	Unciæ of the 8th Term, &c.	
1	1							Unciæ of the ſingle Quantities.
1	2	1						Unciæ of the Square.
1	3	3	1					Unciæ of the Cube.
1	4	6	4	1				Unciæ of the 4th Power.
1	5	10	10	5	1			Unciæ of the 5th Power.
1	6	15	20	15	6	1		Unciæ of the 6th Power.
1	7	21	35	35	21	7	1	Unciæ of the 7th Power, &c.

The Unciæ of the firſt Term are only a Series of Units, whoſe Sum is every where the Unciæ of the ſecond Term. The Unciæ of the ſecond Term are a Series of Numbers in Arithmetick Progreſſion, whoſe Sum is every where the Unciæ of the next Superior Power in the third Term, and may be found by Propoſi-

tion

tion 1. Chap. 6. Part 1. For Inſtance, in the ſeventh Power it will be $\frac{\overline{6+1}\times 6}{2} = 21 =$ the Uncia of the third Term.

The reſt of the Unciæ are a compounded Series, whoſe reſpective Sums may be obtained from the Unciæ of their precedent Terms.

Thus $\frac{21\times 5}{3} = 35$. Then $\frac{35\times 4}{4} = 35$. Again $\frac{35\times 3}{5} = 21$. And $\frac{21\times 2}{6} = 7$ *&c.*

From hence may be deduced this general Rule.

RULE.

If the Index of the firſt Letter of any Term be multiplied into it's own Uncia, and that Product be divided by the Number of Terms to that Place; the Quotient will be the Uncia of the next ſucceeding Term forward.

That is, by the help of thoſe Indices that belong to the ſeveral Powers of the firſt or leading Letter only (as *a*) the true Unciæ of every Term may be eaſily underſtood.

EXAMPLE 2.

Let it be required to compleat all the Terms of the aforeſaid ſeveral Powers, *viz.* $a^7+a^6b+a^5b^2+a^4b^3+a^3b^4+a^2b^5+ab^6+b^7$ with their proper Unciæ.

1. The Index of a^7 the firſt Term will be the Uncia of the ſecond Term. Thus a^7+7a^6b.

2. Then half the ſecond Term's Index into it's Uncia, *viz.* $\frac{7\times 6}{2} = 21$ will be the third Term's Uncia. Thus $a^7+7a^6b+21a^5b^2$ will be the three firſt Terms.

3. Again $\frac{21\times 5}{3} = 35$ is the Uncia of the fourth Term, whence $a^7+7a^6b^2+21a^5b^2+35a^4b^3$ will be the four firſt Terms.

4. And $\frac{35\times 4}{4} = 35$ will be the Uncia of the fifth Term, whence $a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4$ will be the five firſt Terms.

And ſo proceed 'till all the Terms are compleated with their reſpective Unciæ; which will ſtand, thus $a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7$.

Now

Now here it may be further obſerved, that the Unciæ do only increaſe until the Indices of the two Letters become equal, or change Places; and then the reſt of the Unciæ will return or decreaſe in the ſame Order. That is, wherever the Indices of the Letters are alike, there the Unciæ will be alike.

And therefore one needs to find the Unciæ (as before) but to half the Number of Terms in any Power.

If what hath been ſaid and the Work of the *Example* be well underſtood, I preſume it will be found very eaſy to raiſe any Power from a Binomial or Reſidual Root, to what Height you pleaſe; without the Trouble of a continued Involution; and without the Help of ſuch a Table of Powers as is propoſed by Mr *Oughtred* in his Key to the *Mathematicks*, Page 40. and ſince by others.

Now from theſe Conſiderations it was, that I propoſed this Method of raiſing Powers in my Compendium of *Algebra*, Page 57. as wholly New (*viz.* ſo much of it as was there uſeful) having then (I profeſs) neither ſeen the Way of doing it, nor ſo much as heard of it's being done. But ſince the writing of that Tract, I find in Dr *Wallis*'s *Hiſtory of Algebra*, Page 319 and 331, that the Learned Sir *Iſaac Newton* had diſcovered it long before: which the Doctor ſets down in this manner.

Let m be the Exponent of the Power.

Then $\left\{ 1 \times \frac{m-0}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \&c. \right.$

Will be the Series of the Unciæ required; but he doth not tell us how they firſt came to be found out, nor have I met with the leaſt Hint of it in any Author.

Sect. 6. Evolution of whole Quantities.

Evolution is the extracting of Roots from any given Power. That is, it is the Converſe Work to that of Involution, and in ſingle Quantities it is eaſy, if the given Power have ſuch a Root as is required, which may be thus known.

If the given Power have no Numbers prefixed to it, and it's Index can be divided by the Index of the Root required, the Quotient will be the Index of the Root ſought. Thus, if the Cube Root of $aaaaaa$, viz. a^6 were required (the Index of the Cube is 3) then 3) 6 (2. That is, $a^{\frac{6}{3}} = a^2$ the Root required. And ſuch Operations are uſually ſet down

Thus

Thus	1	a^6	$a^6 b^6$	$a^6 b^6 d^6$
1 ww	2	a^3	$a^3 b^3$	$a^3 b^3 d^3$
1 ww	3	a^2	$a^2 b^2$	$a^2 b^2 d^2$
3 ww	4	a	ab	abd

Note, The Figures placed in the Margin after the Sign (ww) of Evolution, denote the Index of the Root to be extracted.

If the given Powers have Co-efficients: (*viz.* Numbers prefixed to them) then you muſt extract their reſpective Roots, as in Vulgar *Arithmetick*.

Thus	1	$81a^4$	$1296a^8b^8$	$20736a^4b^4c^4$
1 ww	2	$9a^2$	$36a^4b^4$	$144a^2b^2c^2$
1 ww	3	$3a$	$6a^2b^2$	$12abc$
or 2 ww	4	$3a$	$6a^2b^2$	$12abc$

But if the Root required cannot be truly extracted out of both the Co-efficients and Indices of the given Power; then it is a Surd, and muſt have the Sign of the Root required prefixed to it.

Thus	1	a^5.	$67a^4$	$216bbbddd$
1 ww	2	$\surd a^5$	$\surd 67a^4$	$\surd 216bbbddd$
1 ww	3	$^3\surd a^5$	$^3\surd 67a^4$	$6bd$

Evolution of compound Quantities or Powers, is a little more troubleſome than that of Single Powers; and would require a great many Words to explain the Manner and Reaſon of forming the ſeveral Canons, that are commonly uſed in extracting the Roots of compound Quantities; eſpecially if the Powers be very high, *&c.* I ſhall therefore for Brevity's ſake omit them, and inſtead thereof propoſe an eaſy Method of diſcovering the Roots of all compound Powers in general. And in order to that, it will be neceſſary to premiſe; that if either the Sum or Difference of ſeveral Quantities be involved to any Power, there will ariſe ſo many ſingle Powers of the ſame heights, as there are different Quantities.

As for inſtance, if $a+b+d$ be ſquared, that is, be involved to the 2d Power, it will be $aa+2ab+2ad+bb+2bd+dd$, here you have aa, bb, and dd. Again, if $a+b+d$ were cubed, *viz.* involved to the third Power, then you will have aaa, bbb, ddd, in it, *&c.*

Whence it follows, that in extracting the Roots of all compound Quantities, there muſt be conſidered,

1. How many different Letters (or Quantities) there are in the given Power.

2. Whether the ſingle Powers of each of thoſe Letters be of an equal Height, and have in them ſuch a ſingle Root as is required: which if they have, extract it as before.

3. Connect thoſe ſingle Roots together with the Sign +, and involve them to the ſame Height with the given Power; that being done, compare the new raiſed Power with the given Power; and if they are alike in all their reſpective Terms, then you have the Root required; or if they differ only in their Signs, the Root may be eaſily corrected with the Sign — as Occaſion requires.

Example 1. Let it be required to extract the Square Root of $cc+2cb-2cd+bb-2bd+dd$. In this Compound Square, there are three diſtinct Powers, *viz.* bb, cc, dd, whoſe ſingle Roots are b, c, d, wherefore I ſuppoſe the Root ſought to be $b+c+d$, or rather $b+c-d$, becauſe in the given Power there is $-2cd$, and $-2bd$, therefore I conclude it is $-d$; then $b+c-d$, being ſquared, produces $bb+2bc-2bd+cc-2cd+dd$, which I find to be the ſame in all it's Terms with the given Power, although they ſtand in a different Poſition; conſequently $b+c-d$ is the true Root required.

Example 2. It is required to extract the Square Root of $a^4-2aabb+b^4$. Here are but two ſingle Powers, *viz.* a^4 and b^4, whoſe Square Roots are aa, and bb. And becauſe in the given Power there is $-2aabb$, therefore I conclude it muſt either be $aa-bb$ or $bb-aa$. Both which, being involved, will produce $a_4-2aabb+b^4$; conſequently the Root ſought may either be $aa-bb$, or $bb-aa$, according to the Nature or Deſign of the Queſtion from whence the given Power was produced.

Example 3. Let it be required to extract the Square Root of $36aaaa+108aa+81$. Here the two ſingle Powers are $36aaaa$, and 81, whoſe Roots are $6aa$ and 9. And becauſe the Signs are all + therefore I ſuppoſe the Root to be $6aa+9$, the which being involved doth produce $36a^4+108aa+81$; conſequently $6aa+9$ is the true Root required.

Example 4. Suppoſe it were required to extract the Cube Root of $125aaa+300aae-450aa+250aee-720ae+64eee+540a-288ee+432e-216$. In this Example there are three diſtinct Powers, *viz.* $125aaa$, $64eee$, and -216. And the Cube Root of $125aaa$ is $5a$; of $64eee$ is $4e$; of -216 is -6. Wherefore I ſuppoſe the Root ſought to be $5a+4e-6$, which being involved to the third Power, does produce the

the ſame with the given Power; conſequently $5a+4e-6$ is the Cube Root required.

But if the new Power, raiſed from the ſuppoſed Root (being involved to it's due Height) ſhould not prove the ſame with the given Power, *viz.* if it hath either more or fewer Terms in it, *&c.* then you may conclude the given Power to be a Surd, which muſt have it's proper Sign prefixed to it, and cannot be otherwiſe expreſſed, until it come to be involved in Numbers.

Example 5. Suppoſe it were required to extract the Cube Root of $27aaa+54baa+8bbb$. Here are two diſtinct and perfect Cubes, *viz.* $27aaa$, and $8bbb$, whoſe Cube Roots are $3a$ and $2b$. Wherefore one may ſuppoſe the Root ſought to be $3a+2b$, which being involved to the third Power, is $27aaa+54baa+36bba+8bb$. Now this new raiſed Power hath one Term (*viz.* $36bba$) more in it than the given Power hath; but this being a perfect Cube, one may therefore conclude the given Power is not ſo, *viz.* it is a Surd, and hath not ſuch a Root as was required, but muſt be expreſſed, or ſet down,

Thus $\sqrt[3]{27aaa+54baa+8bbb}$.

If theſe Examples be well underſtood, the Learner will find it very eaſy by this Method of proceeding to diſcover the true Root of any given Power whatſoever.

CHAP. III.

Of Algebraick Fractions, *or* Broken Quantities.

Sect. 1. Notation *of Fractional Quantities.*

Fractional Quantities are expreſſed or ſet down like Vulgar Fractions in common *Arithmetick*.

Thus $\left\{\frac{a}{b}, \frac{2bc}{d}, \frac{5b-4a}{4d+7b}\right.$ Numerators. Denominators.

How they come to be ſo, ſee *Caſe* 4, in the laſt Chapter, of *Diviſion*. Theſe Fractional Quantities are managed in all reſpects like Vulgar Fractions in common *Arithmetick*.

Sect. 2. *To* **Alter** *or* **Change** *different* **Fractions** *into one Denomination, retaining the ſame Value.*

RULE.

MULTIPLY all the Denominators into each other for a new Denominator, and each Numerator into all the Denominators but it's own for new Numerators.

EXAMPLES.

Let it be required to bring $\frac{a}{b}$ and $\frac{d}{c}$ into one Denomination.

Firſt $a \times c$, and $d \times b$, will be the Numerators, and $b \times c$ will be the common Denominator, *viz.* $\frac{ca}{bc}$ and $\frac{bd}{bc}$ are the two Fractions required: that is, $\frac{ca}{bc} = \frac{a}{b}$, and $\frac{bd}{bc} = \frac{d}{c}$.

Again, let $\frac{b+c}{a+b}$ and $\frac{d-c}{b-d}$ be brought into one Denomination, and they will be $\frac{bb+bc-bd-dc}{ba+bb-da-bd}$, and $\frac{ad-ac+bd-bc}{ba+bb-da-bd}$, &c.

Sect. 3. *To* **Bring** *whole* **Quantities** *into* **Fractions** *of a given Denomination.*

RULE.

MULTIPLY the whole Quantities into the given Denominator for a Numerator, under which ſubſcribe the given Denominator, and you will have the Fraction required.

EXAMPLES.

Let it be required to bring $a+b$ into a Fraction, whoſe Denominator is $d-a$. Firſt $\overline{a+b} \times \overline{d-a}$ is $da+bd-aa-ba$: Then $\frac{da+bd-aa-ba}{d-a}$ is the Fraction required.

Again $b+\frac{a}{d}$ will be $\frac{db+a}{d}$. And $\frac{aa}{d}-a$ will be $\frac{aa-da}{a}$.

Alſo $a+b+\frac{aa+bb}{a-b}$ will be $\frac{2aa}{a-b}$.

When

When whole Quantities are to be ſet down Fraction wiſe, ſubſcribe an Unit for the Denominator. Thus ab is $\frac{ab}{1}$. And $aa-bb$, is $\frac{aa-bb}{1}$, &c.

Sect. 4. *To* Abbreviate *or* Reduce *Fractional Quantities into their loweſt Denomination.*

RULE.

Divide both the Numerator and Denominator by their greateſt common Diviſor, viz. *by ſuch Quantities as are found in both; and their Quotients will be the Fraction in it's loweſt Term.*

Thus $\frac{aac}{dc}$ is $\frac{aa}{d}$. $\frac{abbb}{abc}$ is $\frac{bb}{c}$. And $a+\frac{bdc}{bc}=a+d$.

In ſuch ſingle Fractions as theſe, the common Diviſors (if there be any) are eaſily diſcovered by Inſpection only; but in compound Fractions it often proves very troubleſome, and muſt be done either by dividing the Numerator by the Denominator, until nothing remains, when that can be done: or elſe finding their common Meaſure, by dividing the Denominator by the Numerator, and the Numerator by the Remainder, and ſo on, as in Vulgar Fractions (Sect. 4. Page 51.)

EXAMPLES.

Suppoſe $\frac{aac-aad}{cd-dd}$ were to be reduced lower.

Then $cd-dd)\ aac-aad\ (\frac{aa}{d}$ the Fraction required.

$\underline{aac-aad}$

0 0

In this Example it ſo happens that the Numerator is divided juſt off by the Denominator; but in the next it is otherwiſe, and requires a double Diviſion to find out the common Meaſure, *viz.*

Let it be required to reduce $\frac{aaa-abb}{aa+2ab+bb}$ to it's loweſt Terms.

First $aa+2ab+bb)\ aaa-abb\ (a$

$\underline{aaa+2aab+abb}$

$-2aab-2abb$ the Remainder.

Then $-2aab-2abb)\ aa+2ab+bb\ (-\frac{1}{2b}-\frac{1}{2a}$

$\underline{aa+ab}$

$ab+bb$

$\underline{ab+bb}$

0 0

Hence

Hence it appears that $-2aab-2abb$ is the common Measure; by which $aaa-abb$ being divided.

Viz. $-2aab-2abb)\ aaa-abb\ \left(-\frac{a}{2b}+\frac{1}{2}\right.$

$aaa+aab$

$-aab-abb$

$-aab-abb$

$0\quad 0$

Then $-\frac{a}{2b}+\frac{1}{2}$ is the new Numerator; and ☞ $-\frac{1}{2b}-\frac{1}{2a}$ is the new Denominator. But $-\frac{a}{2b}+\frac{1}{2}=\frac{-2a+2b}{4b}=\frac{-a+b}{2b}$ the Numerator; and $-\frac{1}{2b}-\frac{1}{2a}=\frac{-2a-2b}{4ba}=\frac{-a-b}{2ba}$ the Denominator. Let both be multiplied with $2ba$, and you will have $\frac{-aa+ab}{-a-b}$ the Numerator. the Denominator. Or changing the Signs of all the Quantities, it will be $\frac{aa-ab}{a+b}$ the new Fraction required. That is, $\frac{aa-ab}{a+b}=\frac{aaa-abb}{aa+2ab+bb}$.

Again, let it be required to reduce $\frac{dd-bb}{ddd-bbb}$.

The common Measure of this Fraction will be the easiest found (as appears from Trials) by dividing the Denominator by the Numerator, *&c.* Thus,

$dd-bb)\ ddd-bbb\ (d$

$ddd-bbd$

$+bbd-bbb)\ dd-bb\ \left(\frac{d}{bb}\right.$

$dd-bd$

$+bd-bb\ )bbd-b^3\ (b$

$bbd-b^3$

$0\quad 0$

Hence it appears that $bd-bb$ is the common Measure that will divide both the Numerator and Denominator.

Consequently

Conſequently $bd-bb)\ dd-bb\ (\frac{d}{b}+1$, is the new Numerator.

$$\begin{array}{r} dd-db \\ \hline +db-bb \\ db-bb \\ \hline 0\quad 0 \end{array}$$

And $bd-bb)\ ddd-bbb\ (\frac{dd}{b}+d+b$ the new Denominator.

$$\begin{array}{r} ddd-ddb \\ \hline +ddb-bbb \\ ddb-bbd \\ \hline +bbd-bbb \\ bbd-bbb \\ \hline 0\quad 0 \end{array}$$

Let both be multiplied with b, and then you will have $\frac{d+b}{dd+bd+bb}$ the Numerator, the Denominator, } of the Fraction required.

But if after all Means uſed (as above) there cannot be found one common Meaſure to both the Numerator and Denominator; then is that Fraction in it's leaſt Terms already.

Note, Theſe Operations will be underſtood by a Learner after he hath paſſed thro' *Multiplication* and *Diviſion* of Fractions.

Sect. 5. Addition *and* Subſtraction *of Fractional* Quantities.

THE given Fractions being of one Denomination, or if they are not, make them ſo, per Sect. 4. Then,

RULE.

Add or ſubſtract their Numerators, as Occaſion requires, and to their Sum or Difference ſubſcribe the common Denominator: as in Vulgar Fractions.

Examples in Addition.

1	$\frac{bb}{c}$	$\frac{a+b}{d}$	$\frac{2a-b}{d+c}$	$\frac{1-b+d}{d+a}$
2	$\frac{aa}{c}$	$\frac{2a+c}{d}$	$\frac{2b-a}{d+c}$	$\frac{1+b-d}{d+a}$
1+2	3 $\frac{bb+aa}{c}$	$\frac{3a+b+c}{d}$	$\frac{a+b}{d+c}$	$\frac{2a}{d+a}$

Examples

Examples in **Subſtraction.**

	1	$\frac{bb+ac}{c}$	$\frac{a+b}{d+c}$	$\frac{3a+b+c}{d}$	$\frac{2b}{d+a}$
	2	$\frac{bb}{c}$	$\frac{2b-a}{d+c}$	$\frac{2a+c}{d}$	$\frac{1+b-d}{d+a}$
1—2	3	$\frac{aa}{c}$	$\frac{2a-b}{d+c}$	$\frac{a+b}{d}$	$\frac{b-a+d}{d+a}$

Sect. 6. **Multiplication** *of Fractional Quantities.*

FIRST prepare mixed Quantities (if there be any) by making them improper Fractions, and whole Quantities by ſubſcribing an Unit under them; as per Sect. 3. Then,

RULE.

Multiply the Numerators together for a new Numerator, and the Denominators together for a new Denominator; as in Vulgar Fractions.

Thus	1	$\frac{ab}{c}$	$\frac{3a-2b}{2d+c}$
	2	$\frac{d}{f}$	$\frac{4a+2b}{d}$
1 × 2	3	$\frac{abd}{cf}$	$\frac{12aa-2ab-4bb}{2dd+dc}$

Suppoſe it were required to multiply $2a+\frac{b}{c}-25$ with $36+4c$. Theſe prepared for the Work (per Sect. 3.) will ſtand

Thus	1	$\frac{2ac+b-25c}{c}$
	2	$\frac{3b+4c}{1}$
2 × 2	3	$\frac{6bac+3bb-75bc+8acc+4bc-100cc}{c}$
or	4	$6ba-71b+8ac-100c+\frac{3bb}{c}$ per Sect 4.

N. B.

N. B. Any Fraction is multiplied with it's Denominator, by casting off, or taking the Denominator away. Thus $\frac{b}{a} \times a$ gives b. For $\frac{b}{a} \times \frac{a}{1} = \frac{ba}{a} = b$, &c.

Sect. 7. Division *of Fractional Quantities.*

THE Fractional Quantities being prepared, as directed in the last Section. Then,

RULE.

Multiply the Numerator of the Dividend, into the Denominator of the Divisor, for a new Numerator; and multiply the other two together for a new Denominator; as in Vulgar Fractions.

EXAMPLES.

Let $\frac{abd}{cf}$ be divided by $\frac{ab}{c}$, the Work may stand thus,

$\frac{ab}{c}) \frac{abd}{cf} (\frac{abdc}{abcf} = \frac{d}{f}$ per Sect. 4.

Or thus	1	$\frac{abd}{cf}$	$\frac{a+b}{d}$	$\frac{aaa-bbb}{a+b}$
	2	$\frac{ab}{c}$	$\frac{c-b}{a}$	$\frac{aa-ab+bb}{c}$
1 ÷ 2	3	$\frac{d}{f}$	$\frac{aa+ba}{dc-db}$	$\frac{aaac-bbbc}{aaa+bbb}$

Suppose it were required to divide $aa + \frac{3abb}{a+4b}$ by $a+b$ The Work prepared will stand thus,

$\frac{a+b}{1} (\frac{aaa+4aab+3abb}{a+4b} (\frac{aaa+4aab+3abb}{aa+5ba+4bb}$. But $\frac{aaa+4aab+3abb}{aa+5ba+4bb} = \frac{aa+3b}{a+4b}$ (per Sect. 4.)

When Fractions are of one Denomination, cast off the Denominators, and divide the Numerators. Thus, if $\frac{ab^3}{c}$ were to be divided by $\frac{bb}{c}$ it will be $bb) ab^3 (ab$ the Quotient required.

For $\frac{bb}{c}\big)\frac{ab^3}{c}\big(\frac{ab^3c}{bbc}$. But $\frac{ab^3c}{bbc}=ab$ (per Sect. 4.)

Again, suppose it were required to divide $\frac{a^3-abb}{c-d}$ by $\frac{aa+2ab+bb}{c-d}$. Casting off $c-d$ in both, it will be $aa+2ab+bb)\ aaa-abb\big(\frac{aa-ba}{a+b}$, &c.

Sect. 8. Involution *of Fractional Quantities.*

RULE.

Involve the Number into itself for a new Numerator, and the Denominator into itself for a new Denominator; each as often as the Power requires.

Thus	1	$\frac{b}{a}$	$\frac{3bc}{2ad}$	$\frac{b+d}{a-c}$
1 @^2	2	$\frac{bb}{aa}$	$\frac{9bbcc}{4aadd}$	$\frac{bb+2bd+dd}{aa-2ac+cc}$
1 @^3	3	$\frac{bbb}{aaa}$	$\frac{27bbbccc}{8aaaddd}$	$\frac{bbb+3bbd+3bdd+ddd}{aaa-3aac+3acc-ccc}$

Sect. 9. Evolution *of Fractional Quantities.*

IF the Numerator and Denominator of the Fraction have each of them such a Root as is required (which very rarely happens) then evolve them; and their respective Roots will be the Numerator and Denominator of the new Fraction required.

Thus	1	$\frac{9aabb}{4dd}$	$\frac{aa+2ab+bb}{aa-2ab+bb}$
1 ww^2	2	$\frac{3ab}{2d}$	$\frac{a+b}{a-b}$
Again	1	$\frac{27aaabbb}{8ddd}$	$\frac{aaa+3aab+3abb+bbb}{aaa-3aab+3abb-bbb}$
1 ww^3	2	$\frac{3ab}{2d}$	$\frac{a+b}{a-b}$

Sometimes it so falls out, that the Numerator may have such a Root as is required, when the Denominator hath not; or the Denominator

minator may have ſuch a Root, when the Numerator hath not. In thoſe Caſes the Operations may be ſet down

Thus	1	$\frac{aabb}{ddd}$	$\frac{aaa+4bb-dd}{aa+2ab+bb}$
1 ww	2	$\frac{ab}{\sqrt{ddd}}$	$\frac{\sqrt{aaa+4bb-dd}}{a+b}$

But when neither the Numerator nor the Denominator have juſt ſuch a Root as is required, prefix the radical Sign of the Root to the Fraction; and then it becomes a Surd; as in the laſt Step, which brings me to the Buſineſs of managing Surds.

CHAP IV.

Of Surd Quantities.

THE whole Doctrine of Surds (as they call it) were it fully handled, would require a very large Explanation (to render it but tolerably intelligible); even enough to fill a Treatiſe itſelf, if all the various Explanations that may be of Uſe to make it eaſy ſhould be inſerted; without which it is very intricate and troubleſome for a Learner to underſtand. But now theſe tedious Reductions of Surds, which were heretofore thought uſeful to fit Equations for ſuch a Solution, as was then underſtood, are wholly laid aſide as uſeleſs: Since the new Methods of reſolving all ſorts of Equations render their Solutions equally eaſy, although their Powers are never ſo high. Nay, even ſince the true Uſe of Decimal *Arithmetick* hath been well underſtood, the Buſineſs of Surd Numbers has been managed that Way; as appears by ſeveral Inſtances of that Kind in Dr *Wallis*'s *Hiſtory of Algebra*, from *Page* 23, to 29.

I ſhall therefore, for Brevity ſake, paſs over thoſe tedious Reductions, and only ſhew the young *Algebraiſt* how to deal with ſuch Surd Quantities as may ariſe in the Solution of hard Queſtions.

Sect. 1. Addition *and* Subſtraction *of Surd Quantities.*

Caſe 1. WHEN the Surd Quantities are Homogeneal, (*viz.* are alike) add or ſubſtract the rational

Part, if they are joined to any, and to their Sum, or Difference, adjoin the irrational or Surd.

Examples in Addition.

	1	$5\sqrt{bc}$	$6b\sqrt{ac}$	$b\sqrt{aa+cc}$
	2	$7\sqrt{bc}$	$4b\sqrt{ac}$	$3b\sqrt{aa+cc}$
1+2	3	$12\sqrt{bc}$	$10b\sqrt{ac}$	$4b\sqrt{aa+cc}$

	1	$4d\sqrt[3]{aa}$	$b+\sqrt[3]{aa-cc}$	$bc:\sqrt[5]{aa+d}$
	2	$d\sqrt[3]{aa}$	$c-\sqrt[3]{aa-cc}$	$3bc:\sqrt[5]{aa+d}$
1+2	3	$5d\sqrt[3]{aa}$	$b+c$	$4bc:\sqrt[5]{aa+d}$

Examples in Substraction.

	1	$12\sqrt{bc}$	$10b\sqrt{ac}$	$4b\sqrt{aa+cc}$
	2	$7\sqrt{bc}$	$4b\sqrt{ac}$	$3b\sqrt{aa+cc}$
1−2	3	$5\sqrt{bc}$	$6b\sqrt{ac}$	$b\sqrt{aa+cc}$

	1	$5d:\sqrt[3]{aa}$	$b+c$	$4bc:\sqrt[5]{aa+d}$
	2	$4d:\sqrt[3]{aa}$	$c-\sqrt[3]{aa-cc}$	$3bc:\sqrt[5]{aa+d}$
1−2	3	$d:\sqrt[3]{aa}$	$b+\sqrt[3]{aa-cc}$	$bc:\sqrt[5]{aa+d}$

Case 2. When the Surd Quantities are Heterogeneal, (*viz.* their Indices are unlike) they are only to be added or substracted by their Signs, *viz.* + or −. And from thence will arise Surds either Binomial or Residual.

Examples in Addition.

	1	$\sqrt{bc}$	$4d\sqrt{a}$	$\sqrt[3]{ac-ba}$
	2	$\sqrt{ba}$	$3b\sqrt{ac}$	$\sqrt{ac+ba}$
1+2	3	$\sqrt{bc}:+\sqrt{ba}$	$4d\sqrt{a}:+3b\sqrt{ac}$	$\sqrt[3]{ac-ba}:+\sqrt{ac+ba}$

Examples in Substraction.

	1	$\sqrt{bc}$	$b-d\sqrt{aaa+ca}$
	2	$\sqrt{ba}$	$d-2a\sqrt{bd+dd}$
1−2	3	$\sqrt{bc}-\sqrt{ba}$	$b-d\sqrt{aaa+ca}:-d+2a\sqrt{bd+dd}$

Sect.

Sect. 2. Multiplication *of Surd Quantities.*

Case 1. WHEN the Quantities are pure Surds of the ſame Kind; multiply them together, and to their Product prefix their radical Sign.

EXAMPLES.

	1	$\surd b$	$\surd ba+da$	$\surd aa+bb$
	2	$\surd a$	$\surd ca$	$\surd aa-bb$
1 × 2	3	$\surd ba$	$\surd bcaa+dcaa$	$\surd aaaa-bbbb$

Case 2.. If Surd Quantities of the ſame Kind (as before) are joined to rational Quantities, then multiply the rational into the rational, and the Surd into the Surd, and join their Products together.

EXAMPLES.

	1	$d\surd bc$	$5cd\surd ba+da$	$15\surd ab$
	2	$3b\surd a$	$3a\surd ca$	$5\surd d$
1 × 2	3	$3db\surd bca$	$15cda\surd bcaa+dcaa$	$75\surd abd$

Sect. 3. Diviſion *of Surd Quantities.*

Case 1. WHEN the Quantities are pure Surds of the ſame Kind, and can be divided off, (*viz.* without leaving a Remainder) divide them, and to their Quotient prefix their radical Sign.

EXAMPLES.

	1	$\surd ba$	$\surd bcaa+acaa$	$\surd aaaa-bbbb$
	2	$\surd b$	$\surd ca$	$\surd aa-bb$
1 ÷ 2	3	$\surd a$	$\surd ba+da$	$\surd aa+bb$

Case 2. If Surd Quantities, of the ſame Kind, are joined to rational Quantities; then divide the rational by the rational, if it can be, and to their Quotient join the Quotient of the Surd divided by the Surd with it's firſt radical Sign.

EXAMPLES.

	1	$3db\surd bca$	$15cda\surd bcaa+dcaa$	$75\surd abd$
	2	$3b\surd a$	$3a\surd ca$	$5\surd d$
1 ÷ 2	3	$d\surd bc$	$5cd\surd ba+da$	$15\surd ab$

Note,

Note, If any Square be divided by it's Root, the Quotient will be it's Root.

EXAMPLES.

	1	a	$bb+2bc+cc$	$aaaa-2bbaa+bbbb$
	2	$\sqrt{a}$	$\sqrt{bb+2bc+cc}$	$\sqrt{a^4-2bbaa+b^4}$
$1\div 2$	3	$\sqrt{a}$	$\sqrt{bb+2bc+cc}$	$\sqrt{a^4-2bbaa+b^4}$

Sect. 4. Involution *of Surd Quantities.*

Case 1. WHEN the Surds are not joined to rational Quantities, they are involved to the same Heights as their Index denotes, by only taking away their radical Sign.

EXAMPLES.

	1	$\sqrt{a}$	$\sqrt{bca}$	$\sqrt{aa-bb}$	$\sqrt{5a-da}$
$1\,@^2$	2	a	bca	$aa-bb$	$5a-da$

Case 2. When the Surds are joined to rational Quantities, involve the rational Quantities to the same Height as the Index of the Surd denotes; then multiply those involved Quantities into the Surd Quantities, after their radical Sign is taken away, as before.

EXAMPLES.

	1	$b\sqrt{a}$	$5d\sqrt{ca}$	$3b\sqrt{aa-dd}$
$1\,@^2$	2	bba	$25ddca$	$9bbaa-9bbdd$

	1	$a:\sqrt[3]{bc}$	$3d:\sqrt[3]{aa+bb}$	$da:\sqrt[2]{b}$
$1\,@^3$	2	$aaabc$	$27dddaa+27dddbb$	$dddaaab$

The Reason of only taking away the radical Sign, as in *Case* 1. is easily conceived, if you consider that any Root being involved into it self, produces a Square, &c. And from thence the Reason of those Operations performed by the second Case may be thus stated.

Suppose $b\sqrt{a}=x$. Then $\sqrt{a}=\frac{x}{b}$ *per Axiom* 4. and both Sides of the Equation being equally involved, it will be $a=\frac{xx}{bb}$. Then multiplying both Sides of the Equation into bb, it becomes $bba=xx$ *per Axiom* 3. Which was to be proved.

Again,

Again, Let $5\,d\sqrt{c\,a} = x$: Then $\sqrt{c\,a} = \frac{x}{5\,d}$, and $ca = \frac{x\,x}{25\,d\,d}$.

Alſo from hence it will be eaſy to deduce the Reaſon of multiplying Surd Quantities, according to both the *Caſes*. For

Suppoſe	1	$\sqrt{b} = z$	*Example* 1. *Caſe* 1.
	2	$\sqrt{a} = x$	
1 ☉^{2}	3	$b = z\,z$	
2 ☉^{2}	4	$a = x\,x$	
3×4	5	$b\,a = z\,z\,x\,x$. *per Axiom* 2.	
5 ω^2	6	$\sqrt{b\,a} = z\,x$. which was to be proved.	

Let	1	$d\sqrt{b\,c} = z$	*Example* 1. *Caſe* 2.
	2	$3\,b\sqrt{a} = x$	
$1 \div d$	3	$\sqrt{b\,c} = \frac{z}{d}$	
$2 \div 3\,b$	4	$\sqrt{a} = \frac{x}{3\,b}$	
4×3	5	$\sqrt{a\,b\,c} = \frac{z\,x}{3\,b\,d}$, from what is proved above.	
$5 \times 3\,b\,d$	6	$3\,b\,d\sqrt{b\,c\,a} = z\,x$, &c. for the reſt.	

Diviſion being the Converſe to *Multiplication*, needs no other Proof.

CHAP. V.

Concerning the Nature of Equations *and how to prepare them for a* Solution.

WHEN any Problem or Queſtion is propoſed to be analytically reſolved; it is very requiſite that the true Deſign or Meaning thereof, be fully and clearly comprehended (in all it's Parts) that ſo it may be truly abſtracted from ſuch ambiguous Words as Queſtions of this Kind are often diſguiſed with; otherwiſe it will be very difficult, if not impoſſible, to ſtate the Queſtion right in it's ſubſtituted Letters, and ever to bring it to an Equation by ſuch various Methods of ordering thoſe Letters as the Nature of the Queſtions may require.

Now

Now the Knowledge of this difficult Part of the Work is only to be obtained by Practice, and a careful minding the Solution of ſuch leading Queſtions as are in themſelves very eaſy. And for that Reaſon I have inſerted a Collection of ſeveral Queſtions; wherein there is great Variety.

Having got ſo clear an Underſtanding of the Queſtion propoſed, as to place down all the Quantities concerned in their due Order, *viz.* all the ſubſtituted Letters, in ſuch Order as their Nature requires; the next thing muſt be to conſider whether it be limited or not. That is, whether it admits of more Anſwers than one. And to diſcover that, obſerve the two following Rules.

RULE 1.

When the Number of the Quantities ſought exceed the Number of the given Equations, the Queſtion is capable of innumerable Anſwers.

EXAMPLE.

Suppoſe a Queſtion were propoſed thus; there are three ſuch Numbers, that if the firſt be added to the ſecond, their Sum will be 22. And if the ſecond be added to the third, their Sum will be 46. What are thoſe Numbers?

Let the three Numbers be repreſented by three Letters, thus, call the firſt a, the ſecond e, and the third y.

Then $\left\{\begin{array}{l} a+e=22 \\ e+y=46 \end{array}\right\}$ according to the Queſtion.

Here the Number of Quantities ſought are three; a, e, y, and the Number of the given Equations are but two. Therefore this Queſtion is not limited, but admits of various Anſwers; becauſe for any one of thoſe three Letters you may take any Number at Pleaſure, that is leſs than 22. Which with a little Conſideration will be very eaſy to conceive.

RULE 2.

When the Number of the given Equations (not depending upon one another) are juſt as many as the Number of the Quantities ſought; then is the Queſtion truly limited, viz. *each Quantity ſought hath but one ſingle Value.*

As for inſtance, let the aforeſaid Queſtion be propoſed thus. There are three Numbers (a, e, and y, as before) if the firſt be added to the ſecond, their Sum will be 22; if the ſecond be added

to the third, their Sum will be 46; and if the firſt be added to the third, their Sum will be 36. What are the Numbers? That is, $a+e=22$. $e+y=46$ and $a+y=36$. Now the Queſtion is perfectly limited, each ſingle Quantity having but one ſingle Value, to wit $a=6$, $e=16$, and $y=30$.

N. B. If the Number of the given Equations exceeds the Number of the Quantities ſought; they not only limit the Queſtion, but oftentimes render it impoſſible, by being propoſed inconſiſtent one to another.

Having truly ſtated the Queſtion in it's ſubſtituted Letters, and found it limited to one Anſwer (or at leaſt ſo bounded as to have a certain determinate Number of Anſwers) then let all thoſe ſubſtituted Letters be ſo ordered or compared together, either by adding, ſubſtracting, multiplying, or dividing them, *&c.* according as the Nature of the Queſtion requires, until all the unknown Quantities except one are caſt off or vaniſhed; but therein great Care muſt be taken to keep them to an exact Equality; and when that unknown Quantity, or ſome Power of it (as Square, Cube, *&c.)* is found equal to thoſe that are known; then the Queſtion is ſaid to be brought to an Equation, and conſequently to a Solution, *viz.* fitted for an Anſwer.

But no particular Rules can be preſcribed for the caſting off, or getting away Quantities out of an Equation; that Part of the Art is only to be obtained by Care and Practice. And when that is done, it generally happens ſo, that the unknown Quantity which is retained in the Equation, is ſo mixed and entangled with thoſe that are known, that it often requires ſome Trouble and Skill to bring it (or it's Powers, *&c.)* to one Side of the Equation, and thoſe that are known to the other Side; (ſtill keeping them to a juſt Equality) which the ingenious Mr *Scooten*, in his *Principia Matheſeos Univerſalis*, calls Reduction of Equations.

The Buſineſs of reducing Equations *(as of moſt, if not all Algebraick Operations)* is grounded and depends upon a right Application of the five *Axioms* propoſed in Page 146, and therefore, if thoſe *Axioms* be well underſtood, the Reaſon of ſuch Operations muſt needs appear very plain, and the Work be eaſily performed; as in the following *Sections*.

Sect. 1. *Of Reduction by* Addition.

REDUCTION by *Addition* is grounded upon *Axiom* 1. and is only the tranſpoſing (*viz.* the removing) of any Negative Quantity from either Side of an Equation to the other Side, with the Sign + before it; as in theſe

EXAMPLES.

Suppoſe	1	$a - b = d$		Again,		
Then	2	$a = d + b$		Let	1	$aa - d = c - aa$
For	3	$b = d$		$1 + d$	2	$aa = c - aa + d$
$1 + 3$	4	$a = d + b$		$2 + aa$	3	$2aa = c + d$

Let	1	$3a - 4 = 6 - a$	*Note*, When any abſolute Number is regiſtered in the Margin, you muſt draw a Line over it, to diſtinguiſh it from the other Numbers. As $\bar{4}$ in the 2d Step of this Example.
$1 + \bar{4}$	2	$3a = 6 + 4 - a$	
$2 + a$	3	$4a = 6 + 4 = 10$	

Let	1	$aa - dc - b = dd - 2ba$
$1 + b$	2	$aa - dc = dd - 2ba + b$
$2 + dc$	3	$aa = dd - 2ba + b + dc$
$3 + 2ba$	4	$aa + 2ba = dd + b + dc$

Suppoſe	1	$2da - d = cc - 3baa - aaa$
$1 + aaa$	2	$aaa + 2da - d = cc - 3baa$
$2 + 3baa$	3	$aaa + 3baa + 2da - d = cc$
$3 + d$	4	$aaa + 3baa + 2da = cc + d$, &c.

Sect. 2. *Of Reduction by* Subſtraction.

REDUCTION by *Subſtraction* is grounded upon *Axiom* 2, and is performed by tranſpoſing (or removing) any Affirmative Quantity from either Side of the Equation, to the other Side, with the Sign — before it; as in theſe

EXAMPLES.

Suppoſe	1	$a + b - d$		Let	1	$3a + 4 = 6 + a$
And	2	$b = b$		$1 - a$	2	$2a + 4 = 6$
$1 - 2$	3	$a = d - b$		$2 - \bar{4}$	3	$2a = 6 - 4 = 2$

Suppoſe	1	$aa + dc + b = dd + 2ba$
$1 - 2ba$	2	$aa - 2ba + dc + b = dd$
$2 - dc$	3	$aa - 2ba + b = dd - dc$
$3 - b$	4	$aa - 2ba = dd - dc - b$

Let

Let	1	$aaa+d=cc+3baa+2da$
$1-3baa$	2	$aaa-3baa+d=cc+2da$
$2-2da$	3	$aaa-3baa-2da+d=cc$
$3-d$	4	$aaa-3baa-2da=cc-d$

Sect. 3. *Of Reduction by* Multiplication.

FRACTIONAL Quantities, in any Equation, are brought into whole Quantities by multiplying every Term in the Equation with the Denominators of the Fractions, *per Axiom* 3; as in these

EXAMPLES.

Suppose	1	$\frac{a}{5}=6$
Then	2	$a=6\times5=30$. For $\frac{a}{5}\times5=\frac{5a}{5}=a$

Let	1	$3a=\frac{dc}{2b}$	Suppose	1	$a=\frac{dd}{a-b}$
$1\times2b$	2	$6ba=dc$	$1\times\overline{a-b}$	2	$aa-ba=dd$

Suppose	1	$\frac{aa}{b}+c+f=\frac{dx}{a}$
$1\times b$	2	$aa+bc+bf=\frac{dxb}{a}$
$2\times a$	3	$aaa+bca+bfa=dxb.$

Suppose	1	$\frac{aaa}{aa-bb}=\frac{ba-bb}{a+b}$
$1\times\overline{aa-bb}$	2	$aaa=\frac{baaa-bbaa-bbba+bbbb}{a+b}$
$1\times\overline{a+b}$	3	$aaaa+baaa=baaa-bbaa-bbaa+bbbb$

Sect. 4. *Of Reduction by* Division.

WHEN any Quantity (either known or unknown) is in every Term of an Equation, if the whole Equation be divided by that Quantity, it will be reduced into lower Terms, *per Axiom* 4, as in these following *Examples*.

EXAMPLES.

Suppose			Let		
$1 \div b$	1	$baa+bca=bcd$	$1 \div 1a$	1	$aa=7a$
	2	$aa+ca=cd$		2	$a=7$

Let		
	1	$ffaa+ffcaa-ffa=ffda+ffdda$
$1 \div ff$	2	$aa+caa-a=da+dda$
$2 \div a$	3	$a+ca-1=d+dd$

Or when the unknown Quantity is multiplied (*viz.* joined) with any that is known; let the whole Equation be divided by the known Quantity, that so the unknown may be cleared; as in these

EXAMPLES.

Suppose			Let		
$1 \div \overline{b-c}$	1	$ba-ca=d$	$1 \div \overline{c-d}$	1	$caa-daa=cd-dd$
	2	$a=\frac{d}{b-c}$		2	$aa=\frac{cd-dd}{c-d}=d$

Suppose		
	1	$bbaaa-2bbaa=bda+cba$
$1 \div ba$	2	$baa-2ba=d+c$
$2 \div b$	3	$aa-2a=\frac{d+c}{b}$

Let		
	1	$49daa+42aa=7bca+21ca$
$1 \div 7$	2	$7daa+6aa=bca+3ca$
$2 \div a$	3	$7da+6a=bc+3c$
$3 \div$	4	$a=\frac{bc+3c}{7d+6}$

Sect. 5. *Of Reduction by* Involution.

WHEN there happens to be an Equation between any homogeneal or like Surds, take away the radical Signs from the Quantities, and they will become rational; as in these

EXAMPLES.

Suppose			Let			
$1 \circledcirc 2$	1	$\sqrt{a}=\sqrt{d+c}$	$1 \circledcirc 3$	1	$\sqrt[3]{aa}=\sqrt[3]{db+bc}$	} *per Sect.* 4.
	2	$a=d+c$		2	$aa=db+bc$	} *Chap.* 3.

Or if one Side of the Equation consists of Surd Quantities, and the other Side be rational, then involve the rational Quantities to the

the ſame Power (or Height) with the Index of the Surd, and take away the radical Sign; as in theſe

EXAMPLES.

Let	1	$\surd a=6$	Suppoſe	1	$\surd a=b+c$
$1\,@^{2}$	2	$a=36$	$1\,@^{2}$	2	$a=bb+2bc+cc$

Suppoſe	1	$\sqrt[3]{aa-ba}=d$	Let	1	$\sqrt[5]{aa}=7$
$1\,@^{3}$	2	$aa-ba=ddd$	$1\,@^{5}$	2	$aa=16807.$

Sect. 6. *Of Reduction by* Evolution.

WHEN any ſingle Power of the unknown Quantity is on one Side of an Equation; evolve both Sides of the Equation, according as the Index of that Power denotes, and their Roots will be equal; as in theſe

EXAMPLES.

Suppoſe	1	$aa=36$	Let	1	$aaa=27$
$1\,\omega^{2}$	2	$a=\surd 36=6$	$1\,\omega^{3}$	2	$a=\sqrt[3]{27}=3$, &c.

Suppoſe	1	$aa=bb-dd$	Let		$aaa=b^{3}+3bbc+3bcc+c^{3}$
$1\,\omega^{2}$	2	$a=\surd\overline{bb-dd}$	$1\,\omega^{3}$		$a=b+c$

Or if any compound Power of the unknown Quantity be on one Side of the Equation (that hath a true Root of it's kind) evolve both Sides of the Equation, and it will be depreſſed into lower Terms; as in theſe

EXAMPLES.

Suppoſe	1	$aa+2ba+bb=dd$	$aa-2ba+bb=ddcc$
$1\,\omega^{2}$	2	$a+b=d$	$a-b=dc$

Here follow a few Examples of clearing Equations, wherein all the foregoing Reductions are promiſcuouſly uſed, as Occaſion requires.

EXAMPLE 1.

Suppoſe	1	$\frac{aa+c-d}{4}=\frac{g-aa}{b}$, what is $a=$ to?
$1\times\frac{1}{4}$	2	$aa+c-d=\frac{4g-4aa}{b}$

$2\times b$	3	$baa+bc-bd=4g-4aa$
$3+4aa$	4	$baa+4aa+bc-bd=4g$
$4+bd$	5	$baa+4aa+bc=4g+bd$
$5-bc$	6	$baa+4aa=4g+bd-bc$
$6\div b+4$	7	$aa=\frac{4g+bd-bc}{b+4}$
$7\,ww$	8	$a=\sqrt{\frac{4g+bd-bc}{b+4}}$, as was required.

EXAMPLE 2.

Suppose	1	$\frac{a+354}{a}=\frac{3a}{354-a}$ what is the Value of a?
$1\times a$	2	$a+354=\frac{3aa}{354-a}$
$2\times 353-a$	3	$125316-aa=3aa$
$2+aa$	4	$4aa=125316$
$4\div 4$	5	$aa=31329$
$5\,ww$	6	$a=\sqrt{31329}=177$, the Value of a required.

EXAMPLE 3.

Suppose	1	$\sqrt{\frac{aa+3bb}{4}}-\sqrt{\frac{aa-3bb}{4}}=\sqrt{\frac{baa}{c}}:a=?$
$1\,\mathcal{Q}^2$	2	$\left\{\begin{array}{l}\frac{aa+3bb}{4}-2\sqrt{\frac{aa+3bb}{4}}\times\sqrt{\frac{aa-3bb}{4}}\\ :+\frac{aa-3bb}{4}=\frac{baa}{c}\end{array}\right.$
That is	3	$\frac{aa}{2}-\sqrt{\frac{a^4-9b^4}{4}}=\frac{baa}{c}$
For		$\frac{aa+3bb}{4}+\frac{aa-3bb}{4}=\frac{2aa}{4}=\frac{aa}{2}$
And		$2\sqrt{\frac{aa+3bb}{4}}=\sqrt{\frac{4aa+12bb}{4}}=\sqrt{\frac{aa+3bb}{4}}$
Then		$\sqrt{aa+3bb}\times\sqrt{\frac{aa-3bb}{4}}=\sqrt{\frac{a^4-9b^4}{4}}$
$3+\sqrt{}\ \&c$	4	$\frac{aa}{2}=\frac{baa}{c}+\sqrt{\frac{a^4-9b^4}{4}}$

4—

$4 - \frac{baa}{c}$	5	$\frac{aa}{2} - \frac{baa}{c} = \sqrt{\frac{a^4 - 9b^4}{4}}$
$5 \circledcirc^2$	6	$\frac{a^4}{4} - \frac{ba^4}{c} + \frac{bba^4}{cc} = \frac{a^4 - 9b^4}{4}$
$6 + \frac{ba^4}{c}$	7	$\frac{a^4}{4} + \frac{bba^4}{cc} = \frac{a^4 - 9b^4}{4} + \frac{ba^4}{c}$
$7 \pm$	8	$\frac{bba^4}{cc} + \frac{9b^4}{4} = \frac{ba^4}{c}$
$8 \div b$	9	$\frac{ba^4}{cc} + \frac{9b^3}{4} = \frac{a^4}{c}$
$9 \times cc$	10	$ba^4 + \frac{9ccb^3}{4} = ca^4$
$10 \times \bar{4}$	11	$4ba^4 + 9ccb^3 = 4ca^4$
$11 - 4ba^4$	12	$9ccb^3 = 4ca^4 - 4ba^4$
$12 \div$	13	$aaaa = \frac{9ccb^3}{4c - 4b}$
For		$\overline{4c - 4b} \times a^4 = 4ca^4 - 4ba^4$
$13 \omega^2$	14	$aa = \sqrt{\frac{9ccb^3}{4c - 4b}}$
$14 \omega^2$	15	$a = \sqrt{:} \sqrt{\frac{9ccb^3}{4c - 4b}}$, as was required.

By help of these Reductions (properly applied) the unknown Quantity (a) or it's Powers, are cleared and brought to one Side of an Equation; and if the unknown Quantity (a) chance to be equal to those that are known, the Question is answered: as in the first *Example* of *Sect.* 1 and 2. Or if any single Power of the unknown Quantity (a) is found equal to those that are known, then the respective Root of the known Quantities is the Answer; as in the first four *Examples* of *Sect.* 6, &c.

But when the Powers of the unknown Quantities are either mixed with their Root, as $aa + ba = dd$, &c. or do consist of different Powers, as $aaa + baa = dd$, &c.: Then they are called Affected, or Adfected Equations, which require other Methods to resolve them; *viz.* to find out the Value of (a) as shall be shewed further on.

CHAP.

CHAP. VI.

Of Proportional Quantities; *both* Arithmetical, Geometrical *and* Muſical.

WHAT hath been ſaid of Numbers in *Arithmetical Progreſſion*, Chap. 6. Part I. may be eaſily applied to any Series of Homogeneal or like Quantities.

Sect. 1. *Of* Quantities *in* Arithmetical Progreſſion.

THOSE Quantities are ſaid to be in the moſt ſimple or natural Progreſſion, that begin their Series of increaſe or decreaſe with a Cypher:

Thus { 0 : a : $2a$: $3a$: $4a$: $5a$: $6a$: &c. increaſing,
0 : $-a$: $-2a$: $-3a$: $-4a$: $-5a$: $-6a$: &c. decreaſing.

Or Univerſally, putting a the firſt Term in the Progreſſion, and e the common Exceſs or Difference.

Then { a : $a+e$: $a+2e$: $a+3e$: $a+4e$: $a+5e$: $a+6e$: &c.
a : $a-e$: $a-2e$: $a-3e$: $a-4e$: $a-5e$: $a-6e$: &c.

In the firſt of theſe Series it is evident, that if there be but three Terms; the Sum of the Extreams will be double to the Mean.

As in theſe, 0 : a : $2a$: or, a : $2a$: $3a$: or, $2a$: $3a$: $4a$, &c. *viz.* $2a$: $+0=a+a$: or, $a+3a=2a+2a$, &c.

Alſo, in the ſecond Series, either increaſing or decreaſing, it is evident, that if the Terms be a : $a+e$: $a+2e$, &c. increaſing; then $a+a+2e$, *viz.* $2a+2e$ the Sum of the Extreams, is double to $a+e$ the Mean, or if they be a : $a-e$: $a-2e$, &c. decreaſing; then $a+a-2e$: *viz.* $2a-2e$ the Sum of the Extreams, is double to $a-e$ the Mean. And ſo it will be in any other three of the Terms. 2dly, if there are four Terms; then the Sum of the two Extreams will be equal to the Sum of the two Means; as in theſe, a : $a+e$: $a+2e$: $a+3e$, in the Series increaſing; here $a+a+3e=a+e+a+2e$.

Alſo in theſe, a : $a-e$: $a-2e$: $a-3e$ in the Series decreaſing; here $a+a-3e=a-e+a-2e$, &c. in any other four Terms.

Conſequently, If there are never ſo many Terms in the Series, the Sum of the two Extreams will always be equal to the Sum of

of any two Means, that are equally distant from those Extreams. As in these, $a : a+e : a+2e : a+3e : a+4e : a+5e$: &c. Here $a+a+5e = a+e+a+4e = a+2e+a+3e$, &c. And if the Number of Terms be odd, the Sum of the two Extreams will be double to the middle Term, *&c.* as in Corol. 1. Chap. 6. beforementioned.

CONSECTARY 1.

Whence it follows, (and is very easy to conceive) that if the Sum of the two Extreams be multiplied into the Number of all the Terms in the Series, the Product will be double the Sum of all the Series.

Now for the easier resolving such Questions as depend upon these Progressional Quantities,

Let
- a = the first Term, as before.
- y = the last Term.
- e = the common Excess, *&c.* as before.
- N = the Number of all the Terms.
- S = the Sum of all the Series, *viz.* of all the Terms.

Then will $\overline{a+y} \times N = 2S$, by the precedent Consectary: that is, $Na+Ny = 2S$. Consequently $\frac{Na+Ny}{2} = S$, the Sum of all the Series, be the Terms never so many. Thirdly, in these Series it is easy to perceive, that the common Difference (e) is so often added to the last Term of the Series; as are the Number of Terms, except the first; that is, the first Term (a) hath no Difference added to it, but the last Term hath so many times (e) added to it, as it is distant from the first.

Consequently, the Difference betwixt the two Extreams, is only the common Difference (e) multiplied into the Number of all the Terms less Unity or 1. That is, $\overline{N-1} \times e = y-a$, the Difference betwixt the two Extreams, *viz.* $Ne-e = y-a$.

CONSECTARY 2.

Whence it follows, that if the Difference betwixt the two Extreams be divided by the Number of Terms less 1, the Quotient will be the common Difference of the Series.

To wit, $\frac{y-a}{N-1} = e$.

Now by the Help of these two Consectaries, if any three of the aforesaid five Parts *(viz. a. y. e. N. S.)* be given; the other two may be easily found.

Thus,	1	$\frac{Na+Ny}{2}=S$
And	2	$\frac{y-a}{N-1}=e$ } as before.
$2\times\overline{N-1}$	3	$y-a=Ne-e$
$3+e$	4	$y-a+e=Ne$
$4\div e$	5	$\frac{y-a+e}{e}=N$, the Number of Terms.
$1\times\overline{2}$	6	$Na+Ny=2S$
$6-Na$	7	$Ny=2S-Na$
$7\div N$	8	$\frac{2S-Na}{N}=y$, the last Term.
$6-yN$	9	$Na=2S-Ny$
$9\div N$	10	$\frac{2S-Ny}{N}=a$, the first Term.
$6\div\overline{a+y}$	11	$\frac{2S}{a+y}=N$, the Number of Terms.
5, and 11	12	$\frac{y-a+e}{e}=\frac{2S}{a+y}$, *per Axiom* 5.
$12\times\overline{a+y}$	13	$\frac{yy-aa}{e}+a+y=2S$
$13\div\overline{2}$	14	$\frac{yy-aa}{2e}+\frac{a+y}{2}=2S$, the Sum of all the Series.
$14\times 2e$	15	$yy-aa+ae+ye=2Se$
$15-ae$	16	$yy-aa+ye=2Se-ae$
$16-ye$	17	$yy-aa=2Se-ae-ye$
$17\div$	18	$\frac{yy-aa}{2S-a-y}=e$, the common Difference.
$3+a$	19	$Ne-e+a=y$, the last Term.
$19+e$	20	$Ne+a=y+e$
$20-Ne$	21	$y+e-Ne=a$, the first Term.

&c.

In like Manner you may proceed to find out any of the five Quantities *(a. e. y. N. S.)* otherwise, *viz.* by varying or comparing those Equations one with another, you may produce new Equations

Equations with other Data in them; the which I ſhall here omit purſuing, and leave them for the Learner's Practice.

Sect. 2. *Of* Quantities *in Geometrical Proportion.*

GEOMETRICAL Proportion continued has been already defined in Sect. 2. Chap. 6. Part I. And what is there ſaid concerning Numbers in ∴ may be applied to any Sort of Homogeneal Quantities that are in ∴.

The moſt natural and ſimple Series of Geometrical Proportionals, is when it begins with Unity or 1.

As $1 . a . aa . aaa . aaaa . a^5 . a^6$, &c. in ∴.

For $1 : a :: a : aa :: aa : aaa :: aaa : aaaa$, &c.

Or $a . b . \frac{bb}{a} . \frac{bbb}{aa} . \frac{bbbb}{aaa} . \frac{b^5}{a^4}$, &c. are Terms in ∴

For $a : b :: b : \frac{bb}{a} :: \frac{bb}{a} : \frac{bbb}{aa} :: \frac{bbb}{aa} : \frac{b^4}{a^3} :: \frac{b^4}{a^3} : \frac{b^5}{a^4}$, &c.

That is, when all the middle Terms betwixt the two Extreams are both Conſequents and Antecedents, that Series is in Geometrical Proportion continued. Therefore in every Series of Quantities in ∴ all the Terms except the laſt are Antecedents; and all the Terms except the firſt are Conſequents. But Univerſally putting a the firſt Term in the Series, and e the Ratio, *viz.* the common Multiplier, or Diviſor; then it will be

$a . ae . aee . aeee . aeeee . ae^5 . ae^6$. &c. in ∴

Or $a . \frac{a}{e} . \frac{a}{ee} . \frac{a}{eee} . \frac{a}{eeee} . \frac{a}{e^5}$. &c. are in ∴ Decr.

For $a : ae :: ae : \frac{aaee}{a} = aee$, &c.

And $a : \frac{a}{e} :: \frac{a}{e} : \frac{aa}{aee} = \frac{a}{ee} a : \frac{a}{e} :: \frac{a}{ee} : \frac{a}{eee}$, &c.

I. In any of theſe Series it is evident, that if three Quantities are in ∴, the Rectangle of the two Extreams will be equal to the Square of the Mean; as in theſe, $a . ae . aee$, here $a \times aee = ae \times ae, = aaee$. &c.

Or $a \cdot \frac{a}{e} \cdot \frac{a}{ee}$; here alſo $a \times \frac{a}{ee} = \frac{a}{e} \times \frac{a}{e} = \frac{aa}{ee}$, &c.

II. If four Quantities are in ∺ the Rectangle of the Extreams will be equal to the Rectangle of the Means.

As in theſe, $a \cdot ae \cdot aee \cdot aeee$; here $a \times ae^3 = ae \times aee$.

Or $a \cdot \frac{a}{e} \cdot \frac{a}{ee} \cdot \frac{a}{eee}$; here alſo $a \times \frac{a}{eee} = \frac{a}{e} \times \frac{a}{ee} = \frac{aa}{eee}$, &c.

Conſequently, If there are never ſo many Terms in the Series of ∺, the Rectangle of the Extreams will be equal to the Rectangle of any two Means that are equally diſtant from thoſe Extreams.

As in theſe, $a \cdot ae \cdot aee \cdot aeee \cdot ae^4 \cdot ae^5$
viz. $ae^5 \times a = ae^4 \times ae$. Or $ae^5 \times a = aeee \times aee = aae^5$

III. If never ſo many Quantities are in ∺ it will be, as any one of the Antecedents is to it's Conſequents, ſo is the Sum of all the Antecedents, to the Sum of all the Conſequents.

As in theſe, $\begin{cases} a \cdot ae \cdot aee \cdot aeee \cdot aeeee \cdot ae^5$, &c. increaſing.
$a \cdot \frac{a}{e} \cdot \frac{a}{ee} \cdot \frac{a}{eee} \cdot \frac{a}{eeee} \cdot \frac{a}{e^5}$, &c. decreaſing.

$a : ae :: a + ae + aee + ae^3 + ae^4 : ae + aee + ae^3 + ae^4 + ae^5$

Or $a : \frac{a}{e} :: a + \frac{a}{e} + \frac{a}{ee} + \frac{a}{e^3} + \frac{a}{e^4} : \frac{a}{e} + \frac{a}{e} + \frac{a}{ee} + \frac{a}{e^3} + \frac{a}{e^4} + \frac{a}{e^5}$, *viz.* $a \times \overline{ae + aee + ae^3 + ae^4 + ae^5} = ae \times \overline{a + ae + aee + ae^3 + ae^4}$.

That is, the Rectangle of the Extreams is equal to the Rectangle of the Means; *per* Second of this *Sect.*

Note, The Ratio of any Series in ∺ increaſing is found by dividing any of the Conſequents by it's Antecedent.

Thus, $a) ae (e$ Or $ae (aee (e$, &c.

But if the Series be decreaſing, then the Ratio is found by dividing any of the Antecedents by it's Conſequent.

Thus, $\frac{a}{e}) a (e$. Or $\frac{a}{ee}) \frac{a}{e} (e$, &c.

CON-

CONSECTARY.

These Things being premised, such Equations may be deduced from them, as will solve all such Questions as are usually proposed about Quantities in Geometrical Proportion ∴. *In order to that,*

let $\left\{\begin{array}{l} a = \text{the first Term.} \\ e = \text{the common Ratio} \\ y = \text{the last Term.} \\ S = \text{the Sum of all the Terms.} \end{array}\right\}$ as before.

Tnen $S - y =$ the Sum of all the Antecedents.

And $S - a =$ the Sum of all the Consequents.

Analogy.	1	$a : ae :: S - y : S - a$ *per* III. of this *Sect.*
1 ∵	2	$Sa - aa = aeS - aey$
$2 \div a$	3	$S - a = eS - ey$
$3 + ey$	4	$S + ey - a = eS$
$4 - S$	5	$ey - a = eS - S$
$5 \div e - 1$	6	$\frac{ye - a}{e - 1} = S$, the Sum of all the Series.
$3 \div S - y$	7	$\frac{S - a}{S - y} = e$, the common Ratio.
$5 + a$	8	$ey = eS + a - S$
$8 \div e$	9	$\frac{eS + a - S}{e} = y$, the last Term.
$4 + a$	10	$S + ey = eS + a$
$10 - eS$	11	$S + ey - eS = a$, the first Term.

Note, The ∵ set in the Margin at the second Step, is instead of *ergo*; and imports that the Rectangle of the two Extreams in the first Step, is equal to the Rectangle of the Means. And so for any other Proportion.

Sect. 3. *Of* Harmonical *Proportion.*

HARMONICAL or Musical Proportion is, when of three Quantities (or rather Numbers) the first hath the same Ratio to the third, as the Difference between the first and second, hath to the Difference between the second and third. As in these following.

Suppose a, b, c, in Musical Proportion.

Then	1	$a : c :: b - a : c - b$
1 ∵	2	$cb - ca = ac - ba$

2 +

$2+ca$	3	$cb=2ac-ba$
$3\div\overline{2c-b}$	4	$\frac{cb}{2c-b}=a$, the first Term.
$3+ba$	5	$2ac=cb+ba$
$5\div\overline{c+a}$	6	$\frac{2ac}{c+a}=b$, the second Term.
$5-cb$	7	$2ac-cb=ba$
$7\div\overline{2a-c}$	8	$\frac{ba}{2a-b}=c$, the third Term.

If there are four Terms in Musical Proportion, the first hath the same Ratio to the fourth, as the Difference between the first and second hath to the Difference between the third and fourth.

That is, let a, b, c, d, be the four Terms, &c.

Then	1	$a:d::b-a:d-c$
1 ∴	2	$db-da=da-ca$
$2-da$	3	$db=2da-ca$
$3\div\overline{2d-c}$	4	$\frac{db}{2d-c}=a$
$3\div d$	5	$b=\frac{2da-ca}{d}$
$3+ca$	6	$db+ca=2da$
$6-db$	7	$ca=2da-db$
$7\div a$	8	$c=\frac{2da-db}{a}$
$7\div\overline{2a-b}$	9	$\frac{ca}{2a-b}=d$

CHAP VII.

Of Proportion Disjunct, *and how to turn Equations into* Analogies, &c.

PROPORTION Disjunct, or the Rule of Three in Numbers, is already explained in *Chap.* 7. *Part* 1. And what hath been there said, is applicable to all Homogeneous Quantities, *viz.* of Lines to Lines, &c.

Sect.

SECT. 1.

IF four Quantities, (*viz.* either Lines, Superficies, or Solids) be Proportional: the Rectangle comprehended under the Extreams, is equal to the Rectangle comprehended under the two Means. 16 *Euclid* 6.)

For inſtance, Suppoſe $a \,.\, b \,.\, c \,.\, d$. to repreſent the four Homogeneal Quantities in Proportion, *viz.* $a : b :: c : d$; then will $ad = bc$. For ſuppoſe $b = 2a$, then will $d = 2c$, and it will be $a : 2a :: c : 2c$. Here the Ratio is 2. But $a \times 2c = 2a \times c$. *viz.* $2ca = 2ac$. Or ſuppoſe $b = 3a$ then will $d = 3c$, and it will be $a : 3a :: c : 3c$. Here the Ratio is 3. But $a \times 3c = 3a \times c$. *viz.* $3ca = 3ac$. Or univerſally putting e for the Ratio of the Proportion, *viz.* making $b = ae$, then will $d = ce$, and it will be $a : ae :: c : ce$. But $a \times ce = ae \times c$, viz. $ace = aec$. Conſequently, $ad = bc$ which was to be proved.

Whence it follows, that if any three of the four Proportional Quantities be given, the fourth may be eaſily found; thus,

Let	1	$a : b :: c : d$	
1 ∴	2	$ad = bc$ as before	
$2 \div d$	3	$a = \frac{bc}{d}$	
$2 \div c$	4	$b = \frac{ad}{c}$	
$2 \div b$	5	$c = \frac{ad}{b}$	
$2 \div a$	6	$d = \frac{bc}{a}$	
$2 \div bd$	7	$\frac{a}{b} = \frac{c}{d}$	*Note*, In this Manner *Euclid*, in his 5th Book, expreſſes the Ratio of Proportionals, *viz.* the Ratio of a to b is $\frac{a}{b}$
Or $2 \div ac$	8	$\frac{b}{a} = \frac{d}{c}$	

If four Quantities are Proportionals they will alſo be Proportionals in Alternation, Inverſion, Compoſition, Diviſion, Converſion, and Mixtly. *Euclid* 5. *Def.* 12, 13, 14, 15, 16.

That

That is, if	1	$a : b :: c : d$ be in direct Proportion, as before.
Then	2	$a : c :: b : d$, alternate. For $ad = bc$
And	3	$b : a :: d : c$, inverted. For $ad = bc$
Alſo	4	$a+b : b :: c+d : d$; compounded.
4 ∴	5	$da+bd = bc+bd$, that is, $ad = bc$, as before.
Or	6	$a+c : c :: b+d : d$; alternately compounded.
6 ∴	7	$ad+cd = bd+cd$, that is, $ad = bc$
Again,	8	$a-b : b :: c-d : d$, divided.
8 ∴	9	$ad-bd = bc-bd$, that is, $ad = bc$
Or	10	$a-c : c :: b-d : d$, alternately divided.
10 ∴	11	$ad-cd = bc-cd$, that is, $ad = bc$
And	12	$a : b \pm a :: c : d \pm c$, converted.
12 ∴	13	$ad \pm ac = bc \pm ac$, that is, $ad = bc$
Laſtly,	14	$a+b : a-b :: c+d : c-d$, mixtly.
14 ∴	15	$ac-ad+bc-bd = ac+ad-bc-bd$
15 ±	16	$2bc = 2ad$, that is, $ad = bc$; as at firſt.

Note; What has been here done about whole Quantities in Simple Proportion, may be eaſily performed in Fractional Quantities, and Surds, &c.

For inſtance, If $\frac{ab}{c} : \frac{d-c}{f} :: \frac{d+c}{c}$, and if it be required to find the fourth Term, it will be $\frac{dd-cc}{fc}$ the Rectangle of the Means; which being divided by the firſt Extream, $\frac{ab}{c}$ will become $\frac{ab}{c}) \frac{dd-cc}{fc} (\frac{ddd-ccc}{abfc} = \frac{dd-cc}{abf}$ the fourth Term.

Or if $b : \sqrt{bd+bc} :: \sqrt{bd+bc}$: to a fourth Term. Then is, $\sqrt{bd+bc} \times \sqrt{bd+bc} = bd+bc$ the Rectangle of the Means; and $b) bd+bc (d+c$ the fourth Term. That is, $b : \sqrt{bd+bc} :: \sqrt{bd+bc} : d+c$, &c.

Sect. 2. *Of* Duplicate *and* Triplicate Proportion.

THE Proportions treated of in the laſt Section, are to be underſtood when Lines are compared to Lines, and Superficies to Superficies; or Solids to Solids, *viz.* when each is compared to that of it's like Kind, which is only called Simple Proportion.

But

But when Lines are compared to Superficies, or Lines are compared to Solids, ſuch compariſons are diſtinguiſhed from the former, by the Names of Duplicate, and Triplicate, (&c.) Proportions; ſo that Simple, Duplicate, and Triplicate, *&c.* Proportions are to be underſtood in a different Senſe from Simple, Double, Treble, *&c.* Proportions, which are only as 1, 2, 3, *&c.* to 1; but thoſe of Simple, Duplicate, Triplicate, *&c.* Proportions, are thoſe of $a \,.\, aa \,.\, aaa$., &c. to 1. Or if the Simple Proportions be that of a to b, whoſe Ratio or Exponent is $\frac{a}{b}$ or $\frac{b}{a}$.

Then $\frac{a}{b} \times \frac{a}{b} = \frac{aa}{bb}$ is the Exponent of the Duplicate.

And $\frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^3}{b^3}$ is the Exponent of the Triplicate Proportions, *&c.* } of $\frac{a}{b}$

And if there are three, four, or more Quantities in $\div\div$, as $1 \,.\, a \,.\, aa \,.\, aaa \,.\, a^4 \,.\, a^5$, &c. (as in the firſt Series, Sect. 2. of the laſt Chapter.) Then, that of the firſt to the third, fourth, and fifth, *&c.* (*viz.* 1 to $aa \,.\, aaa \,.\, a^4\ a^5$) is Duplicate, Triplicate, Quadruplicate, *&c.* of the firſt to the ſecond (*viz.* of 1 to a;) and by Inverſion, that of the third, fourth, fifth, is Duplicate, Triplicate, *&c.* of that of the ſecond to the firſt (a to 1) *per Def.* 10. *Eucl.* 5. But the Nature of theſe Proportions will appear more evident, and be eaſier underſtood, when they are applied to Practice, and illuſtrated by Geometrical Figures, further on.

Sect. 3. *How to turn Equations into* Analogies.

FROM the firſt Section of this Chapter, it will be eaſy to conceive how to turn or diſſolve Equations into Analogies or Proportions. For if the Rectangle of two (or more) Quantities, be equal to the Rectangle of two (or more) Quantities; then are thoſe four (or more) Quantities Proportional. By the 16 *Eucl.* 6. That is, if $ab = cd$, then is $a : c :: d : b$, or $c : a :: b : d$, &c. From whence there ariſes this general Rule for turning Equations into Analogies.

RULE.

Divide either Side of the given Equation (if it can be done) into two ſuch Parts, or Factors, as being multiplied together will produce that Side again; and make thoſe two Parts the two Extreams. Then divide the other Side of the Equation (if it can be done) in the ſame Manner as the firſt was, and let thoſe two Parts or Factors be the two Means.

For Inſtance, Suppoſe $ab+ad=bd$. Then $a:b::d:b+d$. or $b:a::b+d:d$, &c. Or taking ad from both Sides of the Equation, and it will be $ab=bd-ad$; then $a:d::b-a:b$, or, $b:d::b-a:a$, &c.

Again, ſuppoſe $aa+2ae=2by+yy$. Here a and $a+2e$ are the two Factors of the firſt Side in this Equation; for $\overline{a+2e}\times a=aa+2ae$.

Again, y and $2b+y$ are the two Factors of the other Side; therefore, $a:y::2b+y:a+2e$, or $2b+y:a+2e::a:y$, &c.

When one Side of any Equation can be divided into two Factors, as before; and the other Side cannot be ſo divided, then make the Square Root of that Side either the two Extreams or the two Means. For Inſtance, Suppoſe $bc+bd=da+g$, then $b:\sqrt{\overline{da+g}}::\sqrt{\overline{da+g}}:c+d$, or $\sqrt{\overline{da+g}}:b::c+d:\sqrt{\overline{da+g}}$, &c.

CHAP. VIII.

Of Subſtitution, *and the Solution of* Quadratick Equations.

Sect. 1. *Of* Subſtitution.

WHEN new Quantities not concerned in the firſt ſtating of any Queſtion, are put inſtead of ſome that are engaged in it, that is called *Subſtitution*. For Inſtance, If inſtead of $\sqrt{\overline{bc-dc}}$ you put z, or any other Letter; that is, make $z=\sqrt{\overline{bc-dc}}$. Or ſuppoſe $aa+ba-ca+da=dc$, inſtead of $b-c+d$ put s, or any other Letter not engaged with the Queſtion, *viz.* $s=b-c+d$, then $aa\mp sa=dc$. That is, if c be greater than

than $b+d$, it is $aa-sa=dc$; but if $b+d$ be greater than c, then it is $aa+sa=dc$.

And this way of ſubſtituting or putting of new Quantities inſtead of others, may be found very uſeful upon ſeveral Occaſions; *viz.* in Order to make ſome following Operations in the Queſtion more eaſy, and perhaps much ſhorter than they would be without it, as you may obſerve in ſome Queſtions hereafter propoſed in this Tract.

And when thoſe Operations, in which the ſubſtituted Quantities were aſſiſting or uſeful, are performed according as the Nature of the Queſtion required, you may then (if there be Occaſion) bring the original or firſt Quantities into the Equation, in the Place (or Places) of thoſe ſubſtituted Quantities, which is called Reſtitution, as you may ſee further on.

Sect. 2. *The Solution of* Quadratick Equations.

WHEN the Quantity ſought is brought to an Equality with thoſe that are known, and is on one Side of the Equation, in no more than two different Powers whoſe Indices are double one to another, thoſe Equations are called Quadratick Equations Adfected; and do fall under the Conſideration of three Forms or Caſes.

Caſe 1. $aa+2ba=dc$.
Caſe 2. $aa-2ba=dc$.
Caſe 3. $2ba-aa=dc$.

And $\begin{cases} a^4+2ba^2=dc. \\ a^4-2ba^2=dc. \\ 2ba^2-a^4=dc. \end{cases}$

Alſo $\begin{cases} a^6+2ba^3=dc. \\ a^6-2ba^3=dc. \\ 2ba^3-a^6=dc. \end{cases}$ And $\begin{cases} a^8+2ba^4=dc. \\ a^8-2ba^4=dc. \\ 2ba^4-a^8=dc. \end{cases}$ &c.

When there happens to be more Terms in one of theſe Kind of Equations than two, and the higheſt Power of the unknown Quantity is multiplied into ſome known Co-efficients; you muſt reduce them by *Diviſion*; as in Sect. 4. of Chap. 5. and for the Fractional Quantities that may ariſe by thoſe Diviſions, ſubſtitute another Quantity doubled.

For Inſtance, let $baa+caa-ca-da=dc+cb$, then $aa-\frac{ca-da}{b+c}=\frac{dc+cb}{b+c}$. Make $\frac{c-d}{b+c}=2x$, and if you pleaſe,

for $\frac{dc+cb}{b+c}$ put z. Then will $aa-2xa=z$ be the new Equation, equal to the other, being now fitted for a Solution.

Now any of these three Forms of Equations being thus prepared for a Solution, may be reduced to simple Powers by casting off the second or lowest Term of the unknown Quantity; which is done by Substitution; thus, always take half the known Co-efficient, and add it to (Case 1.) or subtract it from (Case 2.) it's fellow Factor; and for their Sum, or Difference, substitute another Letter; as in these.

Let	1	$aa+2ba=dc$ Case 1.
Put	2	$a+b=e$
2@^2	3	$aa+2ba+bb=ee$
$3-1$	4	$bb=ee-dc$
$4+dc$	5	$ee=bb+dc$
$5\,ww$	6	$e=\sqrt{bb+dc}$
2 and 6	7	$a+b=\sqrt{bb+dc}$, *per Axiom* 5.
$7-b$	8	$a=\sqrt{bb+dc}:-b$

Again.

Let	1	$aa-2ba=dc$ Case 2.
Put	2	$a-b=e$
2@^2	3	$aa-2ba+bb=ee$
$3-1$	4	$bb=ee-dc$
$4+dc$	5	$ee=dc+bb$
$5\,ww$	6	$e=\sqrt{dc+bb}$
2 and 6	7	$a-b=\sqrt{dc+bb}$
$7+b$	8	$a=b+\sqrt{dc+bb}$

In Case 3. From Half the known Co-efficient subtract it's fellow Factor.

Thus, Let	1	$2ba-aa=dc$
Put	2	$b-a=e$
2@^2	3	$bb-2ba+aa=ee$
$1+3$	4	$bb=dc+ee$
$4-dc$	5	$ee=bb-dc$

$5\,ww$

$5\ \omega^2$	6	$e=\sqrt{bb-dc}$
2 and 6	7	$b-a=\sqrt{bb-dc}$
$7+a$	8	$b=a+\sqrt{bb-dc}$
$8-\surd$, &c.	9	$a=b-\sqrt{bb-ac}$

And this Method holds good in thoſe other Equations, wherein the higheſt Powers are a^4, a^6, a^8, &c. As, for inſtance,

Let	1	$a^6+2b^3=dc$ Caſe 1.
Put	2	$a^3+b=e$
$2\ \textcircled{e}^2$	3	$a^6+2ba^3+bb=ee$
$3-1$	4	$bb=ee-dc$
$4+cd$	5	$ee=bb+dc$
$5\ \omega^2$	6	$e=\sqrt{bb+dc}$
2 and 6	7	$a^3+b=\sqrt{bb+dc}$
$7-b$	8	$a^3=\sqrt{bb+dc}:-b$
$8\ \omega^3$	9	$a=\sqrt[3]{:\sqrt{bb+dc}:-b}$

The ſame may be done with all the Reſt, Care being taken to add, or ſubſtract, according as the Caſe requires.

But all Quadratick Equations may be more eaſily reſolved by compleating the Square, which is grounded upon the Conſideration of raiſing a Square from any Binomial, or Reſidual Root. (See Sect. 5. Chap. 1.) *Viz.* if $a+b$ be involved to a Square, it will be $aa+2ba+bb$; and if $a-b$ be ſo involved, it will be $aa-2ba+bb$. Whence it is eaſy to obſerve, that $aa+2ba=dc$ (Caſe 1), and $aa-2ba=dc$ (Caſe 2), are imperfect Squares, wanting only bb to make them compleat. And therefore it is, that if half the known Co-efficient be involved to the ſecond Power, and the Square be added to both Sides of the Equation, the unknown Side will become a compleat Square.

Thus Let	1	$aa+2ba=dc$	Here half the Co-efficient $2b$ is b, which being ſquared, is bb.
But	2	$bb=bb$	
$1+2$	3	$aa+2ba+bb=dc+bb$	Caſe 1.
$3\ \omega^2$	4	$a+b=\sqrt{dc+bb}$, as before.	

Again.

Again.

Let	1	$aa - 2ba = dc$ Caſe 2.
But	2	$bb = bb$
1+3	3	$aa - 2ba + bb = dc + bb$
$3\,ww$	4	$a - b = \sqrt{dc + bb}$, &c. as before.

But in Caſe 3. you muſt change the Signs of all the Terms in the Equation,

Thus	1	$2ba - aa = dc$ Caſe 3.
1+	2	$aa - 2ba = -dc$
Then	3	$aa - 2ba + bb = bb - dc$, &c.

And this Method of compleating the Square, holds true in thoſe other Equations.

Viz.	1	$aaaa + 2baa = dc$ Caſe 1.
For	2	$bb = bb$, as before.
1+2	3	$aaaa + 2baa + bb = dc + bb$
$3\,ww$	4	$aa + b = \sqrt{dc + bb}$
4−b	5	$aa = \sqrt{dc + bb} : -b$
$5\,ww$	6	$a = \sqrt{} : \sqrt{dc + bb} : -b$, and ſo on for the reſt.

Or let	1	$a^6 + 2baaa = dc$, as before, Caſe 1.
And	2	$bb = bb$
1+2	3	$a^6 + 2baaa + bb = dc + bb$
$1\,ww$	4	$aaa + b = \sqrt{dc + bb}$
4−b	5	$aaa = \sqrt{dc + bb} : -b$
$5\,www$	6	$a = \sqrt[3]{} : \sqrt{dc + bb} : -b$, &c.

COROLLARY.

Hence it is evident, that whatſoever Method is uſed in ſolving theſe (or indeed any other) Equations, the Reſult will ſtill be the ſame, if the Work be true; as you may obſerve from the Operations of this Section: for both theſe Methods here propoſed, give the ſame Theorems in their reſpective Caſes for the Value of (a).

Thus

Thus, when $aa + 2ba = dc$, then
Theorem 1. $a = \sqrt{dc + bb} : -b$
And when $aa - 2ba = dc$. then
Theorem 2. $a = b + \sqrt{dc + bb}$
Again, when $2ba - aa = dc$. then
Theorem 3. $a = b - \sqrt{bb - dc}$

The like *Theorems* may be eaſily raiſed for the reſt.

If the known Co-efficients (of the ſecond or loweſt Term) be any ſingle Quantity, as $aa + ba = dc$, &c. then is $\frac{1}{2}b$ it's Half, and $\frac{1}{4}bb$ will be the Square of that Half; that is, $\frac{1}{2}b \times \frac{1}{2}b = \frac{1}{4}bb$, and then the Work will ſtand

Thus	1	$aa + ba = dc$
1 C □	2	$aa + ba + \frac{1}{4}bb = dc + \frac{1}{4}bb$
2 ω 2	3	$a + \frac{1}{2}b = \sqrt{dc + \frac{1}{4}bb}$
3 $-b\frac{1}{2}b$	4	$a = \sqrt{dc + \frac{1}{4}bb} : -\frac{1}{2}b$, and ſo for the reſt.

Note, C □ placed in the Margin againſt the ſecond Step, ſignifies that the imperfect Square $aa + ba$ in the firſt Step, is there compleated, *viz.* in the ſecond Step.

Now by the help of theſe *Theorems*, it will be eaſy to calculate or find the Value of the unknown Quantity (a) in Numbers.

EXAMPLE 1.

Suppoſe $aa + 2ba = z$. Let $b = 16$, and $z = 4644$.
Then $a = \sqrt{z + bb} : -b$ *per Theorem* 1.
But $z + bb = 4644 + 256 = 4900$, and $\sqrt{4900} = 70$
Conſequently $a = 70 - 16$, viz. $a = 54$.

But every adfected Equation hath as many Roots (or rather Values of the unknown Quantity) either real or imaginary, as are the Dimenſions (*viz.* the Index) of it's higheſt Power; and therefore the Quantity a, in this Equation, hath another Value either Affirmative or Negative; which may be thus found.

The given Equation is $aa + 32a = 4644$, and it's Root $a = 54$.
Let theſe two Equations be made equal or equated to 0, *viz.* to Nothing.

Thus,

Thus, $aa+32a-4644=0$, and $a-54=0$.

Then divide the given Equation by it's firſt Root, and the Quotient will ſhew the ſecond Value of a.

Thus, $a-54=0$) $aa+32a-4644=0$ ($a+86=0$

$aa-54a$

$+86a-4644$

$86a-4644$

(0)

Hence the ſecond Value of a is $=-86$, or $86=-a$ which ſeems impoſſible, *viz.* that an Affirmative Quantity ſhould be equal to a Negative Quantity; yet even by this ſecond Value of a, and the ſame Co-efficient, the true (or firſt) Equation may be formed

Thus, Let	1	$a=-86$
$1\,\square^2$	2	$aa=+7396$, viz. $-86\times-86=+7396$
1×32	3	$32a=-2752$
$2+3$	4	$aa+32a=4644$, as at firſt.

EXAMPLE 2.

Suppoſe	1	$aa-7a=948{,}75$, then *per Theorem* 2.
$1\,C\,\square$	2	$aa-7a+\frac{49}{4}=948{,}75+\frac{49}{4}=961$
$2\,\overline{ww}^2$	3	$a-\frac{7}{2}$ (or 3,5) $=\surd 961=31$
$3+3{,}5$	4	$a=31+3{,}5=34{,}5$

Again, for the ſecond Value of a, let $aa-7a-948{,}75=0$, and $a-34{,}5=0$. Then $a-34{,}5=0$) $aa-7a-948{,}75=0$ ($a+27{,}5=0$. Conſequently this ſecond Value is $a=-27{,}5$ which will form the original Equation, $aa-7a=948{,}75$ if it be ordered as the laſt was.

EXAMPLE 3.

Suppoſe $36a-aa=243$, then *per Theorem* 3. $a=18-\surd\overline{324-243}$, *viz.* half 36 ſquared is 324, &c. that is, $a=18-\surd\overline{81}$; but $\surd 81=9$, therefore $a=18-9=9$. Now this third Form is called an ambiguous Equation, becauſe it hath two Affirmative Values of the unknown Quantity (a), both which may be found without ſuch Diviſion as was uſed before.

before. For in this Case, $a = 18 + \sqrt{81}$, *viz.* $a = 18 + 9 = 27$, or, $a = 18 - 9 = 9$, as before. And both these Values of a are equally true, as to forming the given Equation; *viz.* $36a - aa = 243$. For if $a = 9$, then $aa = 81$, and $36a = 324$; but $324 - 81 = 243$, therefore $a = 9$.

Again, if $a = 27$, then will $aa = 729$, and $36a = 972$: But $972 - 729 = 243$, consequently it may be, $a = 27$. Now either of these Values of a may be found by *Division*, as those were in the other two Cases, one of them being first found by the Theorem. Thus, let $36a - aa - 243 = 0$, and $9 - a = 0$, then

$$\begin{array}{l} 9 - a) \; 36a - aa - 243 = 0 \; (a - 27 = 0 \\ \quad 9a - aa \\ \hline \quad 27a - 0 - 243 \\ \quad 27 \qquad\quad 243 \\ \hline \quad (0) \qquad\quad (0) \end{array}$$

Hence, if $a - 27 = 0$, then $a = 27$, as before.

Notwithstanding all Quadratick Equations of this third Form have two Affirmative Roots (as in this) yet but one of those Roots will give a true Answer to the Question, and that is to be chosen according to the Nature and Limits of the Question, as shall be shewed further on.

SCHOLIUM.

From the Work of the three last Examples, it may be observed; that the Sum of both the Roots will always be equal to the Co-efficient of their respective Equations, with a contrary Sign.

Thus. In *Example* 1. $aa + 32a = 4644$

$$\begin{array}{rl} \text{Here } a = & 54 \\ \text{And } a = & -86 \end{array} \Big\} \text{Add}$$

$$2a = -32$$

In *Example* 2. $aa - 7a = 948{,}75$

$$\begin{array}{rl} \text{Here } a = & 34{,}5 \\ \text{And } a = & -27{,}5 \end{array} \Big\} \text{Add}$$

$$2a = +7$$

In the last *Example*. $36a - aa = 243$

Which was changed into $aa - 36a = -243$

$$\begin{array}{rl} \text{Here } a = & 9 \\ \text{And } a = & 27 \end{array} \Big\} \text{Add}$$

$$2a = 36$$

Hence it is evident, that if either of the Roots be found, the other may be eaſily had without Diviſions.

If the Contents of this Section be well underſtood, it will be eaſy to give a Numerical Solution to any Quadratick Equation, that happens to ariſe in reſolving of Queſtions, &c. And as for giving a Geometrical Conſtruction of them, I think it not proper in this Place; becauſe I here ſuppoſe the Learner wholly ignorant of the firſt Principles of Geometry, therefore I ſhall refer that Work to the next Part.

CHAP. IX.

Of Analyſis, *or the Method of reſolving* Problems *exemplified by Variety of Numerical* Queſtions.

N. B. *HERE I adviſe the Learner to make uſe always of the ſame Letters, to repreſent the ſame* Data *in all Queſtions.*

Viz. { If a repreſent any Number / And e repreſent a leſs Number } or other Quantity,

Then let
- $a+e=s$ their Sum.
- $a-e=d$ their Difference.
- $ae=p$ their Product.
- $\frac{a}{e}=q$ their Quotient.
- $aa+ee=z$ the Sum of their Squares.
- $aa-ee=x$ the Difference of their Squares.

Any two of theſe ſix (s, d, p, q, z, x) being given, thence to find the reſt; which admits of fifteen Variations, or Queſtions.

Queſtion 1. Suppoſe s and d were given, and it were required by them to find a . e . p . q . z . and x .

Let { 1 | $a+e=s$ / 2 | $a-e=d$ } and ſuppoſe { $s=240$ / $d=192$ } Then

1 + 2 | 3 | $2a=s+d=432$

3 ÷ 2 | 4 | $a=\frac{s+d}{2}=216$, here a is found.

1 − 2 | 5 | $2e=s-d=48$.

5 ÷ 2

$5 \div 2$	6	$e = \frac{s-d}{2} = 24$, here e is found.
4×6	7	$ae = \frac{ss-dd}{4} = p = 5184$, here p is found.
$4 \div 6$	8	$\frac{a}{e} = \frac{s+d}{s-d} = q = 9$, here q is found.
$4\,\square^2$	9	$aa = \frac{ss+2sd+dd}{4} = 46656$
$6\,\square^2$	10	$ee = \frac{ss-2sd+dd}{4} = 576$
$9+10$	11	$aa+ee = \frac{ss+dd}{2} = z = 47232$, z found.
$9-10$	12	$aa-ee = sd = x = 46080$, x found.

Question 2. Let s and p be given, to find the rest.

That is	1	$a+e = s. = 240$	Quære $a.e.d.q.z.x.$
	2	$ae = p = 5184$	
$1\,\square^2$	3	$aa+2ae+ee = ss =$	57600
2×4	4	$4ae = 4p =$	20736
$3-4$	5	$aa-2ae+ee = ss-4p =$	36864
$5\,ww^2$	6	$a-e = \sqrt{ss-4p} = d =$	192
$1+6$	7	$2a = s+\sqrt{ss-4p}$	
$7 \div 2$	8	$a = \frac{s+\sqrt{ss-4p}}{2}$, hence $a = 216$	
$1-6$	9	$2e = s-\sqrt{ss-4p}$	
$9 \div 2$	10	$e = \frac{s-\sqrt{ss-4p}}{2}$, hence $e = 24$	
$8 \div 10$	11	$\frac{a}{e} = \frac{s+\sqrt{ss-4p}}{s-\sqrt{ss-4p}} = q = 9$	
$8\,\square^2$	12	$aa = \frac{ss+s\sqrt{ss-4p}}{2} :-p$	
$10\,\square^2$	13	$ee = \frac{ss-s\sqrt{ss-4p}}{2} :-p$	
$12+13$	14	$aa+ee = ss-2p = z = 47232$	
$12-13$	15	$aa-ee = s\sqrt{ss-4p} = x = 46080$	

Queſtion 3. Suppoſe s and q are given, to find the reſt.

Viz. { 1 $a+e=s.=240$; 2 $\frac{a}{e}=q=9$ } Quære $a.e.d.p.z.x.$

$2\times e$	3	$a=qe$
$1-3$	4	$e=s-qe$
$4+qe$	5	$qe+e=s$
$5\div q+1$	6	$e=\frac{s}{q+1}$, for $\overline{q+1}\times e=qe+e$
$1-6$	7	$a=s-\frac{s}{q+1}=\frac{qs}{q+1}$
6×7	8	$ae=\frac{qss}{qq+2q+1}=p$
$7-6$	9	$a-e=\frac{qs-s}{q+1}=d$
$7\,\square^2$	10	$aa=\frac{qqss}{qq+2q+1}$
$6\,\square^2$	11	$ee=\frac{ss}{qq+2q+1}$
$10+11$	12	$aa+ee=\frac{qqss+ss}{qq+2q+1}=z$
$10-11$	13	$aa-ee=\frac{qqss-ss}{qq+2q+1}=x$

Queſtion 4. Let s and z be given, to find the reſt.

Viz. { 1 $a+e=s.=240$; 2 $aa+ee=z.=47232$ } Quære $a.e.d.p.q.x.$

$1\,\square^2$	3	$aa+2ae+ee=ss$
$3-2$	4	$2ae=ss-z$
$2-4$	5	$aa-2ae+ee=2z-ss$
$5\,\omega^2$	6	$a-e=\sqrt{2z-ss}=d$

1 + 6	7	$2a = s + \sqrt{2z - ss}$
7 ÷ 2	8	$a = \frac{s + \sqrt{2z - ss}}{2}$
1 − 6	9	$2e = s - \sqrt{2z - ss}$
9 ÷ 2	10	$e = \frac{s - \sqrt{2z - ss}}{2}$

The rest are found just as in the 2d *Question*; the 8 and 10 Steps here being the very same with the 8 and 10 Steps there.

Question 5. When s and x are given; to find the rest.

Viz.	1	$a + e = s = 240$
	2	$aa - ee = x = 46080$ } Quære $a.e.d.p.q.z.$
2 ÷ 1	3	$a - e = \frac{x}{s} = d$, viz. $a + e) aa - ee (a - e$
1 + 3	4	$2a = s + \frac{x}{s} = \frac{ss + x}{s}$
4 ÷ 2	5	$a = \frac{ss + x}{2s}$
1 − 3	6	$2e = s - \frac{x}{s} = \frac{ss - x}{s}$
6 ÷ 2	7	$e = \frac{ss - x}{2s}$
5 × 7	8	$ae = \frac{ssss - xx}{4ss} = p$
5 ÷ 7	9	$\frac{a}{e} = \frac{ss + x}{ss - x} = q$
5 ◎²	10	$aa = \frac{s^4 + 2ssx + xx}{4ss}$
7 ◎²	11	$ee = \frac{s^4 - 2ssx + xx}{4ss}$
10 + 11	12	$aa + ee = \frac{s^4 + xx}{2ss} = z$

Question

Question 6. Suppose d and p are given, to find the rest.

Viz.	1	$a-e=d=192$ } Quære $a.e.s.q.z.x$.
	2	$ae=p=5184$
$1\,\text{Q}^2$	3	$aa-2ae+ee=dd$
$2\times\overline{4}$	4	$4ae=4p$
$3+4$	5	$aa+2ae+ee=dd+4p$
$5\,\text{ww}^2$	6	$a+e=\sqrt{dd+4p}=s$
$6+1$	7	$2a=d+\sqrt{dd+4p}$
$7\div\overline{2}$	8	$a=\dfrac{d+\sqrt{dd+4p}}{2}$
$6-1$	9	$2e=\sqrt{dd+4p}\sqrt{-d}$
$9\div\overline{2}$	10	$e=\dfrac{\sqrt{dd+4p}\sqrt{-d}}{2}$
$8\div10$	11	$\dfrac{a}{e}=\dfrac{d+\sqrt{dd+4p}}{\sqrt{dd+4p}\sqrt{-d}}=q$
$8\,\text{Q}^2$	12	$aa=\dfrac{dd+2p+d\sqrt{dd+4p}}{2}$
$10\,\text{Q}^2$	13	$ee=\dfrac{dd+2p-d\sqrt{dd+4p}}{2}$
$12+13$	14	$aa+ee=dd+2p=z$
$12-13$	15	$aa-ee=d\sqrt{dd+4p}=x$

Question 7. Let d and q be given, to find the rest.

Viz.	1	$a-e=d=192$ } Quære $a.e.s.p.z.x$.
	2	$\dfrac{a}{e}=q=9$
$2\times e$	3	$a=qe$
$1+e$	4	$a=d+e$
3 and 4	5	$qe=d+e$
$5-e$	6	$qe-e=d$
$6\div\overline{q-1}$	7	$e=\dfrac{d}{q-1}$, for $\overline{q-1}\times e=qe-e$
$1+7$	8	$a=d+\dfrac{d}{q-1}=\dfrac{qd}{q-1}$
$7+8$	9	$a+e=\dfrac{qd+d}{q-1}=s$

7 × 8

7×8	10	$ae = \frac{qdd}{qq-2q+1} = p$
$8 \square^2$	11	$aa = \frac{qqdd}{qq-2q+1}$
$7 \square^2$	12	$ee = \frac{dd}{qq-2q+1}$
$11 + 12$	13	$aa+ee = \frac{qqdd+dd}{qq-2q+1} = z$
$11 - 12$	14	$aa-ee = \frac{qqdd-dd}{qq-2q+1} = x$

Queſtion 8. Suppoſe d and z given, to find the reſt.

Viz.	1	$a-e=d=192$	Quære $a.e.s.p.q.x.$
	2	$aa+ee=z=47232$	
$1 \square^2$	3	$aa-2ae+ee=dd$	
$2 - 3$	4	$2ae=z-dd$	
$2 + 4$	5	$aa+2ae+ae=2z-dd$	
$5 \sqrt{\ }^2$	6	$a+e=\sqrt{2z-dd}=s$	
$1 + 6$	7	$2a=d+\sqrt{2z-dd}$	
$7 \div \overline{2}$	8	$a = \frac{d+\sqrt{2z-dd}}{2}$	
$6 - 1$	9	$2e=\sqrt{2z-dd}-d$	
$9 \div \overline{2}$	10	$e = \frac{\sqrt{2z-dd}-d}{2}$	
8×10	11	$ae = \frac{z-dd}{2} = p$	
$8 \square^2$	12	$aa = \frac{z+d\sqrt{2z-dd}}{2}$	
$10 \square^2$	13	$ee = \frac{z-d\sqrt{2z-dd}}{2}$	
$12 - 13$	14	$aa-ee=d\sqrt{2z-dd}=x$	
$8 \div 10$	15	$\frac{a}{e} = \frac{d+\sqrt{2z-dd}}{\sqrt{2z-dd}-d} = q$	

Queſtion

Queſtion 9. Let d and x be given, to find the reſt.

Viz.	1	$a-e=d=240$	Quære $a.e.s.p.q.z.$
	2	$aa-ee=x=46080$	
$2\div 1$	3	$a+e=\frac{x}{d}=s$, viz. $a-e)\ aa-ee\ (a+e$	
$1+3$	4	$2a=\frac{dd+x}{d}$	
$4\div 2$	5	$a=\frac{dd+x}{2d}$	
$3-5$	6	$e=\frac{x-dd}{2d}$	
5×6	7	$ae=\frac{xx-d^4}{4dd}=p$	
$5\div 6$	8	$\frac{a}{e}=\frac{dd+x}{x-dd}=q$	
$5\,\text{@}^2$	9	$aa=\frac{d^4+2ddx+xx}{4dd}$	
$6\,\text{@}^2$	10	$ee=\frac{xx-2ddx+d^4}{4dd}$	
$9+10$	11	$aa+ee=\frac{d^4+xx}{2dd}=z$	

Queſtion 10. Let p and q be given, to find the reſt.

Viz.	1	$ae=p=5184$	Quære $a.e.d.z.x.$
	2	$\frac{a}{e}=q=9$	
1×2	3	$aa=qp$, for $\frac{ae}{1}\times\frac{a}{e}=\frac{aae}{e}=aa$	
$3\,ww^2$	4	$a=\surd qp$	
$1\div 2$	5	$ee=\frac{p}{q}$, for $\frac{a}{1})\frac{ae}{1}(\frac{aee}{a}=ee$	
$5\,ww^2$	6	$e=\surd\frac{p}{q}$	
$4+6$	7	$a+e=\surd qp+\surd\frac{p}{q}=s$	

$4-6$	8	$a-e=\sqrt{qp}-\sqrt{\frac{p}{q}}=d$
$3+5$	9	$aa+ee=qp+\frac{p}{q}=z$
$3-5$	10	$aa-ee=pq-\frac{p}{q}=x$

Queſtion 11. Let p and z be given, to find the reſt.

Viz.	1	$ae=p=5184$
	2	$aa+ee=z=47232$ } Quære a . e . &c.
1×2	3	$2ae=2p$
$2+3$	4	$aa+2ae+ee=z+2p$
$4\ \omega^2$	5	$a+e=\sqrt{z+2p}=s$
$2-3$	6	$aa-2ae+ee=z-2p$
$6\ \omega^2$	7	$a-e=\sqrt{z-2p}=d$
$5+7$	8	$2a=\sqrt{z+2p}+\sqrt{z-2p}$
$8\div 2$	9	$a=\frac{\sqrt{z+2p}+\sqrt{z-2p}}{2}$
$5-7$	10	$2e=\sqrt{z+2p}-\sqrt{z-2p}$
$10\div 2$	11	$e=\frac{\sqrt{z+2p}-\sqrt{z-2p}}{2}$
$9\div 11$	12	$\frac{a}{e}=\frac{\sqrt{z+2p}+\sqrt{z-2p}}{\sqrt{z+2p}-\sqrt{z-2p}}=q$
$9\ \square^2$	13	$aa=\frac{z+\sqrt{zz-4pp}}{2}$
$11\ \square^2$	14	$ee=\frac{z-\sqrt{zz-4pp}}{2}$
$aa-ee$	15	$aa-ee=\sqrt{zz-4pp}=x$

Queſtion 12. Let p and x be given, to find the reſt.

Viz.	1	$ae=p=5184$
	2	$aa-ee=x=46080$ } Quære a . e . &c.
$1\ \square^2$	3	$aaee=pp$

$2 \circledcirc^2$	4	$aaaa - 2aaee + eeee = xx$
$3 \times \bar{4}$	5	$4aaee = 4pp$
$4 + 5$	6	$aaaa + 2aaee + eeee = xx + 4pp$
6 ɯ^2	7	$aa + ee = \sqrt{xx + 4pp} = z$
$2 + 7$	8	$2aa = x + \sqrt{xx + 4pp}$
$8 \div \bar{2}$	9	$aa = \frac{x + \sqrt{xx + 4pp}}{2}$
9 ɯ^2	10	$a = \sqrt{\frac{x + \sqrt{xx + 4pp}}{2}}$
$7 - 2$	11	$2ee = \sqrt{xx + 4pp} - x$
$11 \div \bar{2}$	12	$ee = \frac{\sqrt{xx + 4pp} - x}{2}$
12 ɯ^2	13	$e = \sqrt{: \frac{\sqrt{xx + 4pp} - x}{2}}$
$10 + 13$	14	$a + e = \sqrt{\frac{x + \sqrt{xx + 4pp}}{2}} + \sqrt{\frac{\sqrt{xx + 4pp} - x}{2}} = s$
$10 - 13$	15	$a - e = \sqrt{\frac{x + \sqrt{xx + 4pp}}{2}} - \sqrt{\frac{\sqrt{xx + 4pp} - x}{2}} = d$
$9 + 12$	16	$aa + ee = \sqrt{xx + 4pp} = z$

Question 13. Having q and z given, to find the rest.

Viz.	1	$\frac{a}{e} = q = 9$ } Quære a . e, &c.
	2	$aa + ee = z = 47232$
$1 \times e$	3	$a = qe$
$3 \circledcirc^2$	4	$aa = qqee$
$2 - 4$	5	$ee = z - qqee$
$4 + qqee$	6	$qqee + ee = z$
$6 \div \overline{qq + 1}$	7	$ee = \frac{z}{qq + 1}$, for $\overline{qq + 1} \times ee = qqee + ee$

2 — 7

2 — 7	8	$aa = z - \frac{z}{qq+1} = \frac{qqz}{qq+1}$
8 ω^2	9	$a = \surd \frac{qqz}{qq+1}$
7 ω^2	10	$e = \surd \frac{z}{qq+1}$
9 + 10	11	$a+e = \surd \frac{qqz}{qq+1} + \surd \frac{z}{qq+1} = s$
9 — 10	12	$a-e = \surd \frac{qqz}{qq+1} - \surd \frac{z}{qq+1} = d$
9 × 10	13	$ae = \surd \frac{qqzz}{q^4+2qq+1} = p$
8 — 7	14	$aa - ee = \frac{qqz-z}{qq+1} = x$

Queſtion 14. When q and x are given, to find the reſt.

Viz.	1	$\frac{a}{e} = q = 9$
	2	$aa - ee = x = 46080$
1 × e	3	$a = qe$
3 $\odot^2$	4	$aa = qqee$
2 + ee	5	$aa = x + ee$
4 and 5	6	$qqee = x + ee$
6 — ee	7	$qqee - ee = x$
7 ÷ $qq-1$	8	$ee = \frac{x}{qq-1}$
2 + 8	9	$aa = x + \frac{x}{qq-1} = \frac{qqx}{qq-1}$
9 ω^2	10	$a = \surd \frac{qqx}{qq-1}$
8 ω^2	11	$e = \surd \frac{x}{qq-1}$
10 + 11	12	$a+e = \surd \frac{qqx}{qq-1} + \surd \frac{x}{qq-1} = s$

Quære a, e, &c.

10 — 11	13	$a - e = \sqrt{\frac{qqx}{qq-1}} - \sqrt{\frac{x}{qq-1}} = d$
10 × 11	14	$ae = \sqrt{\frac{qqxx}{qqqq - 2qq + 1}} = p$
8 + 9	15	$aa + ee = \frac{qqx - x}{qq-1} = z$

Question 15. When z and x are given, to find the rest.

Viz.	1	$aa + ee = z = 47232$ } Quære a . e, &c.
	2	$aa - ee = x = 46080$ }
1 + 2	3	$2aa = z + x$
3 ÷ 2	4	$aa = \frac{z+x}{2}$
1 — 2	5	$2ee = z - x$
5 ÷ 2	6	$ee = \frac{z-x}{2}$
4 ww	7	$a = \sqrt{\frac{z+x}{2}}$
6 ww	8	$e = \sqrt{\frac{z-x}{2}}$
7 + 8	9	$a + e = \sqrt{\frac{z+x}{2}} + \sqrt{\frac{z-x}{2}} = s$
7 — 8	10	$a - e = \sqrt{\frac{z+x}{2}} - \sqrt{\frac{z-x}{2}} = d$
7 × 8	11	$ae = \sqrt{\frac{zz - xx}{4}} = p$
7 ÷ 8	12	$\frac{a}{e} = \frac{\sqrt{z+x}}{\sqrt{z-x}} = q$

These fifteen Questions are proposed in Dr *Pell*'s *Algebra*; but he pursues only the first Question throughout, and breaks off in the other fourteen, after the Values of what I call a and e are found. But I have proceeded in every one of them, to

to find the Values of all the unknown Quantities, becauſe they afford ſuch Variety, as being well obſerved by a Learner, will be found very uſeful in the Solution of moſt Queſtions.

Note, I have choſe to uſe the ſame Numbers for the reſpective Value of each Quantity throughout all the Queſtions, becauſe they will be more ſatisfactory in proving the Work than various Numbers would have been. Not but that any Numbers may be taken at Pleaſure, provided that the Number repreſented by a, be greater than that by e, &c. I have omitted the Numerical Calculations purely for the Learner to practiſe on.

Queſtion 16. There are two Numbers, the Sum of their Squares is 2368; and the greater of them is in Proportion to the leſs, as 6 to 1. What are theſe Numbers?

Let a = the greater Number, e = the leſſer, and $z = 2368$.

Then	1	$aa + ee = z$ }
And	2	$a : e :: 6 : 1$ } by the Queſtion.
2 ∴	3	$1\,a = 6\,e$
3 $\mathcal{Q}^2$	4	$aa = 36\,ee$
1 − 4	5	$ee = z - 36\,ee$
5 + $36\,ee$	6	$37\,ee = z$
6 ÷ 37	7	$ee = \frac{z}{37} = 64$
7 ω^2	8	$e = \sqrt{\frac{z}{37}} = 8$
8 × 6	9	$6\,e = 6\sqrt{\frac{z}{37}} = 48$
3 and 9	10	$a = 48$

Proof.

If $a = 48$
and $e = 8$
$aa = 2304$
$ee = 64$
$aa + ee = 2368$
and $48 : 8 :: 6 : 1$

Queſtion 17. There are three Numbers in continued Proportion, the Sum of the Extreams is 156, and the Mean is 72; What are the two Extreams?

That is, Suppoſe $a \,.\, m \,.\, e$ in ∴, and $m = 72$.

Then {	1	$a + e = 5156 = s$ } by the Queſtion.
	2	$a : m :: m : e$ } Quære $a \,.\, e$. &c.
2 ∴	3	$ae = mm$
1 $\mathcal{Q}^2$	4	$aa + 2ae + ee = ss$
3 × 4	5	$4ae = 4mm$

4 — 5

$4-5$	6	$aa-2ae+ee=ss-4mm$
$6\ \omega^2$	7	$a-e=\sqrt{ss-4mm}$
$1+7$	8	$2a=s+\sqrt{ss-4mm}$
$8\div 2$	9	$a=\frac{s+\sqrt{ss-4mm}}{2}=108$
$1-9$	10	$e=\frac{s-\sqrt{ss-4mm}}{2}=48$

Or $a=48$, $e=108$

Queſtion 18. There are three Numbers in $\div\div$, their Sum is 74, and the Sum of their Squares is 1924; What are thoſe Numbers?

That is, a, e, y are in $\div\div$

Then
	1	$a+e+y=s=74$
	2	$aa+ee+yy=z=1924$
	3	$a:e::e:y$

Quære a, e, y.

$5\ \because$	4	$ay=ee$
$1-e$	5	$a+y=s-e$
$2-ee$	6	$aa+yy=z-ee$
4×2	7	$2ay=2ee$
$6+7$	8	$aa+2ay+yy=z+ee$
$5\ \omega^2$	9	$aa+2ay+yy=ss-2se+ee$
8 and 9	10	$z+ee=ss-2se+ee$
$10+$	11	$2se=ss-z$
$11\div 2s$	12	$e=\frac{ss-z}{2s}=24$
5,	13	$a+y=s-e=50$
$13\ \omega^2$	14	$aa+2ay+yy=2500$
4×4	15	$4ay=4ee=2304$
$14-15$	16	$aa-2ay+yy=196$
$16\ \omega^2$	17	$a-y=\sqrt{196}=14$
$13+17$	18	$2a=50+14=64$
$18\div 2$	19	$a=32$
$13-19$	20	$y=50-32=18$

Or $a=18$, $y=32$

Note, In all Queſtions about continual Proportionals, (either Arithmetical or Geometrical) where three Terms are ſought, the Mean is eaſieſt found firſt (as above) and if all the Terms be Affirmative, then it is equal whether the firſt or laſt Term be the greateſt.

Queſtion

Queſtion 19. There are three Numbers in ÷, their Sum is 76; and if the Sum of the Extreams be multiplied into the Mean, that Product will be 1248; What are thoſe Numbers?

Viz.	1	$a : e :: e : y$
	2	$a+e+y=s=76$ } by the Queſtion.
	3	$ae+ye=p=1248$ }
1 ∴	4	$ay=ee$
1 × e	5	$ae+ee+ye=se$
5 − 3	6	$ee=se-p$
6 − se	7	$ee-se=-p$
7 C □	8	$ee-se+\frac{1}{4}ss=\frac{1}{4}ss-p$
8 ω^2	9	$e-\frac{1}{2}s=\sqrt{\frac{1}{4}ss-p}$
9 + $\frac{1}{2}s$	10	$e=\frac{1}{2}s+\sqrt{\frac{1}{4}ss-p}=$ { 52. per Theor. 3. / 24. Chap. 8.
2 − 10	11	$a+y=52$
4 × 4	12	$4ay=4ee=2304$
11 $\square^2$	13	$aa+2ay+yy=2704$
13 − 12	14	$aa-2ay+yy=400$
14 ω^2	15	$a-y=\sqrt{400}=20$
11 + 15	16	$2a=52+20=72$
16 ÷ 2	17	$a=36$ } Or $a=16$
11 − 17	18	$y=52-36=16$ } and $y=36$

N. B. If you take $e=\frac{1}{2}s+\sqrt{\frac{1}{4}ss-p}=52$ (at the 10th Step) then it will be $76-52=24=a+y$, which is impoſſible, *viz.* that the Mean ſhould be greater than the Sum of the two Extreams. Therefore it muſt be $e=\frac{1}{2}s-\sqrt{\frac{1}{4}ss-p}=24$. (See page 201.)

Queſtion 20. There are three Numbers in Arithmetical Progreſſion, the firſt being added to twice the ſecond, and three times the third, their Sum will be 62; and the Sum of all their Squares is 275; What are thoſe Numbers?

Suppoſe	1	a, e, y, in Arithmetical Progreſſion.
And	2	$a+2e+3y=62$ } by the Queſtion.
	3	$aa+ee+yy=275$ }
Then	4	$a+y=2e$, per Sect. 1. Chap. 6.
2 − 4	5	$2e+2y=62-2e$
5 ÷ 2	6	$e+y=31-e$
6 − e	7	$y=31-2e$
4 − 7	8	$a=4e-31$

$8\,\square^2$	9	$aa = 16\,ee - 248\,e + 961$
$7\,\square^2$	10	$yy = 961 - 124\,e + 4\,ee$
9 + 10	11	$aa + yy = 20\,ee - 372\,e + 1922$
3 — 11	12	$ee = 372\,e - 20\,ee - 1647$
12 + 20 ee	13	$21\,ee = 372\,e - 1647$
13 — 372 e	14	$21\,ee - 372\,e = -1647$
14 ÷ 21	15	$ee - \frac{124}{7}e = -\frac{549}{7}$
15 $C\,\square$	16	$ee - \frac{124}{7}e + \frac{3844}{49} = \frac{3844}{49} - \frac{549}{7} = \frac{1}{49}$
16 ww^2	17	$e - \frac{62}{7} = \surd\frac{1}{49} =$
17 + $\frac{62}{7}$	18	$e = \frac{62}{7} + \frac{1}{7} = 9$, or $8\frac{5}{7}$ the Mean
18 × 4	19	$4\,e = 36$, or $\frac{144}{7}$
8 and 19	20	$a = 36 - 31 = 5$, or $34\frac{4}{7} - 31 = 3\frac{5}{7}$
18 × 2	21	$2\,e = 18$, or $17\frac{3}{7}$
7 and 21	22	$y = 31 - 18 = 13$, or $31 - 17\frac{3}{7} = 13\frac{4}{7}$

Question 21. There are three Numbers in Arithmetical Progression; the Square of the first Term being added to the Product of the other two is 576; the Square of the Mean being added to the Product of the two Extreams, make 612; and the Square of the last Term being added to the Product of the first into the second, is 792: What are those Numbers?

Suppose	1	a, e, y in Arith. Progres. as before.
Then {	2	$aa + ye = 576$ }
	3	$ee + ya = 612$ } by the Question.
	4	$yy + ae = 792$ }
1 ∴	5	$a + y = 2e$, per Sect. 1. Chap. 6.
5 × e	6	$ae + ye = 2\,ee$
2 + 4	7	$aa + ye + yy + ae = 1368$
7 — 6	8	$aa + yy = 1368 - 2\,ee$
3 — ee	9	$ya = 612 - ee$
9 × 2	10	$2\,ya = 1224 - 2\,ee$
8 + 10	11	$aa + 2\,ya + yy = 2592 - 4\,ee$
5 $\square^2$	12	$aa + 2\,ya + yy = 4\,ee$
11 and 12	13	$4\,ee = 2592 - 4\,ee$
13 + 4 ee	14	$8\,ee = 2592$
14 ÷ 8	15	$ee = 324$
15 ww^2	16	$e = \surd 324 = 18$, the Mean
8,	17	$aa + yy = 1368 - 2\,ee = 720$
10,	18	$2\,ya = 1224 - 2\,ee = 576$
17 — 18,	19	$aa - 2\,ya + yy = 720 - 576 = 144$

1 ww^2	20	$a-y=\sqrt{144}=12$
$5+20$	21	$2a=2e+12=48$
$21\div 2$	22	$a=24$
$5-22$	23	$y=2e-24=12$

Or $\begin{cases} a=12 \\ y=24 \end{cases}$

Question 22. It is required to find two such Numbers, that the Sum of their Squares may be $8226\frac{1}{2}$; and their Product being added to the Square of the lesser, may be $6921\frac{1}{2}$

Viz.	1	$aa+ee=8226\frac{1}{2}$
	2	$ae+ee=6921\frac{1}{2}$ Quære a and e
1 — 2	3	$aa-ae=1305$
3 ±	4	$ae=aa-1305$
$4\div a$	5	$e=\frac{aa-1305}{a}$
5 $\mathcal{Q}^2$	6	$ee=\frac{a^4-2610aa+1703025}{aa}$
$1-aa$	7	$ee=8226,5-aa$
6 and 7	8	$\frac{a^4-2610aa+1703025}{aa}=8226,5-aa$
$8\times aa$	9	$a^4-2610aa+1703025=8226,5aa-a^4$
$9+a^4$	10	$2a^4-2610aa+1703025=8226,5aa$
$10\pm$	11	$2a^4-10836,5aa=-1703025$
$11\div 2$	12	$a^4-5418,25aa=-851512,5$ (765
12 $C\,\square$	13	$a^4-5418,25aa+7339358,26562=6487845,$
13 ww^2	14	$aa-2709,125=\sqrt{6487845,765625}-2547,1$
14 + 27 &c.	15	$aa=2709,125+2547,125$ (25
Suppose	16	$aa=2709,125+2547,125=5256,25$
Then	17	$a=\sqrt{5256,25}=72,5$
And 5,	18	$e=\frac{aa+1305}{a}=\frac{5256,25-1305}{72,5}=54,5$
Or let	19	$aa=2709,125-2547,125=162$
10 ww^2	20	$a=\sqrt{162}=12,72$ &c.
Then	21	$e=\frac{162-1305}{12,72}$, which is impossible.
Therefore		$a=72,5$
And		$e=54,5$ as at the 17th and 18th Steps.

This Question may be performed with less Trouble, by substituting Letters for the known Numbers.

Viz. $\begin{cases} aa+ee=z \\ ae+ee=p \end{cases}$ Then let $z-p=d=aa-ae$, &c.

Queſtion 23. It is required to find three ſuch Numbers, that the Sum of the firſt and ſecond, being multiplied with the third, may be 37824; and the Sum of the ſecond and third, multiplied with the firſt, may be 59944; alſo, that the Sum of the firſt and third, being multiplied with the ſecond, may be 52456.

Let a, e, y, repreſent the three Numbers.

Then { 1 | $ay+ey=37824=b$; 2 | $ea+ya=59944=c$; 3 | $ae+ye=52456=d$ } Quære a, e, y.

1+2+3 | 4 | $2ae+2ay+2ye=b+c+d$

Let | 5 | $z=b+c+d$

4÷2 | 6 | $ae+ay+ye=\frac{1}{2}z=\frac{b+c+d}{2}$

6−3 | 7 | $ay=\frac{1}{2}z-d=\frac{z-2d}{2}$

7÷a | 8 | $y=\frac{z-2d}{2a}$

6−2 | 9 | $ye=\frac{1}{2}z-c=\frac{z-2c}{2}$

6−1 | 10 | $ae=\frac{1}{2}z-b=\frac{z-2b}{2}$

10÷a | 11 | $e=\frac{z-2b}{2a}$

8 × 11 | 12 | $ye=\frac{z-2d}{2a}\times\frac{z-2b}{2a}=\frac{zz-2dz-2bz+4bd}{4aa}$

9 and 12 | 13 | $\frac{z-2c}{2}=\frac{zz-2dz-2bz+4bd}{4aa}$

13 × 4aa | 14 | $2zaa-4caa=zz-2dz-2bz+4bd$

14 ÷ | 15 | $aa=\frac{zz-2dz-2bz+4bd}{2z-4c}=55696$

15 ω^2 | 16 | $a=\surd 55696=236$

11 | 17 | $e=\frac{z-2b}{2a}=158$

8 | 18 | $y=\frac{z-2d}{2a}=96$

Queſtion 24. It is required to find two ſuch Numbers, that their Sum being ſubſtracted from the Sum of their Squares, may leave 14, and if their Product be added to their Sum, it may make 14.

Let a and e be put for the Numbers, and let $y=a+e$

Then { 1 | $aa+ee-y=14$; 2 | $ae+y=14$ } by the Queſtion.

1+7

1 + 7	3	$aa+ee=14+y$
2 − y	4	$ae=14-y$
4 × 2	5	$2ae=28-2y$
3 + 5	6	$aa+2ae+ee=42-y$
6 $\omega\omega^2$	7	$a+e=\sqrt{42-y}$
But	8	$a+e=y$, by Subſtitution above.
7 and 8	9	$y=\sqrt{42-y}$
9 $\odot^2$	10	$yy=42-y$
10 + y	11	$yy+y=42$
11 $C\square$	12	$yy+y+\frac{1}{4}=42+\frac{1}{4}=42,25$
12 $\omega\omega^2$	13	$y+\frac{1}{2}=\sqrt{42,25}=6,5$
13 − $\frac{1}{2}$	14	$y=6,5-\frac{1}{2}=6$
Conſequent	15	$a+e=6$, by Reſtitution from above.
3 and 14	16	$aa+ee=14+6=20$
5 and 15	17	$2ae=28-12=16$
16 − 17	18	$aa-2ae+ee=4$
18 $\omega\omega^2$	19	$a-e=\sqrt{4}=2$
15 + 19	20	$2a=8$
23 ÷ 2	21	$a=4$
1 − 21	22	$e=6-4=2$

Proof: If $a=4$, and $e=2$; Then $aa+ee-a-e=14$; And $ae+a+e=14$; According to the Queſtion.

Queſtion 25. Three Men diſcourſing of their Money, ſaith the firſt, if 100 *l.* were added to my Money, it would be as much as both your Money put together; ſaith the ſecond Man, if 100 *l.* were added to my Money, I ſhould have twice as much as both you have; ſaith the third Man, if 100 *l.* were added to my Money, I ſhould have then three times as much Money as both you have: How much Money had each Man?

Let a repreſent the firſt Man's Money, e the ſecond, and y the third.

Then	1	$a+100=e+y$
	2	$e+100=2a+2y$
	3	$y+100=3a+3e$
1 − a	4	$e+y-a=100=s$
2 − e	5	$2a+2y-e=100=s$
3 − y	6	$3a+3e-y=100=s$
4 and 6	7	$e+y-a=3a+3e-y$
7 +	8	$2y=4a+2e$
5 − s	9	$2a-e=s-4a-2e$
9 + 4a − 2e	10	$6a+e=s=100$
4 + 6	11	$2a+4e=2s=200$

Quære a, e, y.

10×4	12	$24a+4e=4s=400$
$12-11$	13	$22a=2s=200$
$13 \div 22$	14	$a=\frac{s}{11}=\frac{100}{11}=9\frac{1}{11}\,l.$
$10-6a$	15	$e=s-6a=100-\frac{600}{11}=\frac{500}{11}=45\frac{5}{11}\,l.$
$8 \div 2$	16	$y=2a+e=\frac{200}{11}+\frac{500}{11}=\frac{700}{11}=63\frac{7}{11}\,l.$

Answer. The { first, second, third } Man had { $9\,l.\ 1\,s.\ 9\frac{9}{11}\,d.$; $45\,l.\ 9\,s.\ 1\frac{2}{11}\,d.$; $64\,l.\ 12\,s.\ 8\frac{1}{11}\,d.$ }

Question 26. Three Men have each such a Sum of Money, that if the first and second Mens Money be added to Half of what the third Man hath; that Sum will be 92 *l.* And if the second and third Mens Money be added to one third Part of the first Man's Money, that Sum will be 92 *l.* Lastly, if one fourth Part of the second Man's Money be added to the first and third Man's Money, that Sum will also be 92 *l.* How much was each Man's Money?

Put a for the 1*st* Man's Money, e for the 2*d*, and y for the 3*d*.

Then {	1	$a+e+\frac{1}{2}y=s$
	2	$\frac{1}{3}a+e+y=s$ } by the Question; and $s=92$
	3	$\frac{1}{4}e+a+y=s$
1 and 2	4	$a+e+\frac{1}{2}y=\frac{1}{3}a+e+y$
$4-e$	5	$a+\frac{1}{2}y=\frac{1}{3}a+y$
$5 \times 2 \times 3$	6	$6a+3y=2a+6y$
$6+$	7	$4a=3y$
2×3	8	$a+3e+3y=3s$
$8-7$	9	$a+3e=3s-4a$
$9-a$	10	$3e=3s-5a$
$10 \div 3$	11	$e=\frac{3s-5a}{3}$
3×4	12	$e+4a+4y=4s=368$
$12-2$	13	$3\frac{2}{3}a+3y=3s=276$
13 and 7	14	$3\frac{2}{3}a+4a=3s=276$
14×3	15	$11a+12a=9s=828$
$15 \div 23$	16	$a=\frac{9s}{23}=\frac{828}{23}=36\,l.$ the 1*st* Man's Money.
11	17	$e=\frac{3s-5a}{3}=\frac{96}{3}=32\,l.$ the 2*d* Man's Money.
$7 \div 3$	18	$y=\frac{4a}{3}=\frac{144}{3}=48\,l.$ the 3*d* Man's Money.

Question

Queſtion 27. Four Men walking abroad, found a Purſe of Shillings only, out of which every one took a Number at an Adventure; afterwards by comparing their Numbers together they found, that if the firſt took 25 Shillings from the ſecond, it would make his Number equal with what the ſecond had then left; if the ſecond took 30 Shillings from the third, his Money would then be triple to what the third had left, and if the third took 40 Shillings from the fourth, his Money would then be double to what the fourth had left; laſtly, the fourth taking 50 Shillings from the firſt, he would then have three times as much as the firſt had left, and 5 Shillings more: It is required to tell how many Shillings each Man had.

Put a for the firſt Sum, e the ſecond, y the third, and u the fourth.

Then	1	$a+25=e-25$
	2	$e+30=3y-90$
	3	$y+40=2u-80$
	4	$u+50=3a-145$
		by the Queſtion.
1 + 25	5	$a+50=e$
2 — 30	6	$3y-120=e$
5 and 6	7	$a+50=3y-120$
7 + 120	8	$a+170=3y$
8 ÷ 3	9	$y=\frac{a+170}{3}$
3 — 40	10	$y=2u-120$
9 and 10	11	$2u-120=\frac{a+170}{3}$
1 + 120	12	$2u=\frac{a+170}{3}+120=\frac{a+530}{3}$
12 ÷ 2	13	$u=\frac{a+530}{6}$
4 — 50	14	$u=3a-195$
13 and 14	15	$3a-195=\frac{a+530}{6}$
15 × 6	16	$18a-1170=a+530$
16 + 17	17	$17a=1700$
17 ÷ 17	18	$a=100$ the 1*ſt*
by the 5	19	$e=150$ 2*d*
by the 9	20	$y=90$ 3*d*
by the 14	21	$u=105$ 4*th*
		Man's Number of Shillings.

Queſtion

10×4	12	$24a+4e=4s=400$
$12-11$	13	$22a=2s=200$
$13 \div 22$	14	$a=\frac{s}{11}=\frac{100}{11}=9\frac{1}{11}\,l.$
$10-6a$	15	$e=s-6a=100-\frac{600}{11}=\frac{500}{11}=45\frac{5}{11}\,l.$
$8 \div 2$	16	$y=2a+e=\frac{200}{11}+\frac{500}{11}=\frac{700}{11}=63\frac{7}{11}\,l.$

Answer. The { first, second, third } Man had { $9\,l.\ 1\,s.\ 9\frac{9}{11}\,d.$; $45\,l.\ 9\,s.\ 1\frac{2}{11}\,d.$; $64\,l.\ 12\,s.\ 8\frac{1}{11}\,d.$ }

Question 26. Three Men have each such a Sum of Money, that if the first and second Mens Money be added to Half of what the third Man hath; that Sum will be 92 *l.* And if the second and third Mens Money be added to one third Part of the first Man's Money, that Sum will be 92 *l.* Lastly, if one fourth Part of the second Man's Money be added to the first and third Man's Money, that Sum will also be 92 *l.* How much was each Man's Money?

Put a for the 1*st* Man's Money, e for the 2*d*, and y for the 3*d*.

Then {	1	$a+e+\frac{1}{2}y=s$
	2	$\frac{1}{3}a+e+y=s$ } by the Question; and $s=92$
	3	$\frac{1}{4}e+a+y=s$
1 and 2	4	$a+e+\frac{1}{2}y=\frac{1}{3}a+e+y$
$4-e$	5	$a+\frac{1}{2}y=\frac{1}{3}a+y$
$5 \times 2 \times 3$	6	$6a+3y=2a+6y$
$6+$	7	$4a=3y$
2×3	8	$a+3e+3y=3s$
$8-7$	9	$a+3e=3s-4a$
$9-a$	10	$3e=3s-5a$
$10 \div 3$	11	$e=\frac{3s-5a}{3}$
3×4	12	$e+4a+4y=4s=368$
$12-2$	13	$3\frac{2}{3}a+3y=3s=276$
13 and 7	14	$3\frac{2}{3}a+4a=3s=276$
14×3	15	$11a+12a=9s=828$
$15 \div 23$	16	$a=\frac{9s}{23}=\frac{828}{23}=36\,l.$ the 1*st* Man's Money.
11	17	$e=\frac{3s-5a}{3}=\frac{264}{5}=32\,l.$ the 2*d* Man's Money.
$7 \div 3$	18	$y=\frac{4a}{3}=\frac{144}{5}=48\,l.$ the 3*d* Man's Money.

Question

Queſtion 27. Four Men walking abroad, found a Purſe of Shillings only, out of which every one took a Number at an Adventure; afterwards by comparing their Numbers together they found, that if the firſt took 25 Shillings from the ſecond, it would make his Number equal with what the ſecond had then left; if the ſecond took 30 Shillings from the third, his Money would then be triple to what the third had left, and if the third took 40 Shillings from the fourth, his Money would then be double to what the fourth had left; laſtly, the fourth taking 50 Shillings from the firſt, he would then have three times as much as the firſt had left, and 5 Shillings more: It is required to tell how many Shillings each Man had.

Put a for the firſt Sum, e the ſecond, y the third, and u the fourth.

Then	1	$a+25=e-25$
	2	$e+30=3y-90$
	3	$y+40=2u-80$
	4	$u+50=3a-145$
		by the Queſtion.
1 + 25	5	$a+50=e$
2 − 30	6	$3y-120=e$
5 and 6	7	$a+50=3y-120$
7 + 120	8	$a+170=3y$
8 ÷ 3	9	$y=\frac{a+170}{3}$
3 − 40	10	$y=2u-120$
9 and 10	11	$2u-120=\frac{a+170}{3}$
1 + 120	12	$2u=\frac{a+170}{3}+120=\frac{a+530}{3}$
12 ÷ 2	13	$u=\frac{a+530}{6}$
4 − 50	14	$u=3a-195$
13 and 14	15	$3a-195=\frac{a+530}{6}$
15 × 6	16	$18a-1170=a+530$
16 +	17	$17a=1700$
17 ÷ 17	18	$a=100$ the 1*ſt*
by the 5	19	$e=150$ 2*d*
by the 9	20	$y=90$ 3*d*
by the 14	21	$u=105$ 4*th* Man's Number of Shillings.

Queſtion

Queſtion 28. Four Men have each a Sum of Money, which being put all together makes 250 Pounds; and if to the firſt Man's Money be added 8 Pounds, it will be juſt as much as the ſecond Man's Money decreaſed by 8 Pounds, and as much as 8 times the third Man's Money, and but as much as one eighth Part of the fourth Man's Money; how much had each Man?

Let a, e, y, u repreſent the four Mens Money.

Then	1	$a+e+y+u=s$
	2	$a+b=e-b$
	3	$yb=\frac{u}{b}=a+b$

by the Queſtion. Let $s=250$ and $b=8$, or any other Number at Pleaſure.

$2+b$	4	$a+2b=e$
$3\div b$	5	$y=\frac{a+b}{b}$, becauſe $yb=a+b$
$3\times b$	6	$u=ba+bb$, for $\frac{u}{b}=a+b$
$4+5+6$	7	$e+y+u=a+2b+\frac{a+b}{b}+ba+bb$
$1-a$	8	$e+y+u=s-a$
7 and 8	9	$a+2b+\frac{a+b}{b}+ba+bb=s-a$
$9\times b$	10	$ba+2bb+a+b+bba+bbb=bs-ba$
$10+$	11	$2ba+bba+a=bs-bbb-2bb-b$
$11\div$	12	$a=\frac{bs-bbb-2bb-b}{bb+2b+1}=16,691358$ &c.
by the 4,	13	$e=a+2b=32,691358$ &c.
by the 5,	14	$y=\frac{a+b}{b}=3,086419$ &c.
by the 6,	15	$u=ba+bb=197,530864$ &c.

That is,

	l.	*s.*	*d.*
$a=$	16 .	13 .	$9,92592$
$e=$	32 .	13 .	$9,92592$
$y=$	3 .	1 .	$8,74056$
$u=$	197 .	10 .	$7,40736$

Conſequently $a+e+y+u=249$. 19 . $11,99976$ which ſhould be juſt 250*l.* the Sum propoſed in the Queſtion. Now what it wants of that Sum, proceeds from the Imperfection of the Decimal Parts being not continued on to more Places, which would have brought it nearer the Truth, though not perhaps exactly ſo. *Sect.* 5. *Chap.* 5. *Part* 1.

Queſtion

Question 29. Several Merchants enter into Partnership, every one put into the Stock 65 times as many Pounds as they were Partners; with that Stock they traded and gained as many Pounds per 100*l.* as they were Partners. Now if 10*l.* 10*s.* be added to, and subtracted from, their Gain, the Product of that Sum and Difference will be 6491*l.* 6*s.* 3*d.*

Quære, How many Merchants there were, &c.

Let	1	a = the Number of Merchants.
1 × 65	2	$65a$ = every one's Sum put into Stock.
2 × a	3	$65aa$ = the whole Stock.
And	4	$100 : a :: 65aa : \frac{65aaa}{100}$, by the Question.
Viz.	5	$\frac{65aaa}{100}$ = the whole Gain.
5 + 10,5	6	$\frac{65aaa}{100} + 10,5$
5 − 10,5	7	$\frac{65aaa}{100} - 10,5$
6 × 7	8	$\frac{4225aaaaaa}{10000} - 110,25 = 6491,3125$, by Quest.
8 × 10000	9	$4225a^6 - 1102500 = 64913125$
9 +	10	$4225a^6 = 66015625$
10 ÷ 4225	11	$a^6 = \frac{66015625}{4225} = 15625$
11 ω^6	12	$a = \sqrt{15625} = 5$ the Number of Merchants.
12 × 65	13	$65a = 325$ the Number of Pounds each put in.

Question 30. Three Merchants join Stocks together; the first Man's Stock was less than the second Man's by 13*l.* the second and third Man's Stock was 175*l.* in trading they gain 48*l.* more than their whole Stock was; the first Man's proportional Part of the Gain was 78. What was each Man's Stock and Part of the Gain.

Let a, e, y represent each Man's Stock.

Then	1	$a + e + y = s$ the whole Stock.
	2	$s + 48 =$ the whole Gain.
And	3	$a + 13 = e$ } by the Question.
	4	$e + y = 175$ }
4 + a	5	$a + e + y = 175 + a$
1 and 5	6	$s = 175 + a$

6 and 2

6 and 2	7	$s + 48 = 223 + a$
But	8	$175 + a : 223 + a :: a : 78$ per Queſtion.
8 $\therefore$	9	$a\,a + 223\,a = 78\,a + 13650$
$9 - 78\,a$	10	$a\,a + 145\,a = 13650$
10 $C\,\square$	11	$a\,a + 145\,a + 5256{,}25 = 18906{,}25$
11 $\underline{ww^2}$	12	$a + 72{,}5 = \checkmark\ 18906{,}25 = 137{,}5$
$12 - 72{,}5$	13	$a = 137{,}5 - 72{,}5 = 65$
3,	14	$e = a + 13 = 78$
$4 - 14$	15	$y = 97$
Then	16	$65 : 78 :: 78 : 93l.\ 12s. = e$'s Gain.
Again	17	$65 : 78 :: 97 : 116l.\ 8s. = y$'s Gain.
Proof	18	$116l.\ 8s. + 93l.\ 12s. + 78l. = 288l.$ the Gain.
	19	$65 + 78 + 97 = 240.$ the whole Stock.
$18 - 19$	20	$288 - 240 = 48$ the Gain more than the Stock.

Queſtion 31. A Father at his Death left his three Sons his Money in this manner; to the eldeſt he gave half of it, wanting 44 Pounds; to the ſecond he gave one third of it, and 14 Pounds more; to the youngeſt he gave the Remainder, which was leſs than the Share of the ſecond Son, by 82 Pounds: What was each Son's Share.

Let a, e, y be the three Shares, and $z =$ the whole Sum.

Then	1	$a + e + y = z$
	2	$a = \frac{1}{2}z - 44$
	3	$e = \frac{1}{3}z + 14$
	4	$y = \frac{1}{3}z + 14 - 82$ } by the Queſtion.
2+3+4	5	$a + e + y = \frac{2z}{3} + \frac{z}{2} - 98$
1 and 5	6	$z = \frac{2z}{3} + \frac{z}{2} - 98$
$6 \times \bar{3}$	7	$3z = 2z + \frac{3z}{2} - 294$
$7 \times \bar{2}$	8	$6z = 4z + 3z - 588$
$8 \pm$	9	$z = 588$, the whole Sum that was left.
2 and 9	10	$a = \frac{588}{2} - 44 = 250$, the eldeſt Son's Share.
3 and 9	11	$e = \frac{588}{3} + 14 = 210$, the ſecond Son's Share.
4 and 9	12	$y = \frac{588}{3} + 14 - 82 = 128$, the youngeſt &c.

Queſtion 32. A Man playing at Hazard or Dice, won the firſt Throw juſt ſo much Money as he had in his Pocket; the ſecond

ſecond Throw he won the Square Root of what he then had, and five Shillings more; the third Throw he won the Square of all he then had; after which his whole Sum was 112*l.* 16*s.* What Money had he when he began to play?

Suppoſe	1	a = his firſt Sum. Then
$1 \times \overline{2}$	2	$2a$ = his Sum after the firſt Throw.
And	3	$5 + \surd 2a$ = the Winnings at the 2*d* Throw.
$2 + 3$	4	$2a + 5 + \surd 2a$ = the Sum after the 2*d* Throw.
4Q^2	5	$4aa + 22a + 25 + 4a\surd 2a : + 10\surd 2a$ = the Winnings at the 3*d* Throw: and therefore
$4 + 5$	6	$4aa + 24a + 30 + 4a\surd 2a + 11\surd 2a = 2256$ Shil.

But to avoid theſe Surd Quantities, let us, inſtead of ſuppoſing a = the firſt Sum, make a ſecond Trial, *viz.*

Let	1	$2aa$ = the firſt Sum.
$1 \times \overline{2}$	2	$4aa$ = the Sum after the firſt Throw.
Then	3	$2a + 5$ = the Sum won at the 2*d* Throw.
$2 + 3$	4	$4aa + 2a + 5$ = his Sum after the 2*d* Throw.
4Q^2	5	$16a^4 + 16a^3 + 44aa + 20a + 25$ = the Winnings at the 3*d* Throw; and therefore
$4 + 5$	6	$16a^4 + 16a^3 + 48aa + 22a + 30 = 2256$ Shil.

Yet again, to avoid theſe high Equations, let us make a third Suppoſition; thus,

Let	1	$\frac{aa}{2}$ = the firſt Sum.
$1 \times \overline{2}$	2	aa = the Sum after the firſt Throw.
Then	3	$a + 5$ = the Winnings at the 2*d* Throw.
$2 + 3$	4	$aa + a + 5$ = the Sum after the 2*d* Throw.
Subſt	5	$e = aa + a + 5$.
5Q^2	6	ee = the Winnings at the 3*d* Throw. Then
$5 + 6$	7	$ee + e = 2256$ Shillings by the Queſtion.
$7\ C\square$	8	$ee + e + 0{,}25 = 2256{,}25$
$8\ \omega^2$	9	$e + 0{,}5 = \surd 2256{,}25 = 47{,}5$
$9 - 0{,}5$	10	$e = 47$
5 and 10	11	$aa + a + 5 = 47$
$11 - \overline{5}$	12	$aa + a = 42$
12, $C\square$	13	$aa + a + 0{,}25 = 42{,}25$
$1\ \omega^2$	14	$a + 0{,}5 = \surd 42{,}25 = 6{,}5$
$14 - 0{,}5$	15	$a = 6$
$15\ \text{Q}^2$	16	$aa = 36$
$16 \div \overline{2}$	17	$\frac{aa}{2} = \frac{36}{2} = 18$ { The Shillings he had in his Pocket when he began to play.

Note, In reſolving of the laſt Queſtion, I have made three different Suppoſitions for the Thing ſought, purely as an Inſtance, to ſhew the young Learner how well he ought to conſider the Nature of the Queſtion, when he firſt ſtates it, and make choice of repreſenting the Thing ſought, ſo as to avoid running it into Surds, if poſſible, *viz.* as in the firſt Suppoſition of $a =$ the firſt Sum, *&c.* Not but that ſuch Equations may be ſolved, as ſhall be ſhew'd in the next Chapter. However, it is moſt like an Artiſt to perform Things of this Nature the neareſt and eaſieſt way they can be done.

Queſtion 33. *Suppoſe there were two equal Circles, whoſe Peripheries* (viz. *Circumferences*) *are divided into* 44310 *equal Parts; and that thoſe Circles were ſo placed upon one Axis, as to move the contrary way to each other; and ſuppoſe one of them to move but one of theſe equal Parts the firſt Day, two Parts the ſecond Day, three Parts the third Day, and ſo on in Arithmetical Progreſſion,* viz. 1, 2, 3, 4, 5, &c. *and the other to move every Day the Cube of thoſe Parts,* 1, 8, 27, 64, 125, &c. *of the ſame Parts; How many Parts, and how many Days muſt each Circle move, before the ſame two Points meet that were together when they began to move?*

In order to give a ready Solution to this Queſtion (or any other in this Kind) it will be convenient to premiſe this *Lemma.*

LEMMA.

The Sum of any Series of Cubes whoſe Roots are in Arithmetical Progreſſion (the firſt Term and common Difference being Unity or 1) is equal to the Square of the Sum of all thoſe Roots. As in theſe

Terms in Arith. *&c.*	their Cubes.
1	1
2	8
3	27
4	64
5	125
6	216 &c.

$21 \times 21 = 441$ Sum of their Cubes.

Let	1	$a =$ the Sum of all the Parts the 1*ſt* Circle moves.
Then	2	$aa =$ the Sum of all the Parts the 2*d* moves.
Conſequen.	3	$aa + a = 44310$ by the Queſt. (*per Lem.*
$2C\square$	4	$aa + a + 0{,}25 = 44310{,}25$

$4w^2$

$4\,\text{ш}^2$	5	$a+0,5=\sqrt{44310,25}=210,5$
$5-\overline{0,5}$	6	$a=210$ { the Number of Parts the first Circle must move.
$6\,\text{ⓞ}^2$	7	$aa=44100$ { the Number of Parts the second Circle moves.

Next to find the Number of Days they moved; there is given the first Term $=1$, the common Difference $=1$, and the Sum of all the Terms $=210$, thence to find the last Term, which in this Case is the same with the Number of all the Terms.

Let $a=1$ the first Term, $e=1$ the common Difference, and $s=210$ the Sum of all the Terms, to find $y=$ the last Term; as per Sect. 1. Chap. 6. Then $yy+ey=2s+aa-ae$ by the 16 Step, Page 186. that is, $yy+y=210\times2=420$ &c. Hence $y=20$ the Number of Days required.

I shall now proceed to give an Example or two of the Method used in arguing about unlimited Questions; *viz.* such Questions which admit of various Answers, such as those in *Alligation Alternate* promised in Page 117.

In order to shorten that Work, it will be convenient for the Learner to know the two Signs of Comparison, $>$ and $<$. The Sign $>$ is of **Greater than**; as $b>a$ signifies that b is greater than a. The Sign $<$ is of **Lesser than**; as $b<d$ signifies that b is lesser than d, &c.

EXAMPLE 1.

Question 34. *A Tobacconist had three Sorts of Tobacco,* viz. *one of* 2 s. 8 d. *the Pound, another of* 20 d. *the Pound, and a third Sort of* 16 d. *the Pound; of these he would make a Mixture to contain* 56 *Pound, that may be sold for* 22 d. *the Pound: How much of each Sort may he take?*

Let $a=$ the Quantity of that worth 32 Pence the Pound, $e=$ that of 20 Pence the Pound, and $y=$ that of 16 Pence the Pound;

Then $a+e+y=56$
And $32a+20e+16y=1232$ } { *viz.* each Quantity multiplied into it's own Price, equals their Sum multiplied into the mean Price.

This Queſtion being thus ſtated, it appears by Rule 1, Page 176, that it is capable of innumerable Anſwers; becauſe for any one of theſe three Letters, a, e, y, there may be taken any Number at Pleaſure, provided it be leſs than 56. But although that may be truly done, yet there are ſeveral Ways of arguing about theſe Sorts of Queſtions, which will limit or bound them to all their proper or poſſible Anſwers in whole Numbers. Thus,

Let	1	$a+e+y=56$
And	2	$32a+20e+16y=1232$ } as above.
$1-a$	3	$e+y=56-a$
$2-32a$	4	$20e+16y=1232-32a$
3×16	5	$16e+16y=896-16a$
$4-5$	6	$4e=336-16a$
$6\div 4$	7	$e=84-4a$; hence $a<21$
$3-7$	8	$5=3a-28$; hence $a>\frac{28}{3}$ or $9\frac{1}{3}$

From the two laſt Steps it appears, that the Quantity ſignified by a, ought to be leſs than 21, and greater than $9\frac{1}{3}$; that is, any Number betwixt $9\frac{1}{3}$ and 21, may be taken for the Value of a: Conſequently there may be eleven Anſwers to this Queſtion in whole Numbers.

Suppoſe $a=10$, then $e=84-40=44$, per 7th Step; and $y=30-28=2$, per 8th Step. Again, if $a=11$, then $e=84-44=40$, per 7th Step, and $y=33-28=5$, per 8th Step: and ſo on for the reſt, which will be as in the following Table.

a	e	y	a	e	y	a	e	y
10	44	2	14	28	14	18	12	26
11	40	5	15	24	17	19	8	29
12	36	8	16	20	20	20	4	32
13	32	11	17	16	23			

Thus it will be eaſy to find out and collect all the limited Anſwers to any Queſtion (of this Kind) wherein there are only three Quantities propoſed to be mixed: but when there are more than three, then the Work requires a little more Trouble; becauſe the ſingle Limits of all the Quantities above two muſt be found; that is, if there are four Quantities concerned in the Queſtion, the Limits of two of them muſt be found; if five Quantities are concerned, then the Limits of three of them muſt be found, &c. As in the following Queſtion.

Queſtion

Queſtion 35. *Suppoſe it were required to mix four Sorts of Wines together*; viz. *one Sort worth* 7 s. 4 d. *the Gallon, another Sort worth* 4 s. 7 d. *the Gallon, a third Sort worth* 3 s. 8 d. *the Gallon, and a fourth Sort worth* 2 s. 9 d. *the Gallon: How much of each Sort may be taken to make a Mixture of* 63 *Gallons, ſo as that the whole Quantity may be ſold for* 5 s. 6 d. *the Gallon, without Loſs*, &c.

Firſt, let all theſe ſeveral Rates, and the mean Rate, be reduced to one Denomination, *viz.* into Pence.

Viz. $\left\{\begin{matrix} 7s.\ 4d = 88d. & 4s.\ 7d. = 55d. \\ 3s.\ 8d. = 44d. & 2s.\ 9d. = 33d. \end{matrix}\right\}$ and $5s.\ 6d. = 66$.

Put a = the Quantity of that worth $88d.$ the Gallon; e = that of $55d.$ the Gallon, y = that of $44d.$ the Gallon, and u = that of $33d.$ the Gallon.

Then	1	$a+e+y+u=63$ by the Queſtion.
And	2	$88a+55e+44y+33u=4158=63\times 66$
$1-a$	3	$e+y+u=63-a$
$2-88a$	4	$55e+44y+33u=4158-88a$
3×33	5	$33e+33y+33u=2079-33a$
$4-5$	6	$22e+11y=2079-55a$
$6\div 11$	7	$2e+y=189-5a$; hence $a < \frac{189}{5}$ or $37\frac{4}{5}$
3×55	8	$55e+55y+55u=3465-55a$
$8-4$	9	$11y+22u=33a-693$
$9\div 11$	10	$y+2u=3a-63$; hence $a > \frac{63}{3}$ or 21

From the 7th and 10th Steps it appears, that the Quantity of that Sort of Wine denoted by a, muſt be leſs than $37\frac{4}{5}$ Gallons, and greater than 21 Gallons: that is, it may be a = any Number of Gallons betwixt 21 and $37\frac{4}{5}$. Whence it follows, that there may be collected 16 Anſwers to this Queſtion from the Limits of a only.

Next to find the Limits of e, y, and u.

Suppoſe	11	$a=22$, then will $5a=110$, and $3a=66$
But	12	$2e+y=189-5a=79$, per 7th Step.
$12-2e$	13	$y=79-2e$; hence $e < \frac{79}{2}$ or $39\frac{1}{2}$
Again	14	$s+y+u=63-a=41$, per 3d Step.
$14-e$	15	$y+u=41-e$
$15-13$	16	$u=e-38$; hence $e > 38$

From the 13th and 16th Steps it appears, that if $a=22$, then $e=39$, $y=79-2e=1$, and $u=e-38=1$.

Again,

Again.

Suppoſe	17	$a=23$, then $5a=115$, and $3a=69$
But	18	$2e+y=189-5a=74$, per 7th Step.
$18-2e$	19	$y=74-2e$; hence $e<\frac{74}{2}=37$
Again	20	$e+y+u=63-a=40$, per 3d Step.
$20-e$	21	$y+u=40-e$
$21-19$	22	$u=e-34$, hence $e>34$

From the 19th and 22d Steps it appears, that if $a=13$, then e may be either 35 or 36.

Once more for a further Illuſtration.

Let	23	$a=24$, then $5a=120$, and $3a=72$
But	24	$2e+y=189-5a=69$, per 7th Step.
$24-2e$	25	$y=69-2e$; hence $e<\frac{69}{2}$ or $34\frac{1}{2}$
Again	26	$e+y+u=63-a=39$, per 3d Step.
$26-e$	27	$y+u=39-e$
$27-25$	28	$u=e-30$, hence $e>30$.

From hence it appears, that if $a=24$, then e may be either 31, 32, 33, or 34, *viz.* it may be any Number betwixt 30 and $34\frac{1}{2}$ by the 25th and 28th Steps; from whence the Values of y and u may be eaſily found.

That is, if
$e=31$. then $y=7$. And $u=1$
$e=32$. $y=5$. $u=2$
$e=33$. $y=3$. $u=3$
$e=34$. $y=1$. $u=4$

Proceeding on in this manner with all the other ſingle Values of a, there may be found above 120 Anſwers to this Queſtion in whole Numbers: and if you pleaſe to put $a=$ Fractions, there may be found an innumerable Set of Anſwers; whereas the Rule of *Alligation* in *Vulgar Arithmetick* affords but only one Anſwer in Fractions, to wit, that of $a=31\frac{1}{2}$, $e=10\frac{1}{2}$, $y=10\frac{1}{2}$, $u=10\frac{1}{2}$; as may be eaſily tried per Rule, *Page* 115, *&c.*

Theſe two Examples being well underſtood (eſpecially if the laſt be thoroughly purſued) may ſuffice to ſhew the Method of limiting the Anſwers to all Sorts of Queſtions of this Kind. I ſhall therefore conclude this Chapter of Queſtions with giving a Solution to the Enigma (or Riddle) propoſed (but not anſwered) by Mr *John Kerſey*, in the Cloſe of the *Appendix* to his *Arithmetick*, which

which affords ſeveral pretty Queſtions, the Solution whereof will diſcover a certain Sentence conſiſting of three Words, which muſt be found by the help of Figures placed (or ſuppoſed to be placed) over the twenty-four Letters of the Alphabet.

Thus { 1 . 2 . 3 . 4 . 5 . 6 . 7 . &c. called Indices.
{ *a* . *b* . *c* . *d* . *e* . *f* . *g* . &c. to the laſt Letter.

So that if the Index to that Letter be once found, the Letter to which it belongs is conſequently known.

The Enigma.

1. If the Difference between the Indices of the ſecond Letter of the ſecond Word, and the third Letter of the firſt Word, be multiplied into the Difference of their Squares, the Product will be 576; and if their Sum be multiplied into the Sum of their Squares, that Product will be 2336, the Index of the ſaid third Letter being the greateſt.

Let	1	a = the greater Index, or that of the 3d Letter.
And	2	e = the leſſer, or that of the 2d Letter.
Then	3	$\overline{a-e} \times \overline{aa-ee} = 576$ } by the Queſtion.
	4	$\overline{a+e} \times \overline{aa+ee} = 2336$ }
3 ×	5	$aaa-aae-aee+eee=576$
4 ×	6	$aaa+aae+aee+eee=2336$
6 − 5	7	$2aae+2ee=1760$
6 + 7	8	$aaa+3aae+3aee+eee=4096$
8 ww^3	9	$a+e=\sqrt[3]{4096}=16$
$4 \div \overline{a+e}$	10	$aa+ee=\frac{2336}{a+e}=\frac{2336}{16}=146$
9 $\odot^2$	11	$aa+2ae+ee=256$
11 − 10	12	$2ae=110$
10 − 12	13	$aa-2ae+ee=36$
13 ww^2	14	$a-e=\sqrt{36}=6$
9 + 14	15	$2a=22$ } *From hence it appears, that the* 3d
15 ÷ 2	16	$a=11$ } *Letter of the* 1ſt *Word is* l, *and*
9 − 16	17	$e=5$ } *the* 2d *Letter of the* 2d *Word is* e.

Note, *In order to ſet down the Letters (as they become found) in their proper Places, it may be convenient to ſupply the vacant Places with Stars.*

Thus { First Word. Second Word. Third Word.
* * *l* * * *e* * * * * * *

2. The

2. The Indices laſt found, are the two Extreams of four Numbers in Arithmetical Progreſſion, the leſſer Mean being the Index of the firſt Letter of the third Word; and the greater Mean is the Index of the fourth and laſt Letter of the firſt Word. *Viz.* 5 . 7 . 9 . 11 are the four Terms in Arithmetical Progreſſion. Whence it appears, that *G* (whoſe Index is 7) is the firſt Letter of the third Word; and that *i* (whoſe Index is 9) is the fourth or laſt Letter of the firſt Word; which being placed down, will ſtand thus,

* * *li* * *e* * *G* * * * * *.

3. The ſecond Letter of the third Word is the ſame with the third Letter of the firſt Word; and the fifth Letter of the third Word is the ſame with the laſt Letter of the firſt Word: whence the Letters will ſtand thus,

* * *li* * *e* * *Gl* * * *i* *.

4. The Sum of the Squares of the Indices of the firſt and ſecond Letters of the firſt Word is 520, and the Product of the ſame Indices is ſeven Ninths of the Square of the greater Index, which is the Index of the ſaid firſt Letter.

Let a = the greater, and e = the leſſer Index.

Then	1	$aa+ee=520$ } according to the *Data*
And	2	$ae=\frac{7}{9}aa$ }
$2\div a$	3	$e=\frac{7}{9}a$
$3\,\mathcal{Q}^2$	4	$ee=\frac{49}{81}aa$
$1-4$	5	$aa=520-\frac{49}{81}aa$
5×81	6	$81aa=42120-49aa$
$6+49aa$	7	$130aa=42120$
$7\div 130$	8	$aa=\frac{42120}{130}=324$
$8\,\omega^2$	9	$a=\surd 324=18$, whoſe Letter is *s*.
3 and 9	10	$e=\frac{7}{9}a=14$, whoſe Letter is *o*.

Hence the Letters will ſtand thus,

Soli * *e* * *Gl* * * *i* *.

5. The Difference between the two laſt Indices, is the Index of the firſt Letter of the ſecond Word, *viz.* $18-14=4$ being the Index of the Letter *D*. Then the Letters will ſtand thus,

Soli De * *Gl* * * *i* *.

6. The

6. The third and laſt Letter of the ſecond Word, alſo the third Letter of the third Word, are the ſame with the ſecond Letter of the firſt Word; hence the Letters will ſtand thus,

*Soli Deo Glo * i *.*

7. The Sum of the Indices of the fourth Letter of the third Word, and the ſixth or laſt Letter of the ſame Word, being added to their Product is 35; and the Difference of their Squares is 288; the Index of the laſt letter being the leaſt.

Put a = the greater, and e = the leſſer Index, as before.

Then	1	$ae + a + e = 35$ } by the *Data*.
And	2	$aa - ee = 288$ }
$1 - a$	3	$ae + e = 35 - a$
$3 \div a + 1$	4	$e = \frac{35 - a}{a + 1}$, for $e \times \overline{a + 1} = ae + e$
$4 \odot^2$	5	$ee = \frac{1225 - 70a + aa}{aa + 2a + 1}$
$2 + 5$	6	$aa = 288 + \frac{1225 - 70a + aa}{aa + 2a + 1}$
$6 \times aa$ &c.	7	$a^4 + 2a^3 + aa = 288aa + 576a + 288$ $+ 1225 - 70a + aa$
$7 +$	8	$a^4 + 2a^3 - 288aa - 506a = 1513$

This laſt Equation being reſolved according to the Method which ſhall be ſhewed in the next Chapter, it will be $a = 17$ it's Letter; and from the 4th Step $e = \frac{35 - a}{a + 1} = 1$, the Index of the Letter a. Then theſe two Letters being placed according to the *Data* above, are all that are required by the Enigma to compleat theſe Words,

Soli Deo Gloria.

CHAP. X.

The Solution of Adfected Equations *in Numbers.*

BEFORE we proceed to the Solution of Adfected Equations, it may not be amiſs to ſhew the Inveſtigation (or Invention) of thoſe Theorems or Rules for extracting the Roots of Simple Powers, made uſe of in Chapter 11. Part 1. I ſhall here make choice of the ſame Letters to repreſent the Numbers both given and ſought, as in my Compendium of *Algebra*.

Viz. Let G always denote the given Reſolvend.
r = any Number taken as near the true Root as may be, whether it be greater or leſs.
e = the unknown Part of the Root ſought by which r is to be either increaſed or decreaſed.

Then if r be any Number leſs than the true Root, it will be $r+e$ = the Root ſought. But if r be taken greater than the true Root, it will then be $r-e$ = the Root ſought. And put D for the Dividend that is produced from G, after it is leſſened and divided by r, &c. (into the Co-efficients of Adfected Equations) according as the Nature of the Root requires. Theſe Things being premiſed, we may proceed to raiſing the Theorems.

SECT. 1.

1. FOR the Square Root, *viz.* $aa=G$. Quære a.

Let	1	$r+e=a$
$1 \circledcirc^2$	2	$rr+2re+ee=aa=G$
$2-rr$	3	$2re+ee=G-rr$. Call it D, viz. $D=G-rr$
Then	4	$\frac{D}{2r+e}=e$ This ſhews the 1ſt Method of extracting the Square Root, Sect. 5. Chap. 11. Part 1.
$3 \div \overline{2}$	5	$re+\frac{1}{2}ee=\frac{G-rr}{2}=D$

Which gives this Theorem $\frac{D}{r+\frac{1}{2}e}=e$.

The Arithmetical Operations of both theſe Theorems, you have in the Examples of Section 2. Page 126. to which I refer the

the Learner, ſuppoſing him by this Time to underſtand them without any more Words than what is there expreſt.

II. To extract the Cube Root; *viz.* $aaa=G$. Quære a.

Let	1	$r+e=a$, ſuppoſing r leſs than the true Root.
$1\,\square^3$	2	$rrr+3rre+3ree+eee=aaa=G$
$2-rrr$	3	$3rre+3ree+eee=G-rrr$
$3\div rrr$	4	$re+ee+\frac{eee}{3r}=\frac{G-rr}{3r}=D$

Let $\frac{eee}{3r}$ be rejected or caſt off, as being of ſmall Value; then it will be, $re+ee=D$, which gives this following

Theorem $\frac{D}{r+e}=e$

By this Theorem or Rule, the 1ſt and 2d Examples in Caſe 1. Page 132. are performed; the which being compared with this Theorem may be eaſily underſtood.

Again, Suppoſe $aaa=G$, as before, and let r be taken greater than the true Root.

Then	1	$r-e=a$	eee being rejected as before.
$1\,\square^3$	2	$rrr-3rre+3ree=a^3=G$	
$2+$	3	$3rre-3ree=rrr-G$	
$3\div 3r$	4	$re-ee=\frac{rrr-G}{3r}=D$	

Which gives this *Theorem* $\frac{D}{r-e}=e$

By this Theorem the third Example in Caſe 2. Page 133. is performed.

III. To extract the *Biquadrat* Root; *viz.* $a^4=G$, Quære a.

Let	1	$r-e=a$	ſuppoſing r leſs than juſt.
$1\,\square^4$	2	$r^4+4rrre+6rree=a^4=G$	rejecting all the Powers of e above ee.
$2-r^4$	3	$4rrre+6rree=G-r^4$	
$3\div 2rr$	4	$2re+3ee=\frac{G-r^4}{2rr}=e$	

Which gives this *Theorem* $\frac{D}{2r+3e}=e$

By this Theorem the Biquadrate Root of any Number may be extracted. But, as I have already ſaid, Page 134. thoſe Extractions may be very well performed by two Extractions of the Square Root. *Vide Example, Page* 135.

IV. To extract the *Surſolid* Root, *viz.* $a^5 = G$. Quære a.

If r be taken leſs than juſt, then $r + e = a$, as before, and $\frac{G - r^5}{5r^4} = D$, which gives this *Theorem* $\frac{D}{r - 2e} = e$. By this Theorem the Surſolid Root, Example 1. Page 136. is extracted. But if r be taken greater than juſt; then $r - e = a$, and $\frac{r^5 - G}{5r^4} = D$, which gives this *Theorem* $\frac{D}{r - 2e} = e$. By this laſt Theorem the Example in Page 137. is performed.

I preſume it needleſs to purſue the raiſing of thoſe Theorems, for extracting the Roots of Simple Powers, any further; becauſe the Method of doing it is general, how high ſoever they are; and therefore it may be eaſily underſtood by what is already done.

SECT. 2.

NOtwithſtanding I have already ſhewed the Solution of Quadratick Equations, two ſeveral Ways, *viz.* by caſting off the loweſt Term; and by compleating the Square, *vide* Section 2. Page 195, *&c.* Yet it may not be amiſs to ſhew, how thoſe Equations may be reſolved into Numbers by this univerſal Method of continued Series; wherein, if the firſt r be taken equal to the firſt true Root, or ſingle Side of the Reſolvend; and every ſingle Value of e (as it becomes found) be ſtill added to it, for a new r, then thoſe Roots may be extracted without repeating a ſecond Operation, as before in the ſingle Powers.

Caſe. 1. Let $aa + 2ba = G$. It is required to find the Value of a.

Put	1	$r + e = a$
1×2	2	$rr + 2re + ee = aa$
$1 \times 2b$	3	$2br + 2be = 2ba$
$2 + 3$	4	$rr + 2br + 2re + 2be + ee = aa + 2ba = G$
$4 - rr$ &c	5	$2re + 2be + ee = G - rr - 2br$
$5 \div 2$	6	$re + be + \frac{1}{2}ee = \frac{1}{2}G - \frac{1}{2}rr - br = D$

Which gives this *Theorem* $\frac{D}{r + b + \frac{1}{2}e} = e$

Suppoſe

Suppoſe $b=364$, and $G=38692865$: If $r=6000$, then $rr=36000000$, and $2br=4368000$. But $36000000+4368000=40368000>38692865=G$. Therefore the firſt $r<6000$. Let $r=5000$, then

1ſt $r=5000$	$19346432,5=\frac{1}{2}G$	
$b=364$	$-1432000,=\frac{1}{2}rr+br$	
1ſt $r+b=5364$	$5026432,5=D$	$(800=e$
$+\frac{1}{2}e=400$	46112	
1 Diviſor 5764	41523	$(60=e$
2d $r+b=6164$	37164	
$+\frac{1}{2}e=30$	4359	$(7=e$
2 Diviſor 6194)	43592,5	867 $=e$
3d $r+b=6224$	(0)	
$+\frac{1}{2}e=3,5$		
3 Diviſor 6227,5		

Firſt $r=5000$, $+e=867$ $\}=5867=a$ as was required

Caſe 2. If $aa-2ba=G$, then proceeding as above, there will ariſe this *Theorem* $\frac{D}{r-b+\frac{1}{2}e}=e$, &c. And in *Caſe* 3, viz. $2ba-aa=G$, you will have this *Theorem* $\frac{D}{b-r-\frac{1}{2}e}$, &c. as above.

I think it needleſs to trouble the Reader with the Work of theſe two Theorems in Numbers; becauſe if the laſt Example of *Caſe* 1. be underſtood, the other will be eaſy. Not but that the Method of compleating the Square is very ready and eaſy, as you may obſerve by the Work in ſeveral Queſtions of this Chapter.

SECT. 3.

IN the Solution of all Adfected Equations, that are above (or higher than) Quadraticks, it will be the beſt way to take $r=$ the next neareſt Root of the Equation: And then it will be $r+e=a$, if r be leſs than juſt; or $r-e=a$ if r be greater than juſt (as at the beginning of this Chapter). And all the Powers of the unknown Part of the Root *(viz. e)* above it's Square *(ee)* are to be rejected or caſt off, as before in raiſing the Theorems for the Simple

Simple Powers. And therefore it is, that to ſupply the want of thoſe Powers (above ee in the Theorem) the Operation muſt be repeated: as in the Example of extracting the Cube Root, Page 133. *viz.* when the Figures in the Root conſiſt of more than three Places (*vide* Page 140 and 141.)

Suppoſe $aaa+ba=G$. Quære a.

Let	1	$r+e=a$ viz. let r be ſuppoſed leſs than juſt.
$1 \circledcirc^3$	2	$rrr+3rre+3ree=aaa$
$1 \times b$	3	$br+be=ba$
$2+3$	4	$rrr+br+3rre+be+3ree=a^3+ba=G$
$4 \div 3r$	5	$\frac{1}{3}rr+\frac{1}{3}b+re+\frac{be}{3r}+ee=\frac{G}{3r}$
$5-$ &c.	6	$re+\frac{be}{3r}+ee=\frac{G}{3r}-\frac{1}{3}rr-\frac{1}{3}b=D$

Which gives this *Theorem* $\frac{D}{r+\frac{b}{3r}+e}=e$

But if r be taken greater than juſt, then it will be $re+\frac{be}{3r}-ee=\frac{1}{2}rr+\frac{1}{3}b-\frac{G}{3r}=D$, which produces this *Theorem* $\frac{D}{r+\frac{b}{3r}-e}=e$

By either of theſe two Theorems the Value of a may be eaſily found. Or rather otherwiſe, as in the following Example.

Let $aaa+24a=587914$. Here $b=24$. Suppoſe the firſt $r=90$, then $r^3=729000>587914$ without the 24×90 being added to it: Therefore $r<90$. Again, Suppoſe $r=80$ then $r^3=512000$, and $24r=1920$. But $512000+1920=513920<587914$, hence >70, but nearer to it than 90. Therefore

it muſt be	1	$r+e=a$ leſs than juſt.
$1 \circledcirc^3$	2	$rrr+3rre+3ree=aaa$
1×24	3	$24r+24e=24a$
2 in Numb.	4	$512000+19200e+240ee=aaa$
3 in Numb.	5	$1920+24e=24a$
$4+5$	6	$513920+19224e+240ee=587914$
$5-513920$	7	$19224e+240ee=73994$
$7 \div 240$	8	$80,1e+ee=308,31=D$
$8 \div$	9	$e=\frac{D}{80,1+e}$

Operation

Operation 80,1
$+e=$ 3,

1 Diviſor 83,1)	308,31	(80, $=r$
$+e=$ 3,6	249,3	3,68 &c. $=e$
2 Diviſor 86,7)	59,01	83,68 &c. $=r+e$
$+e$,67	52,02	
87,37)	6,99 &c.	

Or rather new $r=83,7$ for a ſecond Operation, which being involved and tryed (as above) will be found greater than juſt: therefore it muſt be

	1	$r-e=a$
1 ⊛³	2	$rrr-3rre+3ree=aaa$
1 × 24	3	$24r-24e=24a$
2 in Numb.	4	$586376,253-21017,07e+251,1ee=aaa$
3 in Numb.	5	$2008,8-24e=24a$
4 + 5	6	$588385,053-21041,07e+251,1ee=587914$
6 +	7	$21041,07e-251,1ee=471,053$
7 ÷ 251,1	8	$83,7955e-ee=1,87595778=D$
8 ÷	9	$e=\frac{D}{83,7955-e}$

2d Operation 83,7955
$-e=$,02

1ſt Diviſor 83,7755)	1,87595778	(83,70000000 $=r$
$-e=$,022	1,675510	00,02239331 $=e$
2d Diviſor 83,7535)	,2004477	83,67760669 $=a=r-e$.
$-e=$,0023	,1675070	
3d Diviſor 83,7512)	,03294078	
$-e=$ 3 &c.	,02512536	
83,751	,00781542	
...	00753760	
	27782	
Here the new Diviſors are	25125	
rejected, as inſignificant.	2657	
	2512	
	145	
	83	

All the remaining Examples of extracting Roots (except Page 260.) are left in the Author's own Method; which by this Time, it is preſumed, the Learner will eaſily know how to correct of himſelf, if he takes due Notice of what has been delivered, Page 131, 132, &c.

But

But if more Exactneſs be required, you may make the new $r = 83{,}6776067$, and proceed with it to a third Operation; which will afford twenty-ſeven Places of Figures for the Value of a; that is, every Operation will produce triple the Places of Figures to thoſe of the Precedent r. And this tripling the Places of Figures in the Root, at every Operation, holds good, and is to be obſerved in the Solution of all Adfected Equations (how high ſoever they are) according to this Method of reſolving them. *See Page* 141.

Example 2. Suppoſe $aaa - ba = G$. Quære a. If $r + e = a$, then $re - \frac{\frac{1}{3}be}{r} + ee = \frac{\frac{1}{3}G}{r} + \frac{1}{3}b - \frac{1}{3}rr = D$, which gives this Theorem $\frac{D}{r - \frac{\frac{1}{3}b}{r} + e} = e$. But if $r - e = a$, then $re + \frac{\frac{1}{3}be}{r} + ee = \frac{\frac{1}{3}G}{r} + \frac{1}{3}b - \frac{1}{3}rr = D$, which gives this Theorem $\frac{D}{r + \frac{\frac{1}{3}b}{r} + e}$

Or you may proceed otherwiſe, as in the laſt Example. Let $aaa - 6438a = 104785688$, here $b = 6438$. Suppoſe the firſt $r = 500$, $rrr = 125000000$, and $br = 3219000$, then $125000000 - 3219000 = 121781000$. But $121781000 > 104785688$, therefore $r < 500$. Again, ſuppoſe $r = 400$, $rrr = 64000000$, and $br = 2575200$, then will $64000000 - 2575200 = 6142800$. But $61424800 < 104785688$, hence $r > 400$; conſequently r is betwixt 400 and 500. But 500 is the next neareſt; therefore, let $r = 500$ being greater than juſt.

Then	1	$r - e = a$
1×2	2	$rrr - 3rre + 3ree = aaa$
$1 \times b$	3	$br - be = ba$
2 in Numb.	4	$125000000 - 750000e + 1500ee = aaa$
3 in Numb.	5	$3219000 - 6438e = 6438a$
$4 - 5$	6	$121781000 - 743562e + 1500ee = 104785688$
$6 +$	7	$747562 - 1500ee = 16995312$
$7 \div 1500$	8	$495e - ee = 11330 = D$
$8 \div$	9	$e = \frac{D}{495 - e}$

Operation

Operation 495

$-e=$ 20

1 Diviſor 475) 11330 (500,0 $=r$

$-e=$ 3 950 23,8 $=e$

472 1830 476,2 $=r-e=a$

1416

414,0

377,6

Let new $r=476$ for a 2d Operation, then $r^3=107850176$ and $br=3064488$: but $107850176-3064488=104785688$ the ſame with the Reſolvend. Conſequently $a=476$ juſt.

Example 3. Let $ba-aaa=G$. Quære a. If $r+e=a$, then $\frac{\frac{1}{3}be}{r}-re-ee=\frac{\frac{1}{3}G}{r}+\frac{1}{3}rr-\frac{1}{3}b=D$, which gives this *Theorem* $\frac{D}{\frac{\frac{1}{3}b}{r}-re}=e$. But if $r-e=a$, then $re-\frac{\frac{1}{3}be}{r}-ee=\frac{\frac{1}{3}G}{r}+\frac{1}{3}rr-\frac{1}{3}b=D$, which gives this *Theorem* $\frac{D}{r-\frac{\frac{1}{3}b}{r}-e}=e$.

Or otherwiſe as before in the two laſt Examples. Thus, let $123456a-aaa=12272861$. Here $b=123456$. Suppoſe the firſt $r=200$, then $rrr=8000000$, and $br=24691200$; then $24691200-8000000=16691200$, but $16691200>12272861$, therefore r is here leſs than juſt, becauſe the higheſt Power is $-$, or negative. Again, Suppoſe $r=300$, then $r^3=27000000$, and $br=37036800$, then $37036800-27000000=10036800<12272861$. Conſequently $r<300$, and $r>200$. Let $r=300$, being the next neareſt, but more than juſt.

Ther	1	$r-e=a$
1	2	$rrr-3rre+3ree=aaa$
1 ×	3	$br-be=ba$
2 in Numb	4	$27000000-270000e+900ee$
3 in Numb	5	$37036800-123456e$
5 −	6	$10036800+146544e-900ee=12272861$
6 −	7	$146544e-900ee=2236061$
7 ÷ 90	8	$162e-ee=2484=D$
1 ÷ &	9	$e=\frac{D}{162-e}$

Operation 162
$-e=$ 10

1st Diviſor 152) 2484 (300,0 $=r$
$-e=$ 6 152 16,6 $=e$
2d Diviſor 646 964 283,4 $=r-e=a$.
876
88,0
86,6

Or new $r=283$, which being involved, *&c.* will appear to be the true Root, that is, $a=283$ juſt.

Note, Theſe are uſually called the three Forms of Cubick Equations; and in the Solution of the third or laſt Form, *viz.* $ba-aaa=G$, you may meet with ſome ſeeming Difficulties; eſpecially in making Choice of the firſt r, becauſe this Equation is an ambiguous Equation, and hath two Affirmative Roots, *viz.* a greater and leſſer Root. But having once found either of them, the other may be eaſily obtained by Diviſion only; as in the Quadratick Equations. *Vide* Chap. 8. As for inſtance, in the laſt Example, $a=283$ and $123456a-aaa=12272861$. Make theſe two Equations $=0$, to wit, let $a-283=0$, and $-aaa+123456a-12272861=0$.

Then, $a-283$) $-aaa+123456a-12272861$ ($-aa$
$-aaa+283aa$
$-283aa+123450a$ ($-283a$
$-283aa+80089a$
$+43367a-12272061$ $+43367$
$+43367a-12272861$
(0) (0)

Hence it appears that $-aa-283a+43367=0$. Conſequently $aa+283a=43367$ this Equation being ſolved, $a=110$, 2722 *&c.* which is the leſſer Root of the aforeſaid Equation $ba-aaa-G$, &c. After this Manner all the poſſible and impoſſible Roots of any Equation may be eaſily diſcovered, any one of it's Roots being once found. I ſhall therefore omit inſerting more Examples of that kind.

Suppoſe $aaa+baa+ca=G$. Quære a. Let $b=74$, $c=8729$, and $G=560783$. By Trial (as before) it will be found that the next neareſt $r=40$ being ſomething leſs than juſt.

Therefore

Therefore	1	$r+e=a$
$1\times c$	2	$cr+ce=ca$
$1\bigcirc^2 : \times b$	3	$brr+2bre+bee=baa$
$1\bigcirc^3$	4	$rrr+3rre+3ree=aaa$
2 in Numb.	5	$349160+8729e$
3 in Numb.	6	$118400+5920e+74ee$
4 in Numb.	7	$64000+4800e+120ee$
$5+6+7$	8	$531560+19449e+194ee=560783$
$8-531560$	9	$19449e+194ee=29223$
$9\div 194$	10	$100,2e+ee=153,06=D$
$10\div$	11	$e=\frac{D}{100,2+e}$

Operation 100,2

$+e=$ 1

1st Diviſor 101,2) 153,06 (40,0 = r

$+e=$,5 101,2 1,5 = e

2d Diviſor 101,7) 51,86 41,5 = r + e = a

50,85

1,01

Or new $r=41,5$ for a ſecond Operation, which being duly involved, &c. will be found more than juſt.

Therefore	1	$r-e=a$
Then	2	$cr-ce=ca$
	3	$brr-2bre+bee=baa$
	4	$rrr-3rre+3ree=aaa$

Theſe being turned into Numbers, &c. as above, they will be $20037,75e-198,5ee=390,375$, which being divided by 198,5 the Co-efficient of ee, will become $100,946e-ee=1,966624$, &c. $=D$.

Operation 100,946

$-e=$ 01

1ſt Diviſor 100,936) 1,966624 (41,5000000 = r

$-e=$,009 1,00936 ,0194847 = e

2d Diviſor 100,927) 957264 41,4805153 = r − e = a

908343

* Here I proceed by plain Diviſion without forming new Diviſors.

* 489210

403708

855020

807416

476040

403708

72332 &c.

Let the last Equation in the Enigma, Chap. 9. be here proposed for a Solution. *Viz.* $aaaa+baaa-caa-da=G$; $b=2$, $c=288$, $d=506$, and $G=1513$, Quære a. By Tryals it will be found, that the next nearest $r=20$, being something more than just.

Therefore	1	$r-e=a$
$1\times d$	2	$dr-de=da$
$1\,\text{☉}^2\times c$	3	$crr-2cre+cee=caa$
$1\,\text{☉}^3\times b$	4	$brrr-3brre+3bree=baaa$
$1\,\text{☉}^4$	5	$r^4-4rrre+6rree=aaaa$

These being turned into Numbers, and those duly collected, according as the Signs of the Equation direct, they will become $50680-22374e+2232ee=1513$, which being all divided by 2232 the Co-efficient of ee, will be $10e-ee=22=D$.

$$\text{Then } \frac{D}{10-e}=e.$$

Operation 10
$-e=$ 3

Divisor 7) 22 ($20=r$
$3=e$
21 $17=r-e=a$ just.
1

See the End of Chap. 9.

By what hath been already done about the Solution of these few Equations (being carefully observed) I presume the Learner will easily conceive how to proceed in the Solution of all Kinds of Equations, be they never so high, or adfected; therefore I shall not here propose many various Examples, but only take them as they fall in Course, when I come to the next Part, wherein you will (perhaps) find such Equations with their Solutions as are not common.

CHAP.

CHAP. XI.

Of Simple Intereſt, Annuities, *or Penſions*, &c.

INTEREST, or the Uſe paid for the Loan of Money, is either Simple, or Compound.

Sect. 1. *Of Simple* Intereſt.

SIMPLE Intereſt is that which is paid for the Loan of any Principal or Sum of Money, lent out for ſome Time, at any Rate *per Cent.* agreed on between the Borrower and the Lender; which, according to the late Laws of *England*, ought to be ſix Pounds for the Uſe of 100*l.* for one Year, and twelve Pounds for the Uſe of 100*l.* for two Years: and ſo on for a greater or leſſer Sum, proportionable to the Time propoſed.

There are ſeveral Ways of computing (or anſwering Queſtions about) Simple Intereſt; as by the ſingle and double Rule of Three (See Page 96, &c.) others make uſe of Tables compoſed at ſeveral Rates *per Cent.* as Sir *Samuel Moreland*, in his Doctrine of Intereſt, both ſimple and compound, all performed by Tables; wherein he hath detected ſeveral material Errors committed by Sir *Iſaac Newton*, Mr *Kerſey* upon *Wingate*, and Mr *Clavil*, &c. in the Buſineſs of computing Intereſt, *&c.* by their Tables, too tedious to be here repeated. But I ſhall in this Tract take other Methods, and ſhew that all Computations relating to Simple Intereſt are grounded upon Arithmetick Progreſſion; and from thence raiſe ſuch general Theorems, as will ſuit with all Caſes. In order to that,

Let { P = any Principal or Sum put to Intereſt.
R = the Ratio of the Rate, *per Cent. per Annum.*
t = the Time of the Principal's Continuance at Intereſt.
A = the Amount of the Principal, and it's Intereſt.

Note, The Ratio of the Rate, is only the Simple Intereſt of 1*l.* for one Year, at any given Rate; and is thus found.

Viz. 100 : 6 :: 1 : 0,06 = the Ratio at 6 *per Cent. per Ann.*
Or 100 : 7 :: 1 : 0,07 = the Ratio at 7 *per Cent.* &c.
Again 100 : 7,5 :: 1 : 0,075 = the Ratio at 7 and $\frac{1}{2}$ *per Cent.*

And if the given Time be whole Years; then t = the Number of whole Years: but if the Time given, be either pure Parts of a Year, or Parts of a Year mixed with Years; thoſe Parts muſt be turned into Decimals; and then t = thoſe Decimals, *&c.*

Now

Now the common Parts of a Year may be eaſily turned or converted into Decimal Parts, if it be conſidered

That one { Day is the $\frac{1}{365}$ Part of a Year $=0,00274$ *ferè*.
Month is the $\frac{1}{12}$ Part of a Year $=0,0833333$ &c.
Quarter is the $\frac{1}{4}$ Part of a Year $=0,25$

Theſe Things being premiſed, we may proceed to raiſing the *Theorems*.

Let R = the Intereſt of 1 *l.* for one Year, as before.
Then $2R$ = the Intereſt of 1 *l.* for two Years.
And $3R$ = the Intereſt of 1 *l.* for three Years.
$4R$ = the Intereſt of 1 *l.* for four Years. And ſo on for any Number of Years propoſed.

Hence it is plain, that the Simple Intereſt of one Pound is a Series of Terms in Arithmetick Progreſſion increaſing; whoſe firſt Term and common Difference is R, and the Number of all the Terms is t. Therefore the laſt Term will always be tR = the Intereſt of 1 *l.* for any given Term ſignified by t.

Then { *As one Pound : is to the Intereſt of* 1 l. : : *ſo is any Principal or given Sum : to it's Intereſt.*

That is, $1 : tR :: P : tRP$ = the Intereſt of P. Then the Principal being added to it's Intereſt, their Sum will be $= A$ the Amount required: which gives this general *Theorem*.

Theorem 1. $tRP + P = A$

From whence the three following *Theorems* are eaſily deduced.

Theorem 2. $\frac{A}{tR+1} = P$. *Theorem* 3. $\frac{A-P}{tP} = R$.

Theorem 4. $\frac{A-P}{RP} = t$.

Theſe four *Theorems* reſolve all Queſtions about Simple Intereſt.

Queſtion 1. *What will* 256 l. 10 s. *amount to in three Years, 1 Quarter, 2 Months, and* 18 *Days, at* 6 *per Cent. per Annum.*
Here is given $P = 256,5$; $R = 0,06$; and $t = 3,46599$
For 3 Years $= 3$ Quære A. *per Theorem* 1.
one Quarter $= 0,25$
2 Months $= 0,16667 = 0,08333 \times 2$
18 Days $= 0,04932 = 0,00274 \times 18$
Hence $t = 3,46599 : \times 0,06 = 0,2079594 = tR$
Then $0,2079594 \times 256,5 = 53,341586 = tRP$
And $53,341586 + 256,5 = 309,841586 = tRP + P = A$.
That is, 309,841586 = 309 *l.* 16 *s.* 10 *d.* being the Anſwer required.

Queſtion

Queſtion 2. *What Principal or Sum being put to Intereſt, will raiſe a Stock of* 309 l. 16 s. 10 d. in three Years, one Quarter, two Months, and 18 Days; at 6 *per Cent. per Annum?*

Or the ſame Queſtion otherwiſe ſtated thus.

What is 309 l. 16 s. 10 d. *due* 3 *Years,* 1 *Quarter,* 2 *Months, and* 18 *Days hence, worth in ready Money; abating or diſcounting* 6 *per Cent.* &c.

Here is given $A=309,841586$: $R=0,06$; $t=3,46599$ (found as before) thence to find P. *Per Theorem* 2. Firſt, $3,46599 \times 0,06 = 0,2079594 = t\,R$. Then $t\,R+1 = 1,2079594)\ 309,841586 = A$ ($256,5 = P$; that is, $256,5 = 256$ *l.* 10 *s.* the Anſwer required.

Queſtion 3. *At what Rate or Intereſt, per Cent.* &c. *will* 256 l. 10 s. *amount to* 309 l. 16 s. 10 d. *in three Years, one Quarter, two Months, and* 18 *Days?*

Here is given, $P=256,5$; $A=309,841586$; and $t=3,46599$ to find R. *Per Theorem* 3. Firſt, $309,841586 - 256,5 = 53,341586 = A - P$. Next $3,46599 \times 256,5 = 889,026435 = t\,R$. And $t\,R = 889,026435)\ 53,341586$ ($00,06$ = the Ratio. Then 1 *l.* : 0,06 : : 120 : 6 = the Rate required.

Queſtion 4. *In what Time will* 256 l. 10 s. *raiſe a Stock of (or amount to)* 309 l. 16 s. 10 d. *at* 6 *per Cent.* &c.

Here is given, $P=256,5$; $A=309,841586$, and $R=0,06$ to find t. *Per Theorem* 4. Firſt, $309,841586 - 256,5 = 53,341586 = A - P$. And $256,5 \times 0,06 = 15,39 = P\,R$. Then $15,39)\ 53,341586$ ($3,46599 = t$; that is $t = 3$ Years and ,46599 Decimal Parts of a Year; which may be brought into common Parts of a Year, thus

0,46599	And 0,08333) 0,21599	(2 Months.
0,25 = one Quarter	,16666	
0,21599	0,02074) ,04933	. 18 Days.

Hence $t=3$ Years, one Quarter, 2 Months, and 18 Days; the Anſwer required.

It muſt needs be eaſy to conceive, that what is here done at 6 *per Cent.* may be done at any other Rate of Intereſt, by forming the Ratio (*viz. R.*) accordingly.

SCHOLIUM.

SCHOLIUM.

Although it be according to the Laws and Cuſtom of *England*, to compute Intereſt at the Proportion of 6 *per Cent.* (as above) yet he that takes up Money at Intereſt for any Time leſs than even or compleat Years, pays more Intereſt than ſeems reaſonably due, according to the Rules of Art. As for Inſtance; if 100*l.* be forborn at Intereſt one whole Year, it amounts to 106*l.* But (I ſay) if it be paid at the half Year's End, it ſhould not amount to 103; as appears from this following Proportion.

Let a = the Amounts due at the half Year's End; then it will be $100 : a :: a : 106$ the Amount at the Year's End. *Ergo*, $aa = 10600$, and $a = \sqrt{10600} = 102{,}9563 = 102l.\ 19s.\ 1\frac{1}{2}d.$ which is leſs than 103*l.* by $10\frac{1}{2}d.$ And if it be paid in leſs than half a Year's Time, the Error muſt needs be the greater.

Sect. 2. *Of* Annuities, *or* Penſions *in Arrears, computed at Simple Intereſt.*

ANNUITIES, or Penſions, &c. are ſaid to be in Arrears, when they are payable or due, either Yearly, or Half-yearly, &c. and are unpaid for any Number of Payments. Therefore the Buſineſs is, to compute what all thoſe Payments will amount unto, allowing any Rate of Simple Intereſt for their Forbearance, from the Time each particular Payment became due: Now in order to that,

Put
- u = the Annuity, Penſion, or Yearly Rent, &c.
- t = the Time of it's Continuance, or being unpaid.
- R = the Ratio, or Intereſt of 1*l.* for one Year, as before.
- A = the Amount of the Annuity and it's Intereſt.

Then if u = the firſt Year's Rent, due without Intereſt.

Ru = the Intereſt, $2u$ = the Rent. } due at the End of the ſecond Year.

$2Ru$ = the Intereſt, $3u$ = the Rent. } due at the End of the third Year.

$3Ru$ = the Intereſt, $4u$ = the Rent. } due at the End of the fourth Year.

$4Ru$ = the Intereſt, $5u$ = the Rent. } due at the End of the fifth Year.

And ſo on for any Number of Years. Hence it is evident, that $Ru + 2Ru + 3Ru + 4Ru + 5u = A$ the Sum of all the Rents and their Intereſt, being forborn 5 Years.

From

From whence it follows, that $Ru+2Ru+3Ru+4Ru=A-tu$. Here $t=5$. Divide by u, then $R+2R+3R+4R=\frac{A-tu}{u}$.

Next to find the Sum of this Progreſſion (See Page 185) thus, Let $R+2R+3R+4R$ &c. $=z$, then $1+2+3+4$ &c. $=\frac{z}{R}$. Here the Sum of the firſt and laſt Terms are $4+1=5=t$, and the Numbers of all the Terms is $4=t-1$. Therefore $\frac{t-1}{2}\times t=$ the Sum of all the Terms; that is, $\frac{tt-t}{2}=\frac{z}{R}$: hence $\frac{ttR-tR}{2}=z$. Conſequently $\frac{ttR-tR}{2}=\frac{A-tu}{u}$. Now from this Equation it will be eaſy to deduce the following *Theorems*.

Theorem 1. $\frac{ttRu-tRu+2tu}{2}=A$, or $\frac{ttu-tu}{2}R:+tu=A$.

Theorem 2. $\frac{2A}{ttR-tR+2t}=u$. *Theorem* 3. $\frac{2A-2tu}{ttu-tu}=R$:

Let $\frac{2}{R}-1=x$, then $t=\sqrt{\frac{2A}{Ru}+\frac{xx}{4}}:-\frac{t}{2}x$ *Theorem* 4.

Queſtion 1. *If* 250 l. *yearly Rent (or Penſion,* &c.) *be forborn or unpaid ſeven Years; what will it amount to in that Time, at* 6 per Cent. *for each Payment, as it becomes due?*

Here is given $u=250$, $t=7$, and $R=0{,}06$; to find A. *Per Th.* 1. Firſt $250\times7=1750=tu$. $1750\times7=12250=ttu$. Again $12250-1750=10500=ttu-tu$, and $\frac{10500}{2}\times0{,}06=315$. Laſtly $315+1750=2065=A$; *Viz.* 2065 *l.* is the *Anſw.* required.

But if the Annuity, Rent or Penſion, is to be paid by Quarterly or Half-yearly Payments, *&c.* Then $\frac{0{,}06}{2}=0{,}03=R$ for Half-yearly Payments: and $\frac{0{,}06}{4}=0{,}015=R$ for quarterly; or $0{,}045=R$ for three-quarterly Payments. Example of Half-yearly Payments.

Suppoſe 250 l. per Annum, *to be paid by Half-yearly Payments, were in Arrears, or unpaid for ſeven Years; what would it amount to, allowing* 6 per Cent. per Annum *for each Payment, as it becomes due?*

In this Example there is given $u=125=\frac{250}{2}$; $t=14$ the Number of Payments; and $R=0{,}03=\frac{0{,}06}{2}$; thence to find A.

First $125\times 14=1750=tu$; $1750\times 14=24500=ttu$: again $24500-1750=22750=ttu-tu$; then $\frac{22750}{2}=11375$, and $11375\times 0{,}03=341{,}25$. Lastly $341{,}25+1750=2091{,}25$; that is, $A=2091$ *l.* 5 *s.* the Answer required.

N. B. Hence it may be observed, that Half-yearly Payments are more advantageous than yearly. For 2091 *l.* 5 *s.* > 2065 *l.* by 26 *l.* 5 *s.* consequently, quarterly Payments are more advantageous than Half-yearly Payments.

Question 2. *What yearly Rent, Pension,* &c. *being forborn or unpaid seven Years, will raise a Stock of* 2065 l. *allowing* 6 per Cent. per Annum *for each Payment, as it becomes due?*

Here is given $A=2065$, $t=7$, and $R=0{,}06$; to find u. *Per Theorem* 2. First $7\times 0{,}06=0{,}42=tR$, and $0{,}42\times 7=2{,}94=ttR$. Then $ttR-tR=2{,}52$. Lastly $ttR-tR+2t=16{,}52)\ 4130=2A\ (250=u$; that is, 250 *l. per Annum,* &c. will raise 2065 *l.* the Stock required.

Question 3. *In what Time will* 250 l. *yearly Rent raise a Stock of* 2065 l. *allowing* 6 per Cent. *&c. for the Forbearance of the Payments as they become due?*

Here is given $u=250$, $A=2065$, and $R=0{,}06$; to find t. *Per Theorem* 4. First $\frac{2}{R}=\frac{2}{,06}=33{,}3333$; and $33{,}3333-1=32{,}3333=x=\frac{2}{R}-1$. Then 16,16666 &c. $=\frac{1}{2}x$; 261,3605 &c. $=\frac{1}{4}xx$. Again $\frac{4130}{15}=275{,}333=2A\div Ru$, and $275{,}3333+261{,}3605=536{,}6938=\frac{2A}{Ru}+\frac{1}{4}xx$. Then $\sqrt{536{,}6938}=23{,}1666$. Lastly, $23{,}1666-16{,}1666=7=t$ the Time required.

Question 4. *If* 250 l. *yearly Rent, being forborn seven Years, will amount to* 2065 l. *allowing Simple Interest for every Payment as it becomes due; what must the Rate of the Interest be* per Cent. *&c.*

Here is given $u=250$, $A=2065$, and $t=7$; to find R: *Per Theorem* 3.

Thus $\begin{cases} ttu=12250 \\ tu=1750 \end{cases}$ $\begin{cases} 4130=2A \\ 3500=2tu \end{cases}$

$ttu-tu=10500)\quad 630=2A-2tu\ (0{,}06=R.$

Then 1 : 0,06 : : 100 : 6 the Rate required.

Sect.

Sect. 3. *The* Preſent *Worth of* Annuities *or Penſions, &c. computed at Simple Intereſt.*

THE buſineſs of purchaſing Annuities, or taking of Leaſes, *&c.* for any aſſigned Time, depends upon the true equating of the Principal or Money laid out on the Purchaſe, with the Annuity or Yearly Rent, by allowing (or diſcompting) the ſame Rate of Intereſt to both Parties. Which may be eaſily performed by duly applying the reſpective *Theorems* of the two laſt Sections together; as will fully appear by the following Queſtion.

Queſtion 1. *What is* 75 *l. yearly Rent, to continue nine Years, worth in ready Money, at* 6 per Cent. per Annum *Simple Intereſt?*

1. *Per Theorem* 1. of the laſt Section, find what the propoſed yearly Rent would amount to, if it were forborn 9 Years, at 6 *per Cent.*

Thus $u=75$, $t=9$, and $R=0{,}06$: Quære A.

$ttu=6075$ Then 2) 5400 (2700 } Multiply
$tu=675$ $R=0{,}06$ }

$ttu-tu=5400$ $\left.\begin{matrix}162,\\ +tu=675,\end{matrix}\right\}=837=A$.

2. Then by *Theorem* 2. *Section* 1. find what Principal, being put to Intereſt for the ſame Time, and at the ſame Rate, will amount to 837 *l.* $=A$. Thus $tR=0{,}54=9\times0{,}06$; $tR+1=1{,}54$) 837 (543,5064$=P$: that is, $P=543$ *l.* 10 *s.* $1\frac{1}{2}$ *d*, which is the Worth of 75 *l.* a Year, as was required.

From the Work of theſe two Operations (duly conſidered) it muſt needs be eaſy to conceive, how the two *Theorems* by which they were performed, may be combined in one.

For 1. $\frac{ttRu-tRu+2tu}{2}=A$; and 2. $PtR+P=A$.

Conſequently $PtR+P=\frac{ttRu-tRu+2tu}{2}$ And from this Equation may be deduced the following *Theorems.*

Theorem 1. $\frac{ttRu-tRu+2tu}{2tR+2}=P$, or $\frac{ttR-tr+2t}{2tR+2}\times u=P$.

By this *Theorem* all Queſtions of the ſame Kind with the laſt (*viz.* that above) may be eaſily and readily anſwered at one Operation.

Theorem 2. $\frac{2PtR+2P}{ttR-tR+2t}=u$, or $\frac{tR+1}{ttR-tR+2t}:\times 2P=u$.

Theorem 3. $\frac{2P-2tu}{ttu-tu-2Pt}=R$.

Let $\frac{2}{R}-\frac{2P}{u}-1=x$, then will $tt\pm xt=\frac{2P}{Ku}$.

Which gives this *Theorem* 4. $\sqrt{\frac{2P}{Ku}+\frac{xx}{u}}:\pm\frac{x}{2}=t$.

By the ſecond and fourth *Theorems*, two very uſeful Queſtions may be eaſily anſwered.

1. *As for Inſtance: If it be required to find what Annuity, or yearly Rent,* &c. *may be purchaſed, for any propoſed Sum, to continue any aſſigned Time, allowing any Rate of Intereſt?*

This Queſtion may be anſwered by *Theorem* 2.

2. *Again: If it be required to find how long any yearly Rent, Penſion, or Annuity,* &c. *may be purchaſed (or enjoyed) for any propoſed Sum, at any given Rate of Intereſt?*

All Queſtions of this Kind are eaſily anſwered by *Theorem* 4.

In theſe Queſtions it is ſuppoſed, that the Purchaſe or yearly Rent, is to commence or be immediately entered upon. But if it be required to find the Value or Purchaſe of an Annuity or yearly Rent, *&c.* in Reverſion; that is, when it is not to be entered upon until after ſome Time, or Number of Years are paſt; then you muſt firſt find what the Sum propoſed to be laid out in the Purchaſe, would amount to, if it were put to Intereſt, during the Time the Annuity, *&c.* is not to be put in preſent Poſſeſſion; and make that Amount the Sum for the Purchaſe, proceeding with it as in either of the two laſt Queſtions, *&c.*

Note, *From the firſt Queſtion of this Section it will be eaſy to conceive how to perform the Equation of Payments, between Debtor or Creditor, at any Rate of Intereſt, without doing any Damage to either Party.*

That is, when ſeveral Sums of Money are to be paid, at ſeveral different Times, to find the Time when all the Payments may be truly diſcharged at once: as if one Sum were to be paid at the End of two Months, another at ſix Months, and perhaps a third Sum at eight Months End, *&c.* And if it were required to find the Time when all thoſe Sums may be truly diſcharged at one Payment without Loſs, *&c.*

CHAP.

CHAP. XII.

Of Compound *Intereſt, and* Annuities, *&c.*

COMPOUND Intereſt is that which ariſes from any Principal and it's Intereſt put together, as the Intereſt ſo becomes due; ſo that at every Payment, or at the Time when the Payments became due, there is created a new Principal; and for that Reaſon it is called Intereſt upon Intereſt, or Compound Intereſt.

As for Inſtance; Suppoſe 100*l.* were lent out for two Years, at 6 *per Cent. per Annum*, Compound Intereſt: then at the End of the firſt Year, it will only amount to 106*l.* as in Simple Intereſt. But for the ſecond Year this 106*l.* becomes Principal, which will amount to 112*l.* 7*s.* 2½*d.* at the ſecond Year's End, whereas by Simple Intereſt it would have amounted to but 112*l.*

And altho' it be not lawful to let out Money at Compound Intereſt; yet in purchaſing of Annuities or Penſions, *&c.* and taking Leaſes in Reverſion, it is very uſual to allow Compound Intereſt to the Purchaſer for his ready Money; and therefore it is very requiſite to underſtand it.

Sect. 1. *Of* Compound *Intereſt.*

Let { P = the Principal put to Intereſt. / t = the Time of it's Continuance. / A = the Amount of the Principal and Intereſt. } as before. / R = { the Amount of 1*l.* and it's Intereſt for 1 Year, at any given Rate, which may be thus found.

Viz. 100 : 106 :: 1 : 1,06 = the Amount of 1*l.* at 6 *per Cent.*
Or 100 : 105 :: 1 : 1,05 = the Amount of 1*l.* at 5 *per Cent.*
and ſo on for any other aſſigned Rate of Intereſt.

Then if R = the Amount of 1*l.* for one Year, at any Rate.
R^2 = the Amount of 1*l.* for two Years.
R^3 = the Amount of 1*l.* for three Years.
R^4 = the Amount of 1*l.* for four Years.
R^5 = the Amount of 1*l.* for five Years. Here $t = 5$

For $1 : R :: R : RR :: RR : R^3 :: R^3 : R^4 :: R^4 : R^5$: &c. in $\div$.

That is { As one Pound : is to the Amount of one Pound at one Year's End : : ſo is that Amount : to the Amount of one Pound at two Years End, *&c.*

Whence

Whence it is plain, that Compound Intereſt is grounded upon a Series of Terms, increaſing in Geometrical Proportion continued; wherein t (*viz.* the Number of Years) does always aſſign the Index of the laſt and higheſt Term: *Viz.* the Power of R, which is R^t.

Again, As $1 : R^t :: P : PR^t = A$ the Amount of P for the Time, that $R^t =$ the Amount of 1 *l.*

That is { *As one Pound : is to the Amount of one Pound for any given Time : : ſo is any propoſed Principal (or Sum) to it's Amount for the ſame Time.*

From the Premiſes (I preſume) the Reaſon of the following *Theorems* may be very eaſily underſtood.

Theorem 1. $PR^t = A$, as above.

From hence the two following *Theorems* are eaſily deduced.

Theorem 2. $\frac{A}{R^t} = P$. *Theorem* 3. $\frac{A}{P} = R^t$.

By theſe three *Theorems*, all Queſtions about Compound Intereſt may be truly reſolved by the Pen only, *viz.* without Tables; tho' not ſo readily as by the Help of Tables, calculated on Purpoſe; as will appear farther on.

Queſtion 1. *What will* 256 l. 10 s. *amount to in ſeven Years, at* 6 per Cent. per Annum, *Compound Intereſt?*

Here is given $P = 256{,}5$; $t = 7$; and $R = 1{,}06$ which being involved until it's Index $= t$ (*viz.* 7.) will become $R^7 = 1{,}50363$. Then $1{,}50363 \times 256{,}5 = 385{,}6811 = A = 385$ *l.* 13 *s.* 7 $\frac{1}{2}$ *d.* which is the Anſwer required.

Queſtion 2. *What Principal or Sum of Money muſt be put (or let) out to raiſe a Stock of* 385 l. 13 s. 7 $\frac{1}{2}$ d. *in ſeven Years, at* 6 per Cent. per Annum *Compound Intereſt?*

Here is given $A = 385{,}6811$; $R = 1{,}06$; and $t = 7$; to find P. by *Theorem* 2. Thus $R^t = 1{,}50363)\ 385{,}6811 = A\ (256{,}5 = P$. That is, $P = 256$ *l.* 10 *s.* which is the Principal or Sum, as was required.

Queſtion

Queſtion 3. *In what Time will* 256 l. 10 s. *raiſe a Stock of (or amount to)* 385 l. 13 s. 7½ d, *allowing* 6 per Cent. per Annum *Compound Intereſt?*

Here is given $P=256,5$; $A=385,6811$; $R=1,06$; to find t by the third *Theorem* $R^t=\frac{A}{P}=\frac{385,6811}{256,5}=1,50363$, which being continually divided by $R=1,06$ until nothing remain, the Number of thoſe Diviſions will be $7=t$. Thus 1,06) 1,50363 (1,41852. And 1,06) 1,41852 (1,338225. Again 1,06) 1,338225 (1,262477. And ſo on until it become 1,06) 1,06 (1. which will be at the 7th Diviſion. Therefore it will be $t=7$ the Number of Years required by the Queſtion.

Queſtion 4. *If* 256 l. 10 s. *will amount to (or raiſe a Stock of)* 385 l. 13 s. 7½ d. *in ſeven Years Time; what muſt the Rate of Intereſt be,* per Cent. per Annum?

Here is given $P=256,5$; $A=385,6811$, and $t=7$, Quære R. By *Theorem* 3. $\frac{A}{P}=R^t=1,50363$; as before in the laſt Queſtion. And if $R^t=R^7=1,50363$, then $R=\sqrt[7]{1,5063}$, which may be thus extracted.

Put	1	$r+e=R$, then
$1\,@^7$	2	$r^7+7r^6e+21r^5ee=R^7=1,50363=G$
$2-r^7$	3	$7r^6e+21r^5eeG-r^7$
$3\div 7r^5$	4	$re+3ee=\frac{G-r^7}{7r^5}=D$
$4\div$	5	$e=\frac{D}{r+3e}$; let $r=1$, then $D=0,0719$

Operation $r=1,00$
$+3e=0,18$

Diviſor 1,18) 0,0719 (1,00 = r
708 0,06 = e

11 to be rejected

1,06 = $r+e=R$

Then 1 : 0,06 :: 100 : 6 the Rate *per Cent.* required.

The firſt three Queſtions may be much more eaſily performed by the following Table, which is only the Amounts of one Pound for thirty-nine Years.

That

That is, of R . RR . RRR . R^4 . R^5 . and so on to R^{39}

Years $= t$.	The Amounts of 1 *l.* at 6 *per Cent.* &c. Compound Interest.	Years $= t$.	The Amounts of 1 *l.* at 6 *per Cent.* &c. Compound Interest.	Years $= t$.	The Amounts of 1 *l.* at 6 *per Cent.* &c. Compound Interest.
1	1.06 $= R$	14	2.2609039557	27	4.8223459407
2	1.1236 $= RR$	15	2.3965581931	28	5.1116866971
3	1.191016 $= R^3$	16	2.5403516847	29	5.4183878990
4	1.26247696	17	2.6927727857	30	5.7434911729
5	1.3382255776	18	2.8543391529	31	6.0881006432
6	1.4185191122	19	3.0255995021	32	6.4533866818
7	1.5036302590	20	3.2071354722	33	6.8405898828
8	1.5938480745	21	3.3995636005	34	7.2510252757
9	1.6894789590	22	3.6035374166	35	7.6860867923
10	1.7908476965	23	3.8197496616	36	8.1472519998
11	1.8982985583	24	4.0489346413	37	8.6360871198
12	2.0121964718	25	4.2918707197	38	9.1542523470
13	2.1329282601	26	4.5493829629	39	9.7035074878

The Title of this Table shews it's Construction, and it's Use will easily appear by an Example or two.

EXAMPLE 1.

What will 375 l. 10 s. *amount to in nine Years, at* 6 per Cent. per Annum, &c. ?

The tabular Number against 9 Years is 1,689479 which being multiplied with the Principal 375,5 will produce 634,3993 &c. *viz.* 634 *l.* 8 *s. ferè*, being the Amount or Answer required.

EXAMPLE 2.

What Principal (or Sum) must be put to Interest to raise a Stock of 634 l. 8 s. *in nine Years Time, at* 6 per Cent per Annum, *&c.*

If the proposed Stock (*viz.* 634,4) be divided by the tabular Number that is against the given Number of Years (*viz.* 9.) the Quotient will be the Principal (or Sum) required. *Viz.* against 9 is 1,689479. Then 1,689479) 634,4 (375,5 = 375 *l.* 10 *s.* the Principal (or Sum) required.

EXAMPLE 3.

In what Time will 375 l. 10 s. *raise a Stock of (or amount to)* 634 l. 8 s. *at* 6 per Cent. &c. ?

Divide

Divide the propoſed Stock (*viz.* 634,4) by the given Principal (*viz.* 375,5) and the Quotient will ſhew the tabular Number that ſtands over againſt the Time ſought. Thus 375,5) 634,4 (1,689479 &c. this Number being ſought in the Table, will be found to ſtand againſt 9 Years, which is the Time required.

But if the Quotient cannot be truly found in the Table of Amounts for Years, as above; then take out of that Table the neareſt Number that is leſs, and make it a Diviſor, by which you muſt divide the firſt Quotient; and then ſeek the ſecond Quotient in the Table of Amounts for Days (which is inſerted a little further on) and it will aſſign the Number of Days: as in this Example.

In what Time will 563 l. *amount to* 860 l. *at* 6 per Cent. per Annum, *Compound Intereſt?*

Anſwer. In 7 Years and 99 Days.

Thus 563) 860 (1,52753 which ſhews the Time to be more (or above) ſeven Years; for over againſt 7 Years is 1,50363 which being made the new Diviſor: *Viz.* 1,50363) 1,52753 (1,01589 &c. this Number is the neareſt Amount to 99 Days.

Note, *If the Stock, Principal, and Time be given; the Rate of Intereſt will be beſt found by extracting the Root,* &c. *as before in the fourth Queſtion.*

The next Thing that I ſhall here propoſe, is to make this Table (which is only calculated for the Rate of 6 *per Cent.*) univerſally uſeful for all the Rates of Compound Intereſt, *which I may preſume to ſay, is a new Improvement of my own,* being well ſatisfied it never was publiſhed before; and not only ſo, but I have heard ſeveral very good Artiſts affirm it was impoſſible to be done.

The Method of performing it is briefly thus, Let $x =$ the Difference between $1,06 = R$ the Amount of 1 *l.* for one Year (in the Table) and any other propoſed Amount of 1 *l.* for one Year; which admits of two Caſes.

Caſe. 1. If the propoſed Rate be greater than the $1,06 = R$, then will $R + x =$ the true Amount of 1 *l.* for one Year at that Rate.

Caſe 2. But if the propoſed Rate be leſs than $1,06 = R$, then it will be $R - x =$ the Amount of 1 *l.* &c.

Make $\begin{cases} t - 1 = b, \ t - 2 = c, \ t - 3 = d, \ t - 4 = f, \text{ \&c.} \\ \frac{1}{2} tb = g, \ \frac{1}{2} cg = m, \ \frac{1}{4} dm = n, \ \frac{1}{4} fn = s, \text{ \&c.} \end{cases}$

Then will $R^t + t R^b x + g R^c x^2 + m R^d x^4$ &c. = the Amount of 1 *l.* at the given Rate, for any Time denoted by t, in Case 1. And $R^t - t R^b x + g R^c x^2 - m R^d x^3$ &c. = the Amount of 1 *l.* in Case 2.

Which is no more but this: Let $R+x$ or $R-x$ (which ſoever it is) be involved (as directed in *Sect.* 5. *Chap.* 2.) to the ſame Power or Height as the Index t the given Time in the Queſtion denotes: rejecting all the Powers of x above xxx or $xxxx$ at moſt, as uſeleſs. Then multiply that Power of $R+x$ or $R-x$ into the given Principal, and their Product will be the Amount required.

An Example or two in each Caſe will render all eaſy.

EXAMPLE 1.

Suppoſe it were required to find what 256 l. *would amount to in fifteen Years, at* 8 l. per Cent. per Annum *Compound Intereſt?* Here $t = 15$.

Firſt 100 : 108 :: 1 : 1,08 the Amount of 1 *l.* at 8 *per Cent.* Next 1,08 — 1,06 = 0,02 = x. And $R+x = 1{,}08$ as in Caſe 1. Then $R^{15} + 15 R^{14} x + 105 R^{13} xx + 455 R^{12} xxx$ &c. = the Amount of 1 *l.* for 15 Years, at 8 *per Cent.*

Here $x = 0{,}02$. $xx = 0{,}0004$. and $xxx = {,}000008$

By the Table $R^{15} = 2{,}396558$

And
$15 R^{14} x = 2{,}260904 \times 15 \times {,}02 = 0{,}678271$
$105 R^{13} xx = 2{,}132928 \times 105 \times {,}0004 = 0{,}089583$
$455 R^{12} xxx = 2{,}012196 \times 455 \times {,}000008 = 0{,}007324$

Sum = 3,171730

Then $3{,}171736 \times 256 = 811{,}964416 = A$.

That is, 811 *l.* 9 *s.* 3½ *d. ferè.* Which is the Anſwer required.

EXAMPLE 2.

What will 365 l. *amount to in ſeven Years at four and a half* per Cent. *&c.*

Firſt 100 : 104,5 :: 1 : 1,045 the Amount of 1 *l.* at 4½ *l. per Cent.*

Next 1,06 — 1,045 = 0,015 = x. Conſequently $R - x = 1{,}045$ as in Caſe 2.

Then $R^7 - 7 R^6 x + 21 R^5 xx - 35 R^4 xxx$ &c. = the Amount of 1 *l.* for 7 Years, at 4½ *per Cent.*

Here

Here $x = ,015$; $xx = ,000225$; and $xxx = 000003375$

By the Table $R^7 = +1,503630$

And $\begin{cases} -7R^6x = -0,148944 \\ +21R^5xx = +0,006323 \\ -35R^4xxx = -0,000141 \end{cases}$

$R^7 - 7R^6 + 21R^5xx - 35R^4xxx = 1,360868$

Then $1,360868 \times 365 = 496,71682 = A$.

That is, 496*l.* 14*s.* $3\frac{1}{4}d.$ is the Anſwer required.

If the Reaſon of theſe two Operations be but well underſtood, it will be very eaſy to conceive how to find P, the Principal, by having A, t, and x given (becauſe R and it's Powers are always given by the Table.)

For $R^t \pm tR^bx + gR^cxx \pm mR^dxxx \times P = A$ (as above).

Therefore $\frac{A}{R^t \pm tR^bx + gR^cxx \pm mR^dxxx} = P$.

Or if A, P, and t, be given, x may be found.

For $R^t \pm tR^bx + gR^cxx \pm mR^dxxx = \frac{A}{P}$. This Equation being ſolved (as in Chap. 10.) the Value of x will be found; and then either $R+x$, or $R-x$ will ſhew the Rate of Intereſt, *&c.*

But I ſhall leave the numerical Operations to the Learner's Practice, ſuppoſing enough done to ſhew how all Queſtions of this Kind that are limited by whole Years may be computed.

And if the Time given or ſought be not terminated by whole Years, but by Weeks, Months, Quarters, or Half-Years, *&c.* for reſolving ſuch Queſtions, the beſt way will be to reduce thoſe Parts of a Year into Days; that done, find an Anſwer according to the Demand of the Queſtion (and agreeing to 1*l.* as before) for that Number of Days; and in order to that, it will be requiſite to find the Amount of 1*l.* for one Day *(as in my Compendium of Algebra,* Page 110.) which I ſhall here inſert.

Put a = the Amount ſought, then it will be

$1 : a :: a : aa :: aa : aaa :: aaa : aaaa \div\div$ to a^{365}

That is { *As one Pound is to it's Amount for one Day : : ſo is that Amount : to the Amount of two Days : : and ſo is that of two Days : to that of three Days. And ſo on in* $\div\div$ *to* 365 *Days.*

Then the last of the Terms will be $a^{365} = 1,06$

Put	1	$r + e = a$. And let $r = 1$
$1 \circledcirc^{365}$	2	$r^{365} + 365r^{364}e + 66430r^{363}ee = a^{365} = 1,06$
2 in Numb.	3	$1 + 365e + 66430ee = 1,06$
$3 - \bar{1}$	4	$365e + 66430ee = 0,06$
$4 \div 66430$	5	$,00549e + ee = 0,0000009032 = D$
$5 \div$	6	$e = \dfrac{D}{,00549 + e}$

Operation ,00549
$+e =$,0001

1st Divisor ,00559) 0,0000009032 (1,0000000 = r
0,0001598 = e
1,0001598 = r + e = a

$+e =$,00015
2d Divisor ,00574
$+e =$,000059
3d Divisor ,005799
&c.

559
)3442
2870
)57200
&c.

true to the 7th Figure and only too much by 2 in the 8th, at one Operation.

New $r = 1,00016$ for a second Operation. Then

2 in Numb.	7	$1,06013401407 + 386,887e + 70402,172ee = 1,06$. Hence it appears that $r - e = a$.
Therefore	8	$1,06013401407 - 386,887e + 70402,172ee = 1,06$
$8 \pm$	9	$386,887e - 70402,172ee = 0,00013401407$
$9 \div$	10	$,0054953 - ee = ,0000000019035503$
$10 \div$	11	$e = \dfrac{,0000000019035503}{,0054953 - e}$

Operation ,0054953
$-e =$ 3

1st Divisor ,0054950) 0,0000000019035503 (1,00016 = r
0,000000346417 = e
1,000159653583 = r
$-e$

34
,00549466
$-e =$ 46
2d Divisor ,005494614
$-e =$ 64
3d Divisor ,00549460

164850
)2550503
2197864
)35263900
32967684
)2296216
2197840
98376
54946
&c.

Which being further pursued to a third Operation will give $a = 1,000159653587453$ &c.

This

This Value of *a* is the Amount of 1 *l.* for one Day, from which, if 1 *l.* be ſubſtracted, the Remainder =,000159653587 &c. will be the Intereſt of 1 *l.* for one Day. Conſequently, if any propoſed Principal be multiplied into either of theſe, the reſpective Product will be the Amount or Intereſt of that Principal for one Day, at 6 *per Cent.* &c.

And that the Amount (or Intereſt) of any Principal or Sum may be eaſily computed for any Number of Days leſs than a Year; I have here inſerted the following Table, which with a great deal of Care (and I believe Exactneſs) is calculated from the laſt found (1,000159653587453) Amount of 1 *l.* for one Day. To which alſo is annexed a Table of the Amounts of 1 *l.* for Months.

Days	Amounts of 1 *l.* &c.	Days	Amounts of 1 *l.* &c.	Days	Amounts of 1 *l.* &c.
1	1.0001596536	26	1.0041592879	51	1.0081749166
2	1.0003193326	27	1.0043196055	52	1.0083358753
3	1.0004790372	28	1.0044799487	53	1.0084968597
4	1.0006387673	29	1.0046403175	54	1.0086578699
5	1.0007985229	30	1.0048007120	55	1.0088189057
6	1.0009583039	31	1.0049611320	56	1.0089799673
7	1.0011181105	32	1.0051215776	57	1.0091410545
8	1.0012779426	33	1.0052820488	58	1.0093021675
9	1.0014378002	34	1.0054425457	59	1.0094633062
10	1.0015976834	35	1.0056030682	60	1.0096244707
11	1.0017575920	36	1.0057636164	61	1.0097856608
12	1.0019175262	37	1.0059241901	62	1.0099468767
13	1.0020774859	38	1.0060847895	63	1.0101081184
14	1.0022374712	39	1.0062454146	64	1.0102693858
15	1.0023974820	40	1.0064060653	65	1.0104306789
16	1.0025575184	41	1.0065667416	66	1.0105919978
17	1.0027175803	42	1.0067274436	67	1.0107533424
18	1.0028776677	43	1.0068881712	68	1.0109147128
19	1.0030377808	44	1.0070889245	69	1.0110761090
20	1.0031979193	45	1.0072097035	70	1.0112375309
21	1.0033580850	46	1.0073705082	71	1.0113989786
22	1.0035182732	47	1.0075313385	72	1.0115604521
23	1.0036784885	48	1.0076921945	73	1.0117219513
24	1.0038387294	49	1.0078530762	74	1.0118834764
25	1.0039989958	50	1.0080139835	75	1.0120450272

Days	Amounts of 1 l. &c.	Days	Amounts of 1 l. &c.	Days	Amounts of 1 l. &c.
76	1.0122066038	116	1.0186908655	156	1.0252166658
77	1.0123682062	117	1.0188535031	157	1.0253803453
78	1.0125398344	118	1.0190161667	158	1.0255440509
79	1.0126914885	119	1.0191788563	159	1.0257077827
80	1.0128531683	120	1.0193415719	160	1.0258715406
81	1.0130148739	121	1.0195043134	161	1.0260353247
82	1.0131766054	122	1.0196670809	162	1.0261991349
83	1.0133383627	123	1.0198298745	163	1.0263629713
84	1.0135001458	124	1.0199926934	164	1.0265268338
85	1.0136619547	125	1.0201555389	165	1.0266907225
86	1.0138927895	126	1.0203184110	166	1.0268546374
87	1.0139856501	127	1.0204813084	167	1.0270185784
88	1.0141475365	128	1.0206442319	168	1.0271825456
89	1.0143094488	129	1.0208071814	169	1.0273465389
90	1.0144713869	130	1.0209701569	170	1.0275105585
91	1.0146333511	131	1.0211331585	171	1.0276746046
92	1.0147953408	132	1.0212961861	172	1.0278386764
93	1.0149573565	133	1.0214592397	173	1.0280027746
94	1.0151193981	134	1.0216223193	174	1.0281668989
95	1.0152814655	135	1.0217854250	175	1.0283310494
96	1.0154435589	136	1.0219485567	176	1.0284952262
97	1.0156056781	137	1.0221117144	177	1.0286594291
98	1.0157678232	138	1.0222748982	178	1.0288236583
99	1.0159299941	139	1.0224381081	179	1.0289879137
100	1.0160921910	140	1.0226013440	180	1.0291521953
101	1.0162544138	141	1.0227646060	181	1.0293160231
102	1.0164166624	142	1.0229278940	182	1.0294908372
103	1.0165789370	143	1.0230902081	183	1.0296451975
104	1.0167412375	144	1.0232545483	184	1.0298095841
105	1.0169035638	145	1.0234179146	185	1.0299739969
106	1.0170659161	146	1.0235813069	186	1.0301384359
107	1.0172282944	147	1.0237447253	187	1.0303029012
108	1.0173906985	148	1.0239081699	188	1.0304673928
109	1.0175513086	149	1.0240716405	189	1.0306319206
110	1.0177155846	150	1.0242351372	190	1.0307964557
111	1.0178780665	151	1.0243986600	191	1.0309610251
112	1.0180405744	152	1.0245622089	192	1.0311256216
113	1.0182031083	153	1.0247257830	193	1.0312902445
114	1.0183656680	154	1.0248893851	194	1.0314548937
115	1.0185282578	155	1.0250530124	195	1.0316195692

Days	Amounts of 1 l. &c.	Days	Amounts of 1 l. &c.	Days	Amounts of 1 l. &c.
196	1.0317842709	236	1.0383939484	276	1.0450459680
197	1.0319489990	237	1.0385597318	277	1.0452128133
198	1.0321137534	238	1.0387255415	278	1.0453796853
199	1.0322785341	239	1.0388913778	279	1.0455446584
200	1.0324433410	240	1.0390572405	280	1.0457135092
201	1.0326081742	241	1.0392231298	281	1.0458804611
202	1.0327730339	242	1.0393890454	282	1.0460474397
203	1.0329379198	243	1.0395549876	283	1.0462144449
204	1.0331028321	244	1.0397209563	284	1.0463814768
205	1.0332677706	245	1.0398869515	285	1.0465484353
206	1.0334327355	246	1.0400529732	286	1.0467156206
207	1.0335977268	247	1.0402190214	287	1.0468827325
208	1.0337627444	248	1.0403850961	288	1.0470498711
209	1.0339277883	249	1.0405511973	289	1.0472170363
210	1.0340928586	250	1.0407173250	290	1.0473842283
211	1.0342579552	251	1.0408834793	291	1.0475514469
212	1.0344230782	252	1.0410496601	292	1.0477186923
213	1.0345882275	253	1.0412158674	293	1.0478859643
214	1.0347534033	254	1.0413821012	294	1.0480532631
215	1.0349186054	255	1.0415483616	295	1.0482205885
216	1.0350838338	256	1.0417146485	296	1.0483879407
217	1.0352490887	257	1.0418809620	297	1.0485553196
218	1.0354143699	258	1.0420473021	298	1.0487227252
219	1.0355796775	259	1.0422136687	299	1.0488901576
220	1.0357450115	260	1.0423800618	300	1.0490576166
221	1.0359103719	261	1.0425464815	301	1.0492251025
222	1.0360757587	262	1.0427129278	302	1.0493926150
223	1.0362411719	263	1.0428794007	303	1.0495601543
224	1.0364066116	264	1.0430459001	304	1.0497277204
225	1.0365710776	265	1.0432124261	305	1.0498953132
226	1.0367375701	266	1.0433789787	306	1.0500629327
227	1.0369030889	267	1.0435455579	307	1.0502305790
228	1.0370686342	268	1.0437121637	308	1.0503982521
229	1.0372342059	269	1.0438787961	309	1.0505659519
230	1.0373998041	270	1.0440454551	310	1.0507336786
231	1.0375654287	271	1.0442121407	311	1.0509014320
232	1.0377310798	272	1.0443788529	312	1.0510692121
233	1.0378967573	273	1.0445455918	313	1.0512370191
234	1.0380624612	274	1.0447123572	314	1.0514048529
235	1.0382241916	275	1.0448791493	315	1.0515727134

Days	Amounts of 1 *l.* &c.	Days	Amounts of 1 *l.* &c.	Days	Amounts of 1 *l.* &c.
316	1.0517406008	339	1.0556094165	362	1.0594924636
317	1.0519085150	340	1.0557779484	363	1.0596616154
318	1.0520764559	341	1.0559465071	364	1.0598307942
319	1.0522444237	342	1.0561150927	365	1.06
320	1.0524124183	343	1.0562837053		
321	1.0525804397	344	1.0564523448		
322	1.0527484880	345	1.0566210112		
323	1.0529165631	346	1.0567897045		The Amounts of 1 *l.* at 6 *per Cent.* For Months.
324	1.0530846650	347	1.0569584248		
325	1.0532527937	348	1.0571271720		
326	1.0534209493	349	1.0572959594		
327	1.0535891317	350	1.0574647472	1	1.0048675505
328	1.0537573410	351	1.0576335753	2	1.0097587942
329	1.0539255771	352	1.0578024303	3	1.0146738462
330	1.0540938401	353	1.0579713122	4	1.0196128224
331	1.0542621300	354	1.0581402211	5	1.0245758304
332	1.0544304467	355	1.0583091570	6	1.0295630141
333	1.0545987903	356	1.0584781199	7	1.0345744641
334	1.0547671608	357	1.0586471097	8	1.0396103076
335	1.0549355582	358	1.0588161265	9	1.0446706634
336	1.0551039824	359	1.0589851703	10	1.0497556507
337	1.0552724336	360	1.0591542411	11	1.0548653894
338	1.0554409116	361	1.0593233389	12	1.06

The Use of this Table is in all respects like that of whole Years, in finding the Amount of any given Sum for any proposed Number of Days less than a Year.

EXAMPLE 1.

Suppose it were required to find the Amount of 375 *l.* for 210 *Days, at* 6 per Cent.

The Amount of 1 *l* for 210 Days is 1,0340928 *&c.* per Table. Then $1{,}0340928 \times 375 = 387{,}7848$ *&c.* $= 387\ l.\ 15\ s.\ 8\frac{1}{2}\ d.$ which is the Amount required. And the rest of the Variations may be performed just as in the Examples of whole Years.

But if the Time given consists of Years, and Parts of a Year; as Quarters, Months, *&c.* Then reduce the odd Time or Parts of the Year into Days; and the Answer may then be found at two Operations; as in the following *Example.*

EXAMPLE

Example 2. *Suppoſe it were required to find what* 265 *l. would amount to in five Years and* 135 *Days at* 6 per Cent. *&c.*

First, the Amount of 1 *l.* for { 5 Years is 1,338225, *&c.* / 135 Days is 1,021785, *&c.*

Then 1,338225 × 1,021785 × 265 *l.* = 362,355232, *&c.* being the *Amount* or *Anſwer* required.

Or, if the *Amount* and *Time* are given, to find the *Principal:* Then *multiply* the *Amount* of 1 *l.* for the *Years*, and the *Amount* of 1 *l.* for the odd Days together: And by their *Product* divide the given *Amount*, the *Quotient* will be the *Principal* required.

Example 3. *What Principal will raiſe a Stock of* 362 *l.* 7 *s.* 1 $\frac{1}{4}$ *d. Or* 362,355232 *l. in* 5 *Years and* 135 *Days, at* 6 per Cent. *&c.*

The Amount of 1 for { 5 Years is 1,338225, *&c.* / 135 Days is 1,021785, *&c.*

Then 1,338225 × 1,021785 = 1,367378, &c. the *Diviſor.* Next 1,367378) 362,355232 = *A* (265 *l.* the *Principal* required.

Again, if the *Principal* and its *Amount* are given, to find the *Time*, at 6 *per Cent.* &c. you muſt divide the *Amount* by its *Principal*, and then proceed as in the *Third Example*, Page 256, for the *Anſwer required.*

But if the *Amount* and its *Principal*, with the *Time* of its being at *Intereſt*, are given, to find the *Rate* of *Intereſt:* Then proceed as in the Fourth *Queſtion*, Page 255, *&c.*

Now in order to make this *Table* of *Amounts* for Days uſeful for all *Rates* of *Intereſt (as before in that for Years)* you muſt firſt find the Simple *Intereſt* of 1 *l.* for one Day, both at the given *Rate*, and alſo at 6 *per Cent.* And call their Difference *x.*

Thus, ſuppoſe the given *Ratio* were 8 *per Cent. per Annum*, Firſt 130 : 8 : : 1 : 0,08 And 100 : 6 : : 1 : 0,06 the Two Simple *Intereſts* for one Year.

Then 365) 0,08 (0,00021917, *&c.* the *Simple Intereſt* of 1 *l.* for one Day, at 8 *per Cent.*

And 365) 0,06 (0,00016438, *&c.* the *Simple Intereſt* of 1 *l.* for one Day, at 6 *per Cent.*

Their *Difference* 0,00005479 = *x*, which may do indifferently well for ordinary ſmall *Queſtions:* But where Exactneſs is required, it will be convenient to make Uſe of this *Proportion:*

Viz. { As the Simple Intereſt of 1 *l.* for one Day at 6 *per Cent.* : Is to the Tabular Intereſt of 1 *l.* for one Day : : So is the Simple Intereſt of 1 *l.* for one Day, at any given Rate : To a Fourth Number.

That is, 0,00016438 : 0,00015965 :: 0,00021917 : 0,00021286
Then 0,00021286—0,00015965 = 0,00005321 = *x*.

This *x* being *involved* with the reſpective *Amounts* for Days, in the ſame Manner as was done with thoſe for *Years* (*vide Page* 258) the Reſult will be the *Anſwer* to the *Queſtion*.

Sect. 2. Annuities or Penſions *in Arrear, computed at Compound Intereſt.*

When *Annuities*, &c. are ſaid to be in *Arrear*, ſee *Page* 248. And I ſhall here make Uſe of the ſame Letters to repreſent the ſame Things as before in that *Page*, ſave only that *R* is here equal to the *Amount* of 1 *l.* as in *Section* 1. of this *Chapter*.

Suppoſe *u* = the Firſt *Year's Rent* of any Annuity without *Intereſt*.

Then will $Ru + u =$ { the *Amount* of the Firſt *Year's Rent*, and its *Intereſts* ; More the 2d *Year's Rent*.

And $RRu + Ru + u =$ { the *Amount* of the 1ſt and 2d *Years Rents*, with their *Intereſts* ; More the 3d *Year's Rent*, &c.

Here $RRu + Ru + u = A$, the *Amount* of any *Yearly Rent* or *Annuity*, being forborne *Three Years*. And from hence may be deduced theſe *Proportions* :
Viz. $u : Ru :: Ru : RRu :: RRu : RRRu$, and ſo on in $\div\div$ for any *Number* of *Terms* or *Years* denoted by *t*, wherein the laſt Term will always be uR^{t-1}.
Conſequently, $A - uR^{t-1}$ = the Sum of all the Antecedents
And $A - u$ = the Sum of all the Conſequents in the Series.
And therefore it would be $u : uR :: A - uR^{t-1} : A - u$, Vide *Page* 188.
Ergo $Au - uu = RuA - uuR^{t}$, which, being divided all by *u*, will become $A - u = RA - uR^{t}$.

From this laſt *Æquation* it will be eaſy to raiſe the following *Theorems* :

Theorem 1. $\frac{uR^{t}-u}{R-1} = A$. *Theorem* 2. $\frac{RA-A}{R-1} = u$.

Theorem

Theorem 3. $\left\{\frac{RA+u-A}{u} = R^t\right.$. If this *Æquation* be continually *divided* by R, until nothing *remain*, the *Number* of those *Divisions* will be t. See *Page* 255.

Theorem 4. $\left\{\frac{A}{u}R - R^t = \frac{A-u}{u}\right.$. If this *Æquation* be resolved into *Numbers*, according to the Method proposed in *Sect.* 3. *Chap.* 10. the *Root* will shew the *Value* of R.

Question 1. *If* 30 *l. Yearly Rent, or Annuity*, &c. *be forborne* (i. e. *remain unpaid) Nine Years; what will it amount to, at* 6 per Cent. per Annum, *Compound Interest?*

Here is given $u = 30$, $t = 9$, and $R = 1{,}06$; to find A. per *Theorem* 1.

$R^9 = 1{,}689479$ By the Table of *Amounts* for *Years*
$30 = u$

$R^9u = 50{,}684370$
$-u = 30,$

$R - 1 = 0{,}06)\ 20{,}684370\ (344{,}7395 = 344\ l.\ 14\ s.\ 9\frac{1}{2}\ d. = A$, the Amount required.

Question 2. *What Yearly Rent or Annuity*, &c. *being forborne or unpaid Nine Years, will raise a Stock of* 344 *l.* 14 *s.* $9\frac{1}{2}$ *d.* = 344,7395, *at* 6 per Cent. &c.

Here is given $A = 344{,}7395$, $t = 9$, and $R = 1{,}06$; to find u. per *Theorem* 2.

$AR = 344{,}7395 \times 1{,}06 = 365{,}42387$
$-A = 344{,}7395$

$R^t - 1 = 1{,}689479 - 1 = 0{,}689479)\ 20{,}68437\ (30 = u.$

Question 3. *In what Time will* 30 *l. Yearly Rent raise a Stock or Amount to* 344 *l.* 14 *s.* $9\frac{1}{2}$ *d. allowing* 6 per Cent. *for the Forbearance of Payments?*

Here is given $u = 30$, $A = 344{,}7395$, and $R = 1{,}06$; to find t. per *Theorem* 3.

First $AR + u - A = 365{,}42387 + 30 - 344{,}7395 = 50{,}68437$.
And $u = 30)\ 50{,}68437\ (1{,}689479 = R^t$. Then
$R = 1{,}06)\ 1{,}689479\ (1{,}593848$. And $1{,}06)\ 1{,}593848\ (1\ 50363$; and so on until it become $1{,}06)\ 1{,}06\ (1$. which will be at the *Ninth Division*; therefore $t = 9$.

Or $R^t = 1{,}689479$, being ſought in the *Table* of *Amounts* for *Years*, will be found to ſtand over-againſt 9 *Years*, which is the *Time required.*

Question 4. *If* 30 *l.* per Annum, *being unpaid Nine Years, will amount to* 344 *l.* 14 *s.* $9\frac{1}{2}$ *d. allowing Compound Intereſt for every Payment as it becomes due, What muſt the Rate of Intereſt be* per Cent. *&c.*

Here is given $u = 30$, $A = 344{,}7395$, and $t = 9$; to find R by the laſt of the Four *Æquations*, *Viz.* $\left\{ \frac{A}{u} R - R^t = \frac{A-u}{u} \right.$

First $\frac{A}{u} = \frac{344{,}7395}{30} = 11{,}491317$. And $\frac{A-u}{u} = 10{,}491317$.

Hence there is this *Æquation*; $11{,}491317\ R - R^9 = 10{,}491317$.

Let	1	$r + e = R$, and ſuppoſe $r = 1$
1 & 9	2	$r^9 + 9r^8e + 36\,r^7\,ee = R^9$
$1 \times \frac{z}{u}$, in Numb.	3	$11{,}491317 + 11{,}491317\,e = 11{,}491317\ R$
2 in Numb	4	$1{,}000000 + 9{,}000000e + 36\,ee = R^9$
3—4	5	$10{,}491317 + 2{,}491317e - 36ee = 10{,}491317$
Whence	6	$36ee = 2{,}491317\,e$
$6 \div 36\,e$	7	$e = 0{,}06$, *&c.*

First $r = 1$, $+ e = 0{,}06$ $\left\{ = 1{,}06 = R \right.$ *As may be eaſily try'd by involving it, and ordering it, as the Æquation above directs.*

Section 3. *To find the* Preſent Worth *of Annuities, Penſions, or Leaſes,* &c. *at Compound Intereſt.*

Let $P =$ the *preſent Worth* of any *Annuity*, or *Leaſe*, &c. and the reſt of the Letters as before.

Then, from what has been ſaid in *Section* 3. *Chap.* 11. about *Purchaſing* of *Annuities*, &c. at Simple *Intereſt*, it will be eaſy to form the like *Theorems* here at *Compound Intereſt*, *viz.* by Combining *Theorem* 1. *Page* 266. and *Theorem* 1. *Page* 254. into one *Theorem.*

For $\left\{ \frac{uR^t - u}{R-1} = A \right.$ *The Amount of any Yearly Rent being unpaid any Number of Years. Per* Theorem 1. *of the laſt Section.* Page 266.

And $PR^t = A$ *The Amount of any Principal or Sum being put to Intereſt, for the ſame Number of Years. Per* Theorem 1. *Page* 254.

Hence it follows, That $PR^t = \frac{uR^t - u}{R-1}$,

Viz. $PR^{t\times 1} - PR^t = uR^t - u$ being the very ſame *Æquation* with that in my *Compendium* of *Algebra*, *Page* 112. which is there raiſed from the Conſideration of purchaſing *Annuities*, or taking of *Leaſes*, &c. to be grounded upon a *Rank* or *Series* of *Geometrical Proportionals* continually decreaſing. Thus $\frac{u}{R}$ is the *Firſt* and *Greateſt Term*; R the common *Ratio* of all the *Terms*; and P is the *Sum* of all the *Series*.

That is, $\frac{u}{R} : \frac{u}{RR} :: \frac{u}{RR} : \frac{u}{RRR} :: \frac{u}{RRR} : \frac{u}{R^4} :: \frac{u}{R^4} : \frac{u}{R^5}$, &c. in $\frac{\cdot\cdot}{\cdot\cdot}$ until the laſt *Term* $= \frac{u}{R^t}$. Then will $P - \frac{u}{R^t}$ be the *Sum* of all the *Antecedents*, and $P - \frac{u}{R}$ the *Sum* of all the *Conſequents*. Therefore it will be

$\frac{u}{R} : \frac{u}{RR}$. Or (in the ſame Ratio) $u : \frac{u}{R} :: P - \frac{u}{R^t} : P - \frac{u}{R}$, which produces $PR^{t\times 1} - uR^t = PR^t - u$. As above.

From this *Æquation* may be deduced the following *Theorems*:

Theorem 1. $\left\{ u = \frac{\frac{u}{R^t}}{R-1} = P \right.$. *Theorem* 2. $\left\{ \frac{PR^t \times R : PR^t}{R-1} = u \right.$.

Theorem 3. $\left\{ \frac{u}{P + u - PR} = R^t \right\}$ *Which, being continually divided by* R, *will give t.*

Theorem 4. $\left\{ \frac{u}{P} = \frac{u}{P} R^t + R^t - R^{t\times 1} \right.$. The Reſolving of which *Æquation* will diſcover the Value of R.

Queſtion 1. *What is* 30 *l. Yearly Rent, to continue Seven Years, worth in ready Money, allowing* 6 per Cent. *Compound Intereſt to the Purchaſer?*

Here is given $u = 30$. $t = 7$. And $R = 1{,}06$ to find P. per *Theorem* 1. *Viz.* $\frac{u}{R^t} = \frac{30}{1{,}50303} = 19{,}9517$.

And $30 - 19{,}9517 = 10{,}483 = u - \frac{u}{R^t}$.

Then

Then $R-1=0,06)10,0483(167,4716=P=167\,l.\ 9\,s.\ 5\,d.$ being the Anſwer required.

Queſtion 2. *What Annuity or Yearly Rent, to continue Seven Years, may be purchaſed for* 167 *l.* 9 *s.* 5 *d. allowing* 6 per Cent. *Compound Intereſt to the Purchaſer?*

In this *Queſtion* there is given $P=167,4716$. $t=7$
And $R=1,06$ to find u. By the Second *Theorem.*
Firſt $PR^t\times R=251,8153\times 1,06=266\ 9242$
And $-PR^t=167,4716\times 1,50363)=251,8153$

Then $R^t-1=0,50363)\quad 15,1089\quad (30=u$
That is $u=30\,l.$ the *Anſwer required.*

Queſtion 3. *How long may one have a Leaſe of* 30 *l. Yearly Rent, for* 167 *l.* 9 *s.* 5 *d. allowing* 6 per Cent. *Compound Intereſt to the Purchaſer?*

Here is given $P=167,4716$. $u=30$. And $R=1,06$ to find t. By the Third *Theorem.*
Firſt $P+u=167\ 4716+30=197,4716$
And $-PR=177.5199$

Then $19,9517)30=u(1,50363=R^t$

If this $1,50363=R^t$ be either continually *divided* by $1,06=R$ until nothing *remain* (As before in *Page* 255.) Or if it be ſought in the *Table* of *Amounts* for *Years*, &c. it will diſcover $t=7$ which is the true *Anſwer* required.

Queſtion 4. *Suppoſe one ſhould give* 167 *l.* 9 *s.* 5 *d. for the Purchaſe of a Penſion, or Annuity of* 30 *l.* per Annum, *to continue Seven Years: At what Rate of Intereſt,* per Cent. *would that Purchaſe be made, allowing Compound Intereſt to the Purchaſer?*

In this *Queſtion* there is given, $P=167,4716$. $u=30$ and $t=7$ to find R. Per *Theorem* 4 in this *Æquation* $\left\{\frac{u}{P}=\frac{u}{P}\right.$ $R^t+R^t-R^t\times 1$, which being brought into *Numbers*, and its *Root extracted*, as in the fourth *Queſtion* of the laſt *Section*; the Value of R will be found 1,06, and then it will be 1 : 0,06 : : 100 : 6, the Rate *per Cent.* as was required.

Theſe

Theſe Four *Queſtions* include all the *Varieties* that can be propoſed about purchaſing *Annuities* or *Leaſes*, &c. which are to be either immediately enter'd upon, or in Poſſeſſion at the Time when the *Purchaſe* is made.

But ſuch *Queſtions* as relate to *Annuities*, or a taking of *Leaſes*, &c. in *Reverſion*, muſt be *parted* or *divided* into *two diſtinct Queſtions*, each to be ſeparately conſider'd by itſelf (See *Page* 252.) As in the following *Examples:*

Example 1. *Suppoſe it were requir'd to compute the preſent Worth of* 75 *l. Yearly Rent, which is not to commence or be enter'd upon, until Ten Years hence; and then to continue Seven Years after that Time: at* 6 per Cent. *&c. Compound Intereſt?*

The Firſt Work in this *Queſtion* is, to find what 75 *l. per Annum*, to continue *Seven Years*, is worth in ready Money; as if it were to be immediately enter'd upon: And to perform that, there is given $u = 75$. $R = 1{,}06$. and $t = 7$. to find P. as in the Firſt *Queſtion* of this *Section*.

Thus, $\frac{u}{R^t} = \frac{75}{1{,}503\,63} = 49{,}8793$ And $75 - 49{,}8793 = 25{,}1207$ $= u - \frac{u}{R}$.

Then, $R - 1 = 0{,}06)\ 25{,}1207 = 418{,}6783 = 418$ *l.* 14 *s.* 6$\frac{3}{4}$ *d.* the *Anſwer* to the Firſt Part of the *Queſtion*.

Then the next Work will be, to find what *Principal* or *Sum* being put out Ten *Years*, at 6 *per Cent.* &c. will amount to 418 *l.* 14 *s.* 6$\frac{3}{4}$ *d.* Here is given $A = 418{,}6783$, $R = 1{,}06$, $t = 10$, to find P. Per *Theorem* 2. *Page* 254.

Thus $R^{10} = 1.790847)\ 418{,}6783 = A\ (233{,}7884 = 233$ *l.* 15 *s.* 9 *d.* the *preſent Worth of* 75 *l. per Annum in Reverſion*, &c. As was required.

Example 2. *What Annuity or Yearly Rent, to be enter'd upon Ten Years hence, and then to continue Seven Years, may be purchaſed for* 233 *l.* 15 *s.* 9 *d. Ready Money, at* 6 per Cent. *&c. Compound Intereſt?*

In the 1ſt Work of this *Queſtion* there is given, $P = 233{,}7884$ $R = 1{,}06$. And $t = 10$ (*the Time which the Annuity is not to be enter'd upon*) to find A. Per *Theorem* 1. Page 254.

Thus, $PR^t = 233{,}7884 \times 1{,}790847 = 418{,}6783 = A$ the *Amount*

Amount of 233*l.* 15*s.* 9*d.* put to *Intereſt Ten Years*, at 6 *per Cent.* &c. Then, for the Second Work of the *Queſtion*, there is given $P = 418,6783$. $R = 1,06$. And $t = 7$ *(the Time that the Annuity is to be enjoy'd)* to find u. Per *Theorem* 2. of this *Section*.

Thus $PR^t \times R = 418,6783 \times 1,50363 \times 1,06 = 667,3095$

$-PR^t = 418,6783 \times 1,50363 = 629,5372$

$R^t - 1 = 0,50363)\ 37,7723(75 = u$

That is, $u = 75l.$ the *Yearly Rent* required by the *Queſtion*.

Theſe *Two Examples* of finding P and u do fully ſhew the *Method* that muſt be uſed in *Reſolving* the two *General*, and indeed the moſt uſeful *Queſtions* about *Annuities* or *Leaſes* in *Reverſion*: And if there be Occaſion, either the *Rate*, or the *Time*, viz. R or t, may be found by a due Application of their reſpective *Theorems*.

Note, *That which hath been done in the two laſt Sections about Annuities or Yearly Rents*, &c. *at* 6 per Cent. *may alſo be done for any Rate of Intereſt, by applying the Difference of the Rates* (viz. *x.*) *As directed in the Firſt Section of this Chapter.*

Now becauſe that *Rents* and *Annuities*, &c. are uſually paid either by *Quarterly* or *Half Yearly Payments*, and the Method of computing them by the Pen may be thought a little troubleſome; I have inſerted the following Tables of the *Amounts* of 1 *l.* for each, at 6 *per Cent.*

= *t.* Half Years	Annuities of 1 *l.* at 6 *per Cent.* Compound Intereſt.	= *t.* Half Years	Annuities of 1 *l.* at 6 *per Cent.* Compound Intereſt.	= *t.* Half Years	Annuities of 1 *l.* at 6 *per Cent.* Compound Intereſt.
1	1,0295630141	11	1,3777875592	21	1,8437905523
2	1,06	12	1,4185191122	22	1,8982985583
3	1,0613367949	13	1,4604548127	23	1,9544179853
4	1,1236	14	1,5036302590	24	2,0121964718
5	1,1568170026	15	1,5480821017	25	2,0716830644
6	1,191016	16	1,5938480745	26	2,1329282601
7	1,2262260228	17	1,6409670276	27	2,1959840483
8	1,26247696	18	1,6894789589	28	2,2609039557
9	1,2997995842	19	1,7394250493	29	2,3277430912
10	1,3382255776	20	1,7908476965	30	2,3965581001

Quar-

Quarterly Amounts.

Quarters of a Year = *t*.	Amounts of *1* at 6 *per Cent.* &c. Compound Intereſt.	Quarters of a Year = *t*.	Amounts of *1* at 6 *per Cent.* &c Compound Intereſt.	Quarters of a Year = *t*.	Amounts of *1* at 6 *per Cent.* &c. Compound Intereſt.
1	1,0146738461	21	1,3578024938	41	1,8171263199
2	1,0295630141	22	1,3777875592	42	1,8437905523
3	1,0446706634	23	1,3980050019	43	1,8708660509
4	1,06	24	1,4185191122	44	1,8982985583
5	1,0755542769	25	1,4393342435	45	1,9261538989
6	1,0913367949	26	1,4604548127	46	1,9544179853
7	1,1073509032	27	1,4818853020	47	1,9830968140
8	1,1236	28	1,5036302590	48	2,0121961718
9	1,1400875335	29	1,5256942978	49	2,0417231330
10	1,1568170026	30	1,5480821017	50	2,0716830644
11	1,1737919574	31	1,5707984203	51	2,1020826228
12	1,191016	32	1,5938480745	52	2,1329282601
13	1,2084927856	33	1,6172359557	53	2,1642265211
14	1,2262260228	34	1,6409670276	54	2,1959840483
15	1,2442194748	35	1,6650463253	55	2,2282075801
16	1,26247696	36	1,6894789589	56	2,2609039557
17	1,2810023527	37	1,7142701133	57	2,2940801123
18	1,2997995842	38	1,7394250493	58	2,3277430912
19	1,3188726433	39	1,7649491048	59	2,3619000349
20	1,33822,5776	40	1,7908476965	60	2,3965581931

Either of theſe *Tables* may alſo be made uſeful for any propoſed *Rate* of *Intereſt*, by making the $\frac{1}{2}$ or $\frac{1}{4}$ of the *Difference* of the *Rate* = *x*, &c.

As for Inſtance, ſuppoſe any of the aforeſaid *Queſtions* about *Annuities* or *Rents*, &c. were to be computed at 8 *per Cent. per Ann.*

Then 1,08 — 1,06 = 0,02 = *x* for *Yearly Payments*; as before.

Conſequently 2) 0,02 (0,01 = *x* for *Half Yearly Payments*.

Or 4) 0,02 (0,005 = *x* for *Quarterly Payments*.

Now theſe Values of *x*, although they are not really true, yet they may ſerve indifferently well for ſmall *Rents*; as I have already ſaid, *Page* 265. But if you would work exactly;

Then ✓ 1,08 = 1,0392304845, &c.

— ✓ 1,06 = 1,0295680141, *Vide Table*, *Page* 272.

Difference = 0,0096624704 = *x* for *Half Yearly Payments*.

And $\sqrt{:}\sqrt{}\ 1{,}08 = 1{,}0194263092$, &c.
$-\sqrt{:}\sqrt{}\ 1{,}06 = 1{,}0146738461$. See the *Laſt Table.*

Their Difference $0{,}0047524631 = x$, for *Quarterly Payments.*

Theſe are the true *Values* of x, which being *involved* with their reſpective *Amounts* (as before for *Years*, &c.) according as the *Queſtion* requires, the *Reſult* will be the *Anſwer* at 8 *per Cent.* &c. The like may be done for any other *Rate*, either *Greater* or *Leſs* than 6.

Now, although the Method uſed here (and in *Page* 257 and 258, &c.) be really true (by which the *Tables* calculated only for 6 *per Cent.* are made effectual for all Rates of *Compound Intereſt*) yet it was rather propos'd to ſhew what may poſſibly be performed by the Pen, without a great many *Tables* of ſeveral *Rates*, than intended for common Practice.

For it muſt needs be confeſs'd, that *Tables*, calculated on Purpoſe for any deſigned *Rate* of *Intereſt*, are much more ready and uſeful in common Practice. And therefore ſince the Legiſlative Power hath thought fit to reduce the *Rate* of *Intereſt*, and hath ſettled it by an Act of Parliament, at 5 *per Cent.* I've therefore been at the Trouble *(which was not a little)* to calculate the following *Tables* for that *Rate*; but don't think it convenient to take the *Tables* at 6 *per Cent.* out of the Book, becauſe the Examples are all ſuited to them; and not only ſo, but they may be found uſeful in the taking of *Leaſes* for Houſes, *&c.* For in thoſe Caſes, the *Purchaſer* is allowed more *Intereſt* for his Purchaſe Money, than the common *Rate* paid upon the Loan of Money.

Here

Here follow New *Tables* of the *Amounts* of one Pound at the *Rate* of 5 *per Cent. per Annum* Compound Intereſt. For *Years*, *Half Years*, *Quarters*, *Months*, and *Days*.

I. The *Table* of the *Yearly Amounts* of 1 *l.* &c.

= *t.* Years	The Amounts of 1 *l.* &c.	= *t.* Years	The Amounts of 1 *l.* &c.	= *t.* Years	The Amounts of 1 *l.* &c.
1	1,05 = *R*	14	1,97993160	27	3,73345632
2	1,1025 = *RR*	15	2,07892318	28	3,92012914
3	1,157625 = R^3			29	4,11613599
4	1,21550625	16	2,18287459	30	4,32194239
5	1,27628156	17	2,29201832		
		18	2,40661923	31	4,53803949
6	1,34009564	19	2,52695019	32	4,76494147
7	1,40710042	20	2,65329770	33	5,00318854
8	1,47745544			34	5,25334797
9	1,55132822	21	2,78596259	35	5,51601536
10	1,62889463	22	2,92526072		
		23	3,07152375	36	5,79181613
11	1,71033936	24	3,22509994	37	6,08140694
12	1,79585633	25	3,38635494	38	6,38547729
13	1,88564914	26	3,55567269	39	6,70475115

II. The *Table* of the *Half Yearly Amounts* of 1 *l.* &c.

= *t.* Half Yrs.	The Amounts of 1 *l.* &c.	= *t.* Half Yrs.	The Amounts of 1 *l.* &c.	= *t.* Half Yrs.	The Amounts of 1 *l.* &c.
1	1,02469507	11	1,30779943	21	1,66912030
2	1,05	12	1,34009564	22	1,71033936
3	1,07592983	13	1,37318940	23	1,75257632
4	1,1025	14	1,40710042	24	1,79585633
5	1,12972632	15	1,44184887	25	1,84020513
6	1,57625	16	1,47745544	26	1,88564914
7	1,18621264	17	1,51394132	27	1,93221539
8	1,21550625	18	1,55132822	28	1,97993160
9	1,24552327	19	1,58963838	29	2,02882616
10	1,27628156	20	1,62889463	30	2,07892818

III. The *Table* of the *Quarterly Amounts* of 1 *l.* &c.

= *t.* Quarters	The Amounts of 1 *l.* &c.	= *t.* Quarters	The Amounts of 1 *l.* &c.	= *t.* Quarters	The Amounts of 1 *l.* &c.
1	1,01227223	21	1,29194439	41	1,64388480
2	1,02469507	22	1,30779943	42	1,66912031
3	1,03727037	23	1,32384905	43	1,68960414
4	1,05	24	1,34009564	44	1,71033936
5	1,06288585	25	1,35654161	45	1,73132904
6	1,07592983	26	1,37318940	46	1,75257632
7	1,08913389	27	1,39004151	47	1,77408435
8	1,1025	28	1,40710042	48	1,79585633
9	1,11603014	29	1,42436869	49	1,81789549
10	1,12972632	30	1,44184887	50	1,84020513
11	1,14359059	31	1,45954358	51	1,86278836
12	1,157625	32	1,47745544	52	1,88564914
13	1,17183164	33	1,49558712	53	1,90879027
14	1,18621264	34	1,51394132	54	1,93221539
15	1,20077012	35	1,53252076	55	1,95592799
16	1,21550625	36	1,55132822	56	1,97993160
17	1,23042323	37	1,57036641	57	2,00422978
18	1,24552327	38	1,58963838	58	2,02882616
19	1,26080862	39	1,60914680	59	2,05372439
20	1,27628156	40	1,62889463	60	2,07892818

IV. The *Table* of the *Monthly Amounts* of 1 *l.* &c.

= *t.* Months	The Amounts of 1 *l.* &c.	= *t.* Months	The Amounts of 1 *l.* &c.	= *t.* Months	The Amounts of 1 *l.* &c.
1	1,00407412	5	1,02053728	9	1,03727037
2	1,00816485	6	1,02469507	10	1,04149634
3	1,01227223	7	1,02886981	11	1,04573953
4	1,01639636	8	1,03306155	12	1,05

NOTE: The *Amount* of one *Pound*, for one *Day*, is 1,00013368072 25, *&c.* (found as that in *Page* 260) but, in the following *Table*, I take only Nine of thoſe *Figures*, as being ſufficient in Practice, for computing the *Intereſt* of any *Sum* not exceeding One Hundred Millions of *Pounds*.

V. The

V. The *Table* of the *Daily Amounts* of 1 *l.* &c.

Days *t.*	The Amounts of 1 *l.* &c.	Days *t.*	The Amounts of 1 *l.* &c.	Days *t.*	The Amounts of 1 *l.* &c.
1	1,00013368	36	1,00482376	71	1,00953587
2	1,00026739	37	1,00495810	72	1,00967082
3	1,00040109	38	1,00509245	73	1,00980579
4	1,00053483	39	1,00522681	74	1,00994079
5	1,00066858	40	1,00536119	75	1,01007579
6	1,00080235	41	1,00549559	76	1,01021083
7	1,00093614	42	1,00563000	77	1,01034587
8	1,00106994	43	1,00576443	78	1,01048093
9	1,00120377	44	1,00589888	79	1,01061602
10	1,00133761	45	1,00603335	80	1,01075112
11	1,00147147	46	1,00616784	81	1,01088623
12	1,00160535	47	1,00630234	82	1,01102137
13	1,00173924	48	1,00643687	83	1,01115652
14	1,00187315	49	1,00657141	84	1,01129169
15	1,00200708	50	1,00670597	85	1,01142688
16	1,00214103	51	1,00684055	86	1,01156209
17	1,00227500	52	1,00697514	87	1,01169732
18	1,00240899	53	1,00710975	88	1,01183256
19	1,00254299	54	1,00724438	89	1,01196783
20	1,00267701	55	1,00737903	90	1,01210311
21	1,00281105	56	1,00751370	91	1,01223841
22	1,00294510	57	1,00764839	92	1,01237372
23	1,00307918	58	1,00778309	93	1,01250906
24	1,00321327	59	1,00791781	94	1,01264441
25	1,00334738	60	1,00805255	95	1,01277978
26	1,00348151	61	1,00818731	96	1,01291517
27	1,00361565	62	1,00832208	97	1,01305058
28	1,00374982	63	1,00845687	98	1,01318600
29	1,00388400	64	1,00859168	99	1,01332145
30	1,00401820	65	1,00872651	100	1,01345691
31	1,00415242	66	1,00886136	101	1,01359239
32	1,00428665	67	1,00899623	102	1,01372788
33	1,00442091	68	1,00913111	103	1,01386340
34	1,00455518	69	1,00926601	104	1,01399893
35	1,00468947	70	1,00940093	105	1,01413448

Days t. =	The Amounts of 1 l. &c.
106	1,01427005
107	1,01440564
108	1,01454125
109	1,01467687
110	1,01481252
111	1,01494818
112	1,01508386
113	1,01521955
114	1,01535527
115	1,01549100
116	1,01562675
117	1,01576252
118	1,01589831
119	1,01603412
120	1,01616994
121	1,01630578
122	1,01644164
123	1,01657752
124	1,01671349
125	1,01684933
126	1,01698527
127	1,01712122
128	1,01725719
129	1,01739317
130	1,01752918
131	1,01766521
132	1,01780125
133	1,01793731
134	1,01807338
135	1,01820948
136	1,01834559
137	1,01848173
138	1,01861788
139	1,01875405
140	1,01889024
141	1,01902644
142	1,01916267
143	1,01929891
144	1,01943517
145	1,01957145

Days t. =	The Amounts of 1 l. &c.
146	1,01970775
147	1,01984406
148	1,01998039
149	1,02011675
150	1,02025312
151	1,02038950
152	1,02052591
153	1,02066234
154	1,02079878
155	1,02093524
156	1,02107172
157	1,02120822
158	1,02134473
159	1,02148127
160	1,02161782
161	1,02175439
162	1,02189098
163	1,02202758
164	1,02216421
165	1,02230085
166	1,02243751
167	1,02257419
168	1,02271089
169	1,02284761
170	1,02298434
171	1,02312109
172	1,02325787
173	1,02339466
174	1,02353147
175	1,02366829
176	1,02380514
177	1,02394200
178	1,02407888
179	1,02421578
180	1,02435270
181	1,02448964
182	1,02462659
183	1,02476356
184	1,02490055
185	1,02503756

Days t. =	The Amounts of 1 l. &c.
186	1,02517459
187	1,02531164
188	1,02544870
189	1,02558578
190	1,02572288
191	1,02586000
192	1,02599714
193	1,02613430
194	1,02627147
195	1,02640866
196	1,02654588
197	1,02668310
198	1,02682015
199	1,02695762
200	1,02709490
201	1,02723221
202	1,02736953
203	1,02750686
204	1,02764422
205	1,02778160
206	1,02791899
207	1,02805640
208	1,02819384
209	1,02833129
210	1,02846875
211	1,02860624
212	1,02874375
213	1,02888127
214	1,02901881
215	1,02915637
216	1,02929395
217	1,02943154
218	1,02956916
219	1,02970679
220	1,02984445
221	1,02998212
222	1,03011980
223	1,03025751
224	1,03039524
225	1,03053298

Days t.	The Amounts of 1 l. &c.	Days t.	The Amounts of 1 l. &c.	Days t.	The Amounts of 1 l. &c.
226	1,03067074	266	1,03619636	306	1,04175160
227	1,03080852	267	1,03633488	307	1,04189086
228	1,03094632	268	1,03647342	308	1,04203015
229	1,03108414	269	1,03661197	309	1,04216944
230	1,03122197	270	1,03675055	310	1,04230876
231	1,03135983	271	1,03688914	311	1,04244810
232	1,03149770	272	1,03702775	312	1,04258245
233	1,03163559	273	1,03716638	313	1,04272683
234	1,03177350	274	1,03730503	314	1,04286622
235	1,03191143	275	1,03744370	315	1,04300563
236	1,03204938	276	1,03758239	316	1,04314506
237	1,03218734	277	1,03772109	317	1,04328451
238	1,03232533	278	1,03785982	318	1,04342397
239	1,03246333	279	1,03799856	319	1,04356346
240	1,03260135	280	1,03813732	320	1,04370297
241	1,03273939	281	1,03827609	321	1,04384249
242	1,03287744	282	1,03841489	322	1,04398203
243	1,03301552	283	1,03855371	323	1,04412159
244	1,03315361	284	1,03869254	324	1,04426117
245	1,03329173	285	1,03883139	325	1,04440077
246	1,03342986	286	1,03897027	326	1,04454038
247	1,03356801	287	1,03910916	327	1,04468002
248	1,03370617	288	1,03924817	328	1,04481967
249	1,03384436	289	1,03938699	329	1,04495934
250	1,03398157	290	1,03952594	330	1,04509903
251	1,03412079	291	1,03966491	331	1,04523874
252	1,03425903	292	1,03980389	332	1,04537847
253	1,03439729	293	1,03994289	333	1,04551822
254	1,03453557	294	1,04008191	334	1,04565798
255	1,03467387	295	1,04022095	335	1,04579777
256	1,03481218	296	1,04036001	336	1,04593757
257	1,03495052	297	1,04049908	337	1,04607739
258	1,03508887	298	1,04063818	338	1,04621723
259	1,03522724	299	1,04077729	339	1,04635709
260	1,03536563	300	1,04091642	340	1,04649697
261	1,03550404	301	1,04105557	341	1,04663686
262	1,03564247	302	1,04119474	342	1,04677678
263	1,03578091	303	1,04133393	343	1,04691671
264	1,03591938	304	1,04147314	344	1,04705667
265	1,03605786	305	1,04161236	345	1,04719664

Days = t.	The Amounts of 1 l. &c.	Days = t.	The Amounts of 1 l. &c	Days = t.	The Amounts of 1 l. &c.
346	1,04733663	353	1,04831708	360	1,04929845
347	1,04747764	354	1,04845722	361	1,04943872
348	1,04761666	355	1,04859738	362	1,04957901
349	1,04775671	356	1,04873756	363	1,04971932
350	1,04789677	357	1,04887775	364	1,04985965
351	1,04803686	358	1,04901797	365	1,04999999
352	1,04817696	359	1,04915820	366	1,05

I think it is needless to say any Thing of the Use of these *Tables*, because I take it for granted, that whoever understands the Work of the foregoing *Examples*, at 6 *per Cent.* cannot but know how to make Use of these *Tables* at 5 *per Cent.* as Occasion requires.

Thus far concerning *Annuities*, or *Leases*, &c. that are *limited* by any assigned *Time*; and 'tis only such that can be computed by *Theorems* or certain *Rules*. However, it may not perhaps be *unacceptable* to insert a brief Account of some *Estimates* that have been reasonably made, by *two* very ingenious *Persons*, about the Proportion or Difference of *Mens Lives*, according to their several *Ages*; which may be of good Use in computing the *Values* of *Annuities*, or taking of *Leases* for *Lives*, &c.

Sir *William Petty*, in his Discourse made before the *Royal Society* (*Anno* 1674) concerning the Use of DUPLICATE PROPORTION, in the Life of *Man* and its Duration, saith, that it's found by *Experience* there are more Persons living of between 16 and 26 *Years Old*, than of any other *Age* or *Decade* of Years in the whole Life of *Man* (*viz.* 70 or 80 *Years.*) *His Reason for that Assertion I shall omit*; but supposing it true, he thence infers, that the *Roots* of every *Number* of *Mens Ages* under 16 (whose *Root* is 4) compared with the said Number 4, doth shew the *Proportion* of the Likelihood of such Mens reaching the *Age* of 70 Years.

As for *Example*, 'tis 4 *Times* more likely, that one of 16 *Years Old* should live to 70, than a New-born *Babe*: 'Tis 3 *Times* more likely, that one of 9 *Years Old* should attain the *Age* of 70, than the said *Infant*, &c.

On the other Hand, 'tis 5 to 4, that one of 25 *Years Old* will die before one of 16: And 6 to 5, that one of 36 will die before one of 25. And so on according to the *Roots* of any other declining *Age*, compared with the (4, 6) the *Root* of 21, which is the *Year* of *Perfection* according to the Sense of our *Law*, and the *Age* for whose Life a *Lease* is most *valuable*.

2. *The*

2. *The ingenious* and great *Mathematician*, Doctor *Edmund Halley* (*in Philosoph. Transact. Numb.* 196) doth, with great *Industry* and *Skill*, draw an *Estimate* of the *Proportion* of *Mens* Lives, from the *Monthly Tables* of the *Births* and *Funerals* in *Breslaw*, the *Capital City* of the *Province* of *Silesia*; or, as the *Germans* call it, *Schlesia*. Whence he proves that it's 80 to 1, a Person of 25 *Years Old* will not die in a *Year*: That it is $5\frac{1}{2}$ to 1, that a Man of 40 will live 7 *Years*: That a Man of 30 *Years Old* may reasonably expect to live 27 or 28 *Years*, &c.

Now from these and the like *Proportions* (he justly infers) that the *Price* of *Insurance* upon *Lives* ought to be regulated, there being a great Difference between the Life of a Man of 20, and one of 50. For *Example*: 'Tis 100 to 1, that a Man of 20 dies not in a *Year*, and but 38 to 1, for a Man of 50 *Years* of *Age*. And upon these also depends the *Valuation* of *Annuities* for Lives; for it is plain, that the *Purchaser* ought to pay only such a Part of the *Value* of any *Annuity*, as he hath *Chances* that he is living.

And for that Purpose he hath taken the Pains (*which was not a little*) to compute the following *Table* (that shews the *Value* of *Annuities*) for every Fifth Year of *Age* to the 70th.

Age.	*Year's Purchase.*	*Age.*	*Year's Purchase.*	*Age.*	*Year's Purchase*
1	10,28	25	12,27	50	9,21
5	13,40	30	11,72	55	8,51
10	13,44	35	11,12	60	7,60
15	13,33	40	10,57	65	6,54
20	12,78	45	9,91	70	5,32

The same *ingenious Gentleman* proceeds on, and shews how to *estimate* or find the *Value* of *Two Lives*, and then of *Three Lives*, which being too long a Discourse to be recited here, I have, for Brevity's Sake, omitted it; and shall only add this serious Observation,

Viz. How unjustly we repine at the Shortness of our *Lives*, and think ourselves wrong'd if we attain not to *Old Age*; whereas it appears, that the *One Half* of those, that are Born, *die* in Seventeen Years Time. For by the aforesaid Bills of Mortality at *Breslaw*, it was found, that 1238 were in that Time reduced to 616. So that, instead of murmuring at what we call a *Short Life*, we ought to account it as a great Blessing that we have *surviv'd*, perhaps by many Years, that *Period* of *Life* whereat the one Half of the whole Race of Mankind does not arrive.

Sect. 4. *Of Purchaſing* Free-hold, *or* Real Eſtates; *at Compound Intereſt.*

All *Free-hold* or *Real Eſtates*, are ſuppoſed to be purchaſed or bought to continue for *ever* (viz. *without any limited Time*); therefore the Buſineſs of computing the true *Value* of ſuch *Eſtates* is grounded upon a *Rank* or *Series* of *Geometrical Proportions* continually *decreaſing*, *ad Infinitum*.

Thus, let P, u, R, denote the ſame *Data* as in the laſt *Section*. Then the *Series* will be, $\frac{u}{R}$, $\frac{u}{RR}$, $\frac{u}{R^3}$, $\frac{u}{R^4}$, $\frac{u}{R^5}$, and ſo on in $\frac{\cdot\cdot}{\cdot\cdot}$ until the laſt *Term* $= 0$. Then will $P - 0$ (viz. P) be the ſum of all the *Antecedents*. And $P - \frac{u}{R}$ will be the *Sum* of all the *Conſequents*; therefore it will be $u : \frac{u}{R} :: P : P - \frac{u}{R}$ which produces $P\,R - u = P$.

This *Equation* affords the following *Theorems*.

Theorem 1. $P\,R - P = u$. *Theorem.* 2. $\left\{\frac{u}{R-1} = P.\right.$

Theorem. 3. $\left\{\frac{P+u}{P} = R.\right.$

Example. *Suppoſe a Free-hold Eſtate of* 75 *l. Yearly Rent, were to be ſold; what is it worth, allowing the Buyer* 6 per Cent. *&c. Compound Intereſt for his Money?*

In this *Queſtion* there is given $u = 75$. $R = 1{,}06$ to find P. Per *Theorem* 2. Thus $R - 1 = 0{,}006)\ 75 = u\ (1250\ l. = P.$ the *Anſwer* required. And ſo on for any of the reſt, as Occaſion requires. But if the *Rent* is to be paid, either by *Quarterly* or *Half Yearly Payments*;

Then $R = \surd\ 1{,}06$ for *Half Yearly*
And $R = \surd : \surd\ 1{,}06$ for *Quarterly* } *Payments at* 6 *per Cent.*

Or
$R = \quad 1{,}08$ for *Yearly*
$R = \surd\ 1{,}08$ for *Half Yearly*
$R = \surd : \surd\ 1{,}08$ for *Quarterly*
} *Payments at* 8 *per Cent.*

The like is to be underſtood for any other propoſed *Rate* of *Intereſt*, either *greater* or *leſs* than 6 *per Cent.*

The Application of theſe *Theorems* to Practice is ſo very eaſy, that it's needleſs to inſert more *Examples*.

A N

AN INTRODUCTION TO THE Mathematicks.

PART III.

CHAP. I.

Of Geometrical Definitions, &c.

Sect. I. Of *Lines* and *Angles.*

A POINT *hath no Parts:* That is, a *Geometrical Point* is not any *Quantity*, but only an *assignable Place* in any *Quantity*, denoted by a *Point:* As at *A.* and *B.* } *A.* *B.*

Such a Place may be conceived so infinitely small, as to be void of Length, Breadth, and Thickness; and therefore a Point may be said to have no *Parts.*

2. A LINE is called a *Quantity* of one *Dimension*, because it may have any supposed *Length*, but no *Breath* nor *Thickness*, being made or represented to the *Eye*, by the *Motion* of a *Point.*

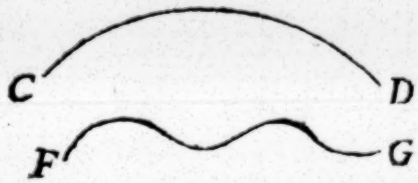
That is, if the *Point* at *A*, be *moved (upon the same Plane)* to the *Point* at *B*, it will describe a *Line* either *right* or *circular* (viz. *crooked)* according to its *Motion.*

Therefore the *Ends* or *Limits* of a *Line* are *Points.*

3. A RIGHT LINE, is that *Line* which lieth *even* or *straight* betwixt those *Points* that *limit* its *Length*, being the *shortest Line* that can be drawn between any *Two Points.* As the *Line A B.* } *A.*——————*B.*

Therefore, between any two Points, there can lie or be drawn but one right Line.

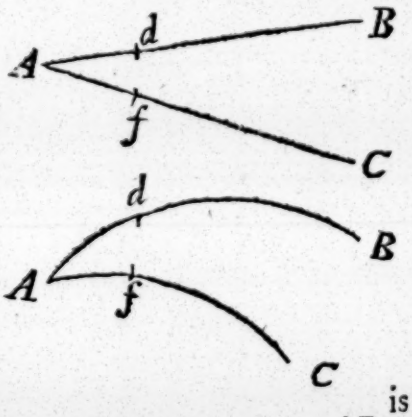
4. A CIRCULAR, *crooked* or OBLIQUE *Line*, is that which *lies* bending between thoſe *Points* which *limit* its *Length*, as the *Lines CD* or *FG*. &c.



Of theſe Kinds of Lines there are various Sorts; but thoſe of the Circle, Parabola, Ellipſis, and Hyperbola are of moſt general Uſe in Geometry; of which a particular Account ſhall be given further on.

5. PARALLEL LINES, are thoſe that lie *equally diſtant* from one another in all their *Parts*, viz. ſuch *Lines* as being *infinitely extended (upon the ſame Plane)* will never *meet*: As the *Lines AB* and *ab*: or *CD* and *cd*.

6. LINES not PARALLEL, but INCLINING (*viz. leaning*) one towards another, whether they are *Right Lines*, or *Circular Lines*, will (if they are *extended*) *meet and* make an *Angle*; the *Point* where they *meet* is called the *Angular Point*, as at *A*. And according as ſuch *Lines* ſtand, *nearer* or *further* off each other, the *Angle* is ſaid to be *leſſer* or *greater*, whether the *Lines* that *include* the *Angle* be *long* or *ſhort*. That is, the *Lines Ad* and *Af* include the ſame *Angle* as *AB* and *AC* doth; notwithſtanding that *AB* is *longer* than *Ad*, &c.



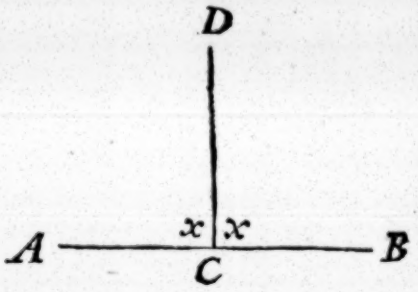
7. All ANGLES *including* between *Right Lines* are called *Right-lin'd Angles*; and thoſe *included* between *Circular Lines* are called *Spherical Angles*. But all *Angles*, whether *Right-lin'd* or *Spherical*, fall under one of theſe *Three Denominations*.

Viz. { A **Right Angle.** / An **Obtuſe Angle.** / An **Acute Angle.** }

8. A RIGHT-ANGLE is that which is *included* betwixt *Two Lines*, that *meet* one another *Perpendicularly*.

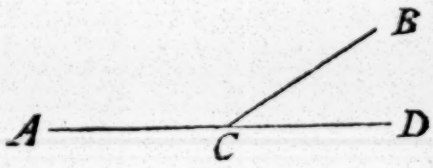
That

That is, when a *Right Line*, as *DC*, *meets* with another *Right-Line*, as *AB*, ſo directly as that it neither *inclines* nor *declines* to one *Side* more than the other, but make the *Angles* on both *Sides* of it *equal*, as at *x*, *x*; then are thoſe *Angles* called *Right-Angles*; and the *Lines* ſo *meeting* are ſaid to be *Perpendicular* to each other.



That is, *AC*, and *CB*, are *Perpendicular* to *DC*, as well as *D*, *C* is to *either* or both of them.

9. An OBTUSE ANGLE is that which is *greater* than a *Right Angle*. Such is the *Angle included* between the *Lines AC* and *CB*.



10. An ACUTE ANGLE is that which is *leſs* than a *Right Angle:* As the *Angle included* between the *Lines CB* and *CD*.

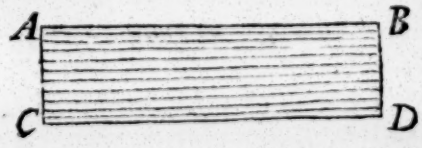
Theſe *Two Angles* are generally called OBLIQUE *Angles*.

Sect. 2. Of a Circle, &c.

Before a Circle and its Parts are defined, it will be convenient to give a brief Account of Superficies in general.

1. A SUPERFICIES or SURFACE is the Upper, or very *Out-ſide* of any *viſible* Thing. But by *Superficies* in GEOMETRY, is meant only ſo much of the *Out-ſide* of any Thing as is *incloſed* within a *Line* or *Lines*, according to the Form or *Figure* of the Thing *deſigned*; and it is *produced* or *formed* by the *Motion* of a *Line*, as a *Line* is deſcribed by the *Motion* of a *Point*; thus:

Suppoſe the *Line AB* were *equally moved (upon the ſame Plane)* to *CD*; then will the *Points* at *A* and *B deſcribe the two Lines AC* and *BD*; and by ſo doing they will form (and *incloſe*) the SUPERFICIES or *Figure ABCD*, being a *Quantity of Two Dimenſions*, viz. it hath *Length* and *Breadth*, but not *Thickneſs*. *Conſequently the Bounds or Limits of a Superficies are Lines.*

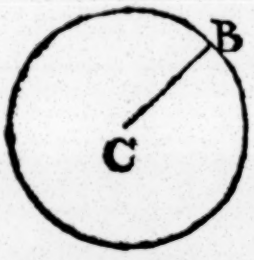


Note,

Note, *The Superficies of any Figure, is uſually called its* AREA.

2. A CIRCLE is a *plain regular Figure*, whoſe *Area is bounded* or *limited* by one continued *Line*, called the CIRCUMFERENCE or PERIPHERY of the *Circle*, which may be thus *deſcribed* or *drawn*.

Suppoſe a *Right Line*, as *C B*, to have one of its *Extream Points*, as *C*, ſo fix'd upon any *Plane*, as that the other *Point* at *B* may *move* about it; then if the *Point* at *B* be *moved round* about (*upon the ſame Plane*) it will *deſcribe a Line equally diſtant* in all its *Parts* from the *Point C*, which will be the *Circumference* or *Periphery* of that *Circle*; the *Point C* will be its CENTER, and the contained *Space* will be its *Area*, *and the Right Line C B, by which the Circle is thus deſcribed, is called* RADIUS.



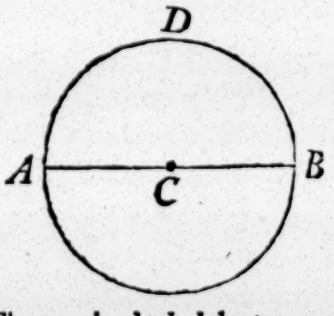
Conſectary.

From hence 'tis evident, that an infinite Number of Right Lines may be drawn from the Center of any Circle to touch its Periphery, which will be all equal to one another, becauſe they are all Radius's. And with a little Conſideration it will be eaſy to conceive, that no more than two equal Right Lines can be drawn from any Point within a Circle to touch its Periphery, but from the Center only. (9. *e.* 3.)

3. EQUAL CIRCLES are thoſe which have *equal Radius's*; for it's plain by the laſt *Definition*, that one and the ſame *Radius* (as *C B*) muſt needs *deſcribe equal Circles*, how many ſoever they are.

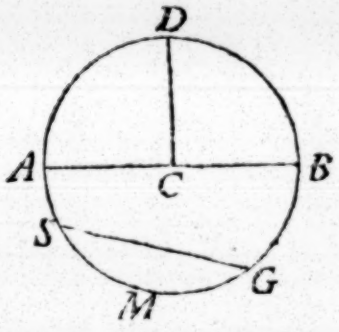
4. The *Diameter* of a *Circle*, is *twice* its *Radius joined* into one *Right Line* as *A B* drawn through the *Center C*, and *ending* at the *Periphery* on each *Side*.

That is the *Diameter divides* the *Cirtle* into *Two* equal Parts.



5. A *Semicircle* (viz. *Half a Circle*) is a *Figure included* between the *Diameter*, and *Half* the *Periphery cut off* by the *Diameter*; as *A D B*.

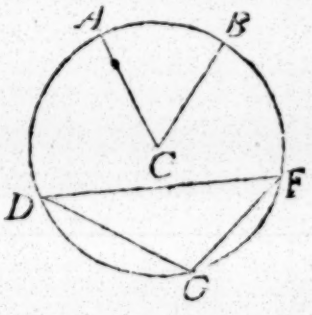
6. A QUADRANT is *Half* a *Semicircle*, viz. one *Quarter* of a *Circle*; and 'tis made by the *Radius* (as *DC*) *ſtanding Perpendicular* upon the *Diameter* at the *Center C*, *cutting* the *Periphery* of the *Semicircle* in the *Middle*, as at *D*. *Therefore a Quadrant, or half the Semicircle, is the Meaſure of a Right Angle.*



7. A CHORD LINE, or the *Subtenſe* of an *Arch*, is any *Right Line that cuts* the *Circle* into *Two unequal Parts, as the Line S G*; and is always *leſs* than the *Diameter*.

8. A SEGMENT of a *Circle*, is a *Figure included* betwixt the *Chord* and that *Arch* of the *Periphery* which is *cut off* by the *Chord*: And it may either be *greater* or *leſs* than a *Semicircle*; as the *Figure S DG*, or *S MG*.

9. A SECTOR is a *Figure included* between *Two Radius*'s of the *Circle*, and that *Arch* of its *Periphery* where they *touch*, as the *Figure ACB*: And the *Arch AB* is the *Meaſure* of the *ngle* at *C*, *included* betwixt the *Radius*'s *AC* and *BC*.



Note, *All Angles of Sectors are called Angles at the Center of a Circle.*

10. An ANGLE in the *Segment* of a *Circle* is that which is *included* between *Two Chords* that *flow* from *one* and the *ſame Point* in the *Periphery*, as at *D*, and *meet* with the *Ends* of another *Chord Line*, as at *F* and *G*.

That is, the *Angles* at *D*, at *F* and at *G*, are called *Angles* at the *Periphery*, or *Angles ſtanding* on the *Segment* of a *Circle*.

Sect. 3. *Of* TRIANGLES.

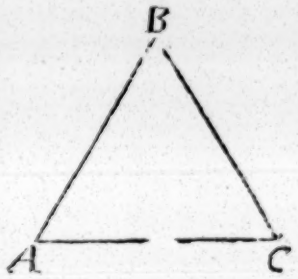
There are two Kinds of Triangles, viz. *Plain and Spherical*; *but I ſhall not give any Definition of the Spherical, becauſe they more immediately relate to Aſtronomy.*

1. A PLAIN TRIANGLE is a *Figure* whoſe *Area* is *contained* within the *Limits* of *Three Right Lines* called *Sides*, *including Three Angles*: And it may be *divided*, and takes its *Name*, either according to its *Sides* or *Angles*.

1. By

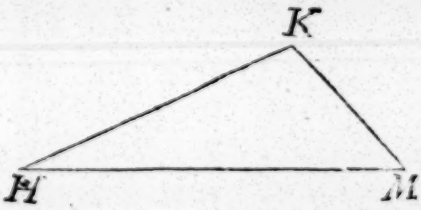
1. By its SIDES.

2. An EQUILATERAL TRIANGLE is that which hath all its *Three Sides* equal; as the *Figure A B C*.
That is, $AB = BC = AC$.



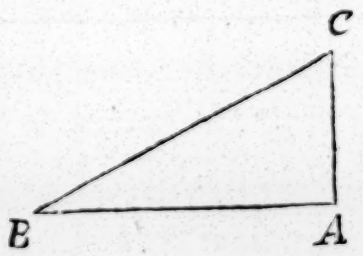
3. An ISOSCELES TRIANGLE, is that which hath *only Two* of its *Sides equal*, as the *Figure B D G:* That is, $BD = DG$; but the *Third Side B G* may be either *greater* or *leſs*, as Occaſion requires.

4. A SCALENE TRIANGLE, is that which hath all its *Three Sides unequal*;
ſuch as the *Figure H K M*.



2. By its ANGLES.

5. A RIGHT-ANGLED *Triangle*, is that which hath one *Right Angle*; that is, when *Two* of its *Sides* are *Perpendicular* to each other, as *C A* is ſuppoſed to be to *B A*. Therefore the *Angle* at *A*, is a *Right Angle*, per *Defin.* 8. *Sect.* 1.



Note, *The longeſt Side of every Right-angled Triangle (as B C) is called the Hypothenuſe, and the longeſt of the other Two Sides which include the Right Angle (as B A) is called the Baſe: The Third Side (as C A) is called the Cathetus or Perpendicular.*

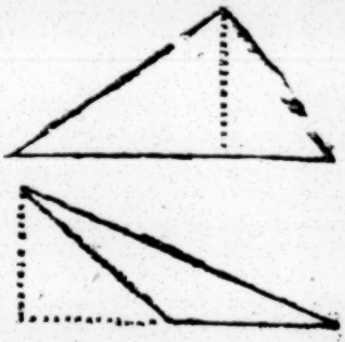
6. An OBTUSE-ANGLED *Triangle*, is that which hath *one* of its *Angles Obtuſe*, and it's called an *Amblygonium Triangle*. Such is the *Third Triangle H K M*.

7. An ACUTE ANGLED TRIANGLE, is that which hath all its *Angles Acute*, and it's called an *Oxygonium Triangle*; ſuch are the *Firſt* and *Second Triangles A B C* and *B D G*.

Note, *All Triangles that have not a Right Angle, whether they are Acute, or Obtuſe, are, in general Terms, called Oblique Triangles,*

gles, without any other Diſtinction, as before. And the longeſt Side of every oblique Triangle is uſually called the Baſe ; *the other two are only called* Sides *or* Legs.

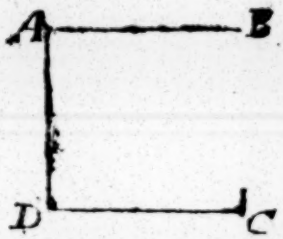
8. The ALTITUDE or HEIGHT of any *Plain Triangle*, is the *Length* of a *Right Line* let fall *perpendicular* from any of its *Angles*, upon the *Side* oppoſite to that *Angle* from whence it falls ; and may be either within, or without the *Triangle*, as *Occaſion requires*, being denoted by the *Two prick'd Lines*, in the *annexed Triangles*.



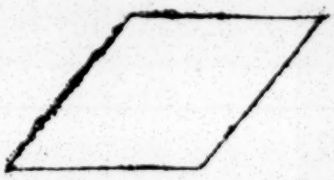
Sect. 4. Of Four-ſided Figures.

1. A SQUARE is a plain *regular Figure*, whoſe *Area* is limited by *Four* equal *Sides*, all *perpendicular* one to another.

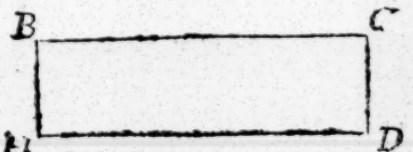
That is, when $AB = BC = CD = DA$, and the *Angles A, B, C, D* are all *equal*, then it's uſually called a *Geometrical Square*.



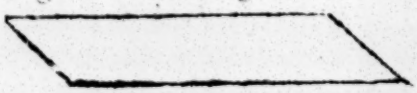
2. RHOMBUS, or *Diamond-like Figure*, is that which hath *Four* equal *Sides*, but no *Right-angle*. That is, a *Rhombus* is a *Square* mov'd out of its *right Poſition*, as the *annexed Figure*.



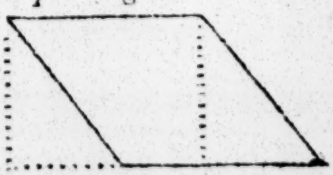
3. A RECTANGLE, or a *Right-angled Parallelogram* (often called an *Oblong*, or *long Square*) is a *Figure* that hath *four Right-angles* and its *two oppoſite Sides* equal, *viz.* $BC = HD =$ and $BH = CD$.



4. A RHOMBOIDES, is an *Oblique-angled Parallelogram* ; that is, it is a *Parallelogram moved* out of its *right Poſition*, like the *annexed Figure*.

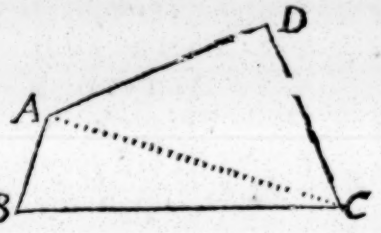


5. The ALTITUDE or *Height* of any *Oblique angled Parallelogram*, *viz* either of the *Rhombus* or *Rhomboides*, is a *Right line* let fall *perpendicular* from any *Angle* upon the *Side oppoſite* to that *Angle* ; and may either be within or without the *Figure* : As the *prick'd Lines* in the *annexed Figure*.



6. Every *Four-ſided Figure*, different from thoſe before-mentioned, is called a TRAPEZIUM.

That is, when it has neither *oppoſite Sides*, nor *oppoſite Angles equal*; as the *Figure ABCD*.



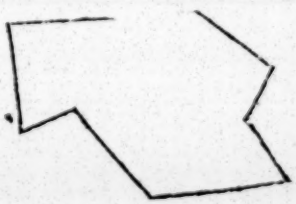
7. A *Right-line*, drawn from any *Angle* in a *Four-ſided Figure* to its *oppoſite Angle*, is called a DIAGONAL *Line*, and will *divide* the *Area* of the *Figure* into *two Triangles*, being *denoted* by the *prick'd Line AC* in the laſt *Figure*.

8. All *Right-lin'd Figures*, that have more than *four Sides*, are call'd *Polygons*, whether they be *regular* or *irregular*.

9. A REGULAR POLYGON is that which hath all its *Sides equal*, ſtanding at *equal Angles*, and is named according to the *Number* of its *Sides* (or *Angles*). That is, if it have *five equal Sides*, it is called a PENTAGON; if *ſix equal Sides*, it is call'd a HEXAGON; if *ſeven*, 'tis a HEPTAGON; if *eight*, 'tis an OCTAGON, &c.

Note, *All* Regular Polygons *may be inſcrib'd in a Circle*; *that is, their Angular Points, how many ſoever they have, will all juſt touch the Circle's Periphery*.

10. An IRREGULAR POLYGON is that *Figure* which hath many *unequal Sides* ſtanding at *unequal Angles* (like unto the *annexed Figure*, or otherwiſe); and of ſuch Kind of *Polygons* there are *infinite Varieties*, but they may all be *reduced* to *regular Figures* by drawing *Diagonal Lines* in them; as ſhall be ſhew'd farther on.



Theſe are the moſt *general* and *uſeful Definitions* that concern *plain* or *ſuperficial Geometry*.

As for thoſe which relate to *Solids*, I thought it convenient to omit giving any Account of them in this Place, becauſe they would rather *puzzle* and *amuſe* the *Learner*, than *improve* him, until he has gain'd a *competent Knowledge* in the moſt uſeful *Theorems* concerning *Superficies*; for then thoſe *Definitions* may be more eaſily underſtood, and will help him to form a *clearer Idea* of their *reſpective Solids*, than 'tis *poſſible* to conceive of them before; and therefore I have reſerv'd thoſe *Definitions* until we come to the Fifth Part.

Sect.

Sect. 5. Of ſuch Terms *as are generally uſed in Geometry.*

Whatſoever is propoſed in *Geometry* will either be a PROBLEM or a THEOREM.

Both which *Euclid includes* in the *general Term* of *Propoſition.*

A PROBLEM is that which *propoſes* ſomething to be *done*, and relates more immediately to *practical* than *ſpeculative Geometry*; That is, it's generally of ſuch a *Nature*, as to be performed by ſome known or *Commonly-receiv'd* Rules, without any Regard had to their *Inventions* or *Demonſtrations.*

A THEOREM is when any *Commonly-receiv'd* Rule, or any *New Propoſition* is required to be *demonſtrated*, that ſo it may from thence forward become a *certain* Rule, to be *rely'd* upon in Practice when Occaſion requires it. And therefore ſeveral Rules are often call'd *Theorems*, by which *Operations* in *Arithmetick*, and *Concluſions* in *Geometry*, are *perform'd.*

Note, *By* DEMONSTRATION *is underſtood the higheſt Degree of* Proof *that human Reaſon is capable of attaining to, by a Train of Arguments deduced or drawn from ſuch plain Axioms, and other Self-evident Truths, as cannot be denied by any one that conſiders them.*

A COROLLARY, *or* CONSECTARY, is ſome *Conſequent Truth* drawn or gain'd from any *Demonſtration.*

A LEMMA is the *Demonſtration* of ſome *Premiſes* laid down or propoſed as *preparative* to obviate and ſhorten the Proof of the *Theorem* under Conſideration.

A SCHOLIUM is a brief *Commentary* or *Obſervation* made upon ſome *precedent Diſcourſe.*

N. B. *I adviſe the young Geometer to be very perfect in the* Definitions, *viz. Not to reſt ſatisfied with a bare Remembrance of them; but, that he endeavour to gain a clear* Idea *or* Underſtanding *of the Things defined; and for that Reaſon I have been fuller in every* Definition *than is uſual.*

And, that he may know from whence moſt of the following Problems *and* Theorems *contain'd in the Two next Chapters are collected, I have all along cited the* Propoſition *and* Book *of* Euclid's Elements *where they may be found.*

As for Inſtance; at Problem 1. *there is* (3 *e.* 1.) *which ſhews that it is the* Third Propoſition *in* Euclid's First Book. *The like muſt be underſtood in the* Theorems.

CHAP. II.

The First Rudiments, or *Leading* and *Preparatory* Problems, in Plain Geometry.

IN order to perform the following Problems, *the young Geometer ought to be provided with a thin ſtreight* Ruler, *made either of* Braſs *or* Box-wood, *and two Pair of very good* Compaſſes, *viz. one Pair call'd* Three-pointed Compaſſes, being very uſeful for *drawing of* Figures *or* Schemes, *either with Black Lead or Ink*; *and one Pair of* plain Compaſſes *with very fine Points, to meaſure and ſet off Diſtances*; *alſo he ſhould have a very good Steel* Drawing-Pen: *And then he may proceed to the Work with this Caution*; *that he ought to make himſelf Maſter of one* Problem *before he undertakes the next: That is, he ought to underſtand the* Deſign, *and, as far as he can, the Reaſon of every* Problem, *as well as how to do it*; *and then a little Practice will render them very eaſy, they being all grounded upon theſe following* Poſtulates.

Poſtulates or Petitions.

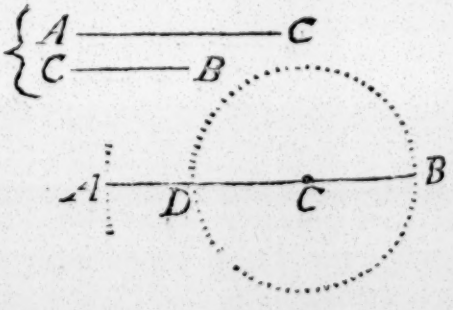
1. That a *Right-line* may be *drawn* from any one given *Point* to another.

2. That a *Right-line* may be *produced*, *increaſed*, or made *longer* from either of its Ends.

3. That upon any given *Point* (or *Center*) and with any given *Diſtance* (*viz.* with any RADIUS) a *Circle* may be deſcribed.

PROBLEM I.

Two Right-lines being given, to find their Sum and Difference (3. *e.* 1.)



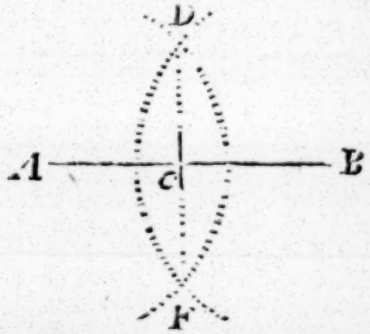
Let the given *Lines* be Make the *ſhorteſt Line*, as *C B*, *Radius*, and with it *deſcribe a Circle:* From its Center *C* ſet off the other *Line A C*, and *join A C B* with a *Right line.* Then will $AB = AC + CB$; and $AD = AC - CB$; as was required.

PRO

PROBLEM II.

To biſect, or divide a Right-line given (as A B) into two equal Parts (10. *e.* 1.)

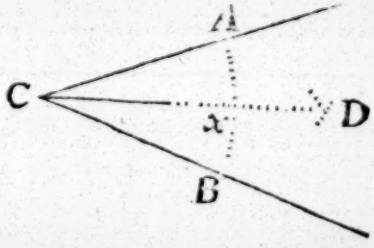
From both *Ends* of the given *Line* (*viz. A* and *B* with any *Radius greater* than *half* its *Length*, *deſcribe Two Arches* that may *croſs* each other in *two Points*, as at *D* and *F*; then *join* thoſe *Points D F* with a *Right-line*, and it will *biſect* the *Line A B* in the Middle at *C*; *viz.* it will make *A C* = *C B*; as was required.



PROBLEM III.

To biſect a Right-lin'd Angle given, into two equal Angles. (9. *e.* 1.)

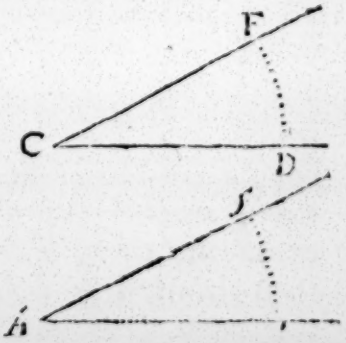
Upon the *Angular Point*, as at *C*, with any *convenient Radius*, *deſcribe* an *Arch* as *A B*; and from thoſe *Points A* and *B*, *deſcribe two* equal *Arches croſſing* each other, as at *D*; then *join* the *Points C* and *D* with a *Right-line*, and it will *biſect* the *Arch A B*, and conſequently the *Angle*; as was required.



PROBLEM IV.

At a Point A, in a Right-line given A B, to make a Right-lin'd Angle equal to a Right-lin'd Angle given C. (23. *e.* 1.)

Upon the given *Angular Point C* deſcribe an *Arch*, as *F D*, (making *C D* any *Radius* at *Pleaſure*) and with the ſame *Radius deſcribe* the like *Arch* upon the given *Point A*, as *f d*; that is, make the *Arch f d* equal to the *Arch F D*; Then *join* the *Points A* and *f* with a *Right-line*, and it will form the Angle requir'd.

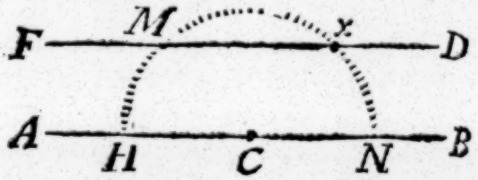


PRO-

PROBLEM V.

To draw a Right-line, as F D, parallel to a given Right-line A B, that ſhall paſs thro' any aſſigned Point, as at x, viz. *at any Diſtance required.* (31. *e.* 1.)

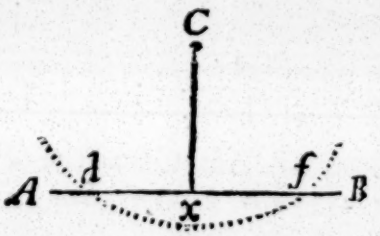
Take any convenient Point in the given *Line*, as at *C*, (the farther off *x* the better;) make *C x Radius*, and with it upon the *Point C*, deſcribe a *Semi-circle*, as *H M x N*; then make the *Arch H M* equal to the *Arch x N*; thro' the *Points M* and *x* draw the *Right-line F D*, and it will be *parallel* to the *Line A C*, as was requir'd.



PROBLEM VI.

To let fall a Perpendicular, *as C x, upon a given Right-line A B, from any aſſign'd Point that is not in it, as from C.* (12. *e.* 1.)

Upon the given Point *C deſcribe* ſuch an *Arch* of a *Circle* as will croſs the given Line *A B* in two *Points*, as at *d* and *f*; Then *biſect* the *Diſtance* between thoſe *two Points d f* (per Probl. 2.) as at *x*. *Draw* the *Right-line C x*, and it will be the *Perpendicular* requir'd.



PROBLEM VII.

To erect or raiſe a Perpendicular *upon the End of any given Right-line, as at B; or upon any other Point aſſign'd in it.* (11. *e.* 1.)

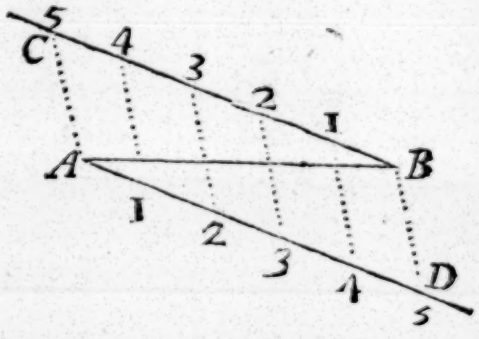
Upon any *Point (taken at an Adventure)* out of the given *Line* as at *C*, *deſcribe* ſuch a *Circle* as will *paſs* through the *Point* from whence the *Perpendicular* muſt be *raiſed*, as at *B*, (viz. *make C B Radius*): And from the *Point* where the *Circle cuts* the given *Line*, *as at A*, *draw* the *Circle's Diameter A C D*; then from the *Point D* draw the *Right-line D B*, and it will be the *Perpendicular* as was requir'd.

PRO-

PROBLEM VIII.

To divide any given Right-line, as A B, into any propoſed Number of equal Parts. (10. *e.* 6.)

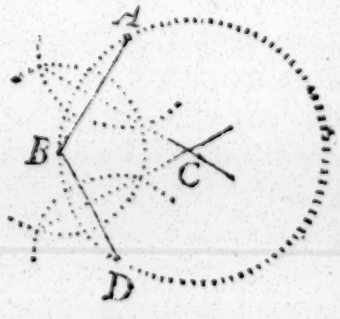
At the extream Points (or Ends) of the given Line, as at *A* and *B*, make *two* equal *Angles* (by *Prob.* 4.) continuing their Sides *A D* and *B C* to any ſufficient *Length*; then upon thoſe Sides, beginning at the Points *A* and *B*, ſet off the propoſed *Number* of equal Parts *(ſuppoſe 'em* 5.*)* If *Right-lines* be drawn *(croſs the given Line)* from one Point to the other, as in the annexed Figure, thoſe *Lines* will divide the given *Line A B* into the *Number* of equal Parts required.



PROBLEM IX.

To deſcribe a Circle that ſhall paſs (or cut) thro' any Three Points given, not lying in a Right line, as at the Points A B D.

Join the Points *A B* and *B D* with *Right-lines*; then biſect both thoſe *Lines (per Problem* 2.*)* the Point where the biſecting Lines meet, as at *C*, will be the *Center* of the *Circle* required.



The Work of this Problem being well underſtood, 'twill be *eaſy* to perform the two following, without any *Scheme*, viz.

1. *To find the Center of any Circle given.* (1. *e.* 3.)

By the laſt Problem 'tis plain, that if *three* Points be any where taken in the given *Circle's Periphery*, as at *A*, *B*, *D*, the *Center* of that Circle may be found as before.

2. *If a Segment of any Circle be given, to compleat or deſcribe the whole Circle.*

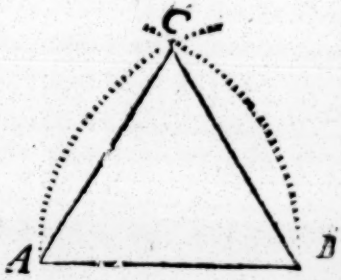
This may be done by taking any *three Points* in the given *Segment's* Arch, and then proceed as before.

PRO-

PROBLEM X.

Upon a Right-line given, as AB, to deſcribe an Equilateral Triangle. (1. *e.* 1.)

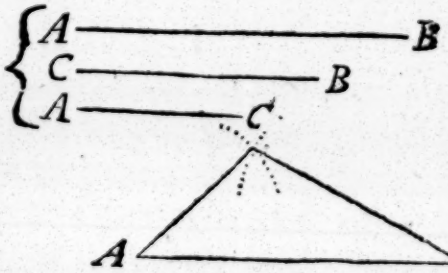
Make the given Line *Radius*, and with it, upon each of its *extream Points* or *Ends*, as at *A* and *B*, deſcribe an *Arch*, *viz.* *AC* and *BC*; then join the *Points AC* and *BC* with *Right-lines*, and they will make the *Triangle* requir'd.



PROBLEM XI.

Three Right-lines being given, to form them into a Triangle, (provided any two of them, taken together, be longer than the Third) (22. *e.* 1.)

Let the given *Lines* be

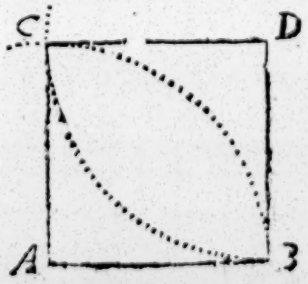


Make either of the *ſhorter Lines* (as *AC*) *Radius*, and upon either End of the *longeſt Line* (as at *A*) deſcribe an Arch; then make the other Line *CB Radius*, and upon the other End of the *longeſt Side* (as at *B*) deſcribe another Arch, to croſs the Firſt Arch (as at *C*): Join the *Points CA* and *CB* with *Right-lines*, and they will form the *Triangle* required.

PROBLEM XII.

Upon a given Right-line, as AB, to form a Square. (46. *e.* 1.)

Upon one End of the given *Line*, as at *B*, erect the *Perpendicular BD*, equal in Length with the given *Line*, *viz.* make *BD* = *AB*; that being done, make the given Line *Radius*, and upon the *Points A* and *D* deſcribe equal Arches to croſs each other, as at *C*; then join the *Points CA* and *CD* with *Right-lines*, and they will form the *Square* required.

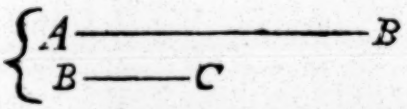


PRO-

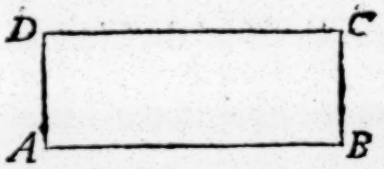
PROBLEM XIII.

Two unequal Right-lines being given, to form or make of them a Right-angled Parallelogram.

Let the given *Lines* be



Upon one End of the *longeſt Line*, as at *B*, erect a *Perpendicular* of the ſame Length with the *ſhorteſt Line B C*; then from the *Point C* draw a Line parallel and of the ſame Length, to *A B*; *viz.* make *D C* = *A B*: Join *D A* with a *Right-line*, and it will form the *Oblong* or *Parallelogram* required.



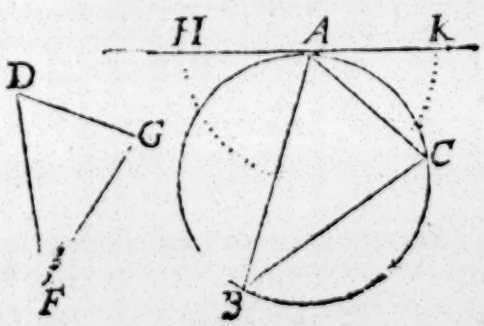
As for *Rhombus*'s and *Rhomboides*'s, to wit, *Oblique-angled* Parallelograms, they are made, or deſcrib'd, after the ſame Manner with the two laſt Figures; only inſtead of erecting the Perpendiculars, you muſt ſet off their given Angles, and then proceed to draw their Sides parallel, *&c.* as before.

PROBLEM XIV.

In any given Circle, to inſcribe or make a Triangle, whoſe Angles ſhall be equal to the Angles of a given Triangle; as the Triangle F D G, (2. *e.* 4)

Note, *Any Right-lin'd Figure is ſaid to be inſcrib'd in a Circle, when all the Angular Points of that Figure do juſt touch the Circle's Periphery.*

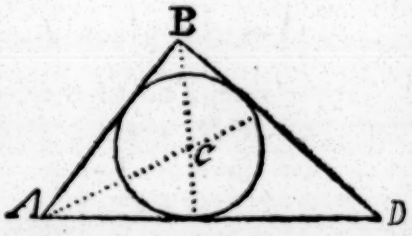
Draw any *Right-line* (as *H K*) ſo as juſt to touch the Circle, as at *A*; then make the Angle *K A C* equal to any one Angle of the given Triangle, as *D F G*; and the Angle *H A B* equal to another Angle of the Triangle, as *D G F*; then will the Angle *B A C* be equal to the Angle *F D G*. Join the *Points B* and *C* with a Right-line, and 'twill form the Triangle required.



PROBLEM XV.

In any given Triangle, as A B D, to deſcribe a Circle that ſhall touch all its Sides. (4. *e.* 4.)

Biſect any two Angles of the Triangle, as *A* and *B*, and where the biſecting Lines meet (as at *C*) will be the Center of the Circle required; and its *Radius* will be the neareſt Diſtance to the Sides of the Triangle.



PROBLEM XVI.

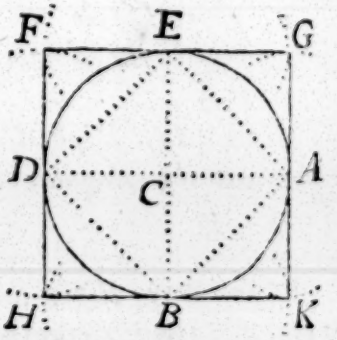
To deſcribe a Circle about any given Triangle. (5. *e.* 4.)

This *Problem* is perform'd in all Reſpects like the Ninth, *viz.* by biſecting any *Two Sides* of the given Triangle; the Point, where thoſe biſecting Lines meet, will be the Center of the Circle required.

PROBLEM XVII.

To deſcribe a Square about any given Circle. (7. *e.* 4.)

Draw two *Diameters* in the given Circle (as *D A* and *E B*) croſſing at Right Angles in the Center *C*; and, with the Circle's *Radius C A*, deſcribe from the *extream Points* of thoſe Diameters, *viz. A*, *B*, *D*, *E*, croſs Arches, as at *F*, *G*, *H*, *K*; then join thoſe *Points* where the Arches croſs with *Right-lines*, and they will form the *Square* required.



PROBLEM XVIII.

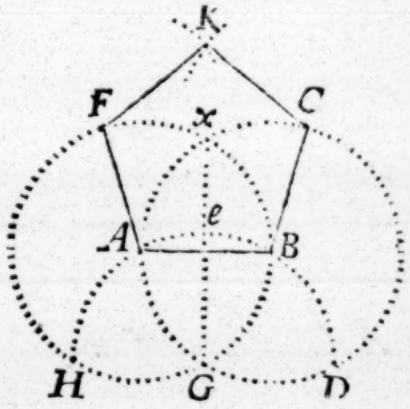
In any given Circle, to deſcribe the largeſt Square it can contain. (6. *e.* 4.)

Having drawn the *Diameters*, as *D A* and *E B*, biſecting each other at Right angles in the Center *C*, (as in the laſt Scheme); then join the *Points A*, *B*, *D*, and *E*, with *Right-lines*, viz. *A B*, *B D*, *D E*, *E A*, and they will be Sides of the *Square* required.

PROBLEM XIX.

Upon any given Right-line, as A B, to describe a regular Pentagon, *or Five-sided* Polygon.

Make the given Line *Radius*, and upon each End of it describe a Circle; and through those Points where the Circles cross each other (as at *G x*) draw the Right-line *G e x*: Upon the Point *G* with the same *Radius* describe the Arch *H A e B D*, and laying a Ruler upon the Points *D*, *e*, mark where it *crosses* the other Circle, as at *F*. Again, lay the Ruler upon the Points *H*, *e*, and mark where it *crosses* the other Circle, as at *C*: Then from the Points *F* and *C* *(with the same Radius as before)* describe cross Arches, as at *K*: Join the Points *A F*, *F K*, *K C*, and *C B*, with Right-lines, and they will form the *Pentagon* required, *viz.* *A F* = *F K* = *K C* = *C B* = *A B*; and the Angles at *A*, *B*, *C*, *K*, *F* will be equal.

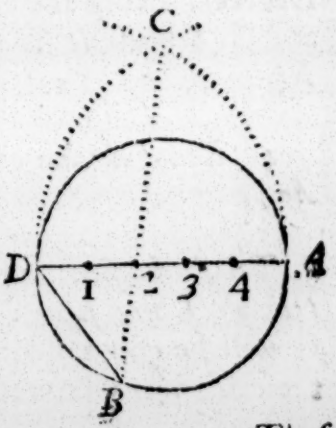


PROBLEM XX.

In any given Circle, to describe a regular Pentagon. (11. *e.* 4. & 10. *e.* 3.)

Or, in general Terms, to describe any regular Polygon *in a Circle.*

Draw the Circle's Diameter *D A*, and divide it into as many equal Parts as the proposed *Polygon* hath Sides; then make the whole Diameter a *Radius*, and describe the two Arches *C A* and *C D*. If a Right-line be drawn from the Point *C*, through the Second of those equal Parts in the Diameter, as at 2, it will assign a Point in the opposite Semicircle's *Periphery*, as at *B*. Join *D B* with a Right-line, and it will be the Side of the *Pentagon* required.



Theſe Twenty *Problems* are ſufficient to exerciſe the young Practitioner, and bring his Hand to the right Management of a *Ruler* and *Compaſſes*, wherein I would adviſe him to be very ready and exact.

As to the Reaſon why ſuch Lines muſt be ſo drawn as directed at each *Problem*, that, I preſume, will fully and clearly appear from the following *Theorems*; and therefore I have *(for Brevity's Sake)* omitted giving any *Demonſtrations* of them in this Chapter, deſiring the Learner to be ſatisfied with the bare Knowledge of doing them only, until he hath fully conſidered the Contents of the next Chapter; and then I doubt not but all will appear very plain and eaſy.

CHAP. III.

A Collection of moſt uſeful Theorems *in plain Geometry* Demonſtrated.

Note, *In order to ſhorten ſeveral of the following Demonſtrations, it will be neceſſary to premiſe, that*

1. THE *Periphery* (or Circumference) of every Circle *(whether great or ſmall)* is ſuppos'd to be divided into 360 equal Parts, called *Degrees*; and every one of thoſe *Degrees* are divided into 60 *equal Parts*, called *Minutes*, &c.

2. All *Angles* are meaſured by the Arch of a Circle deſcrib'd upon the *Angular Point (See Defin.* 9. *Page* 287.) and are eſteem'd *greater* or *leſs*, according to the Number of Degrees contain'd in that *Arch*.

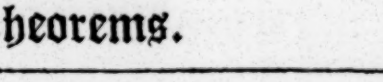
3. A *Quadrant*, or *Quarter-part* of any Circle, is always 90 Degrees, being the *Meaſure* of a *Right-angle (Defin.* 6. *P.* 287.) and a *Semicircle* is 180 Degrees, being the *Meaſure* of *two Right-angles*.

4. *Equal Arches* of a *Circle*, or of equal Circles, *meaſure* equal *Angles*.

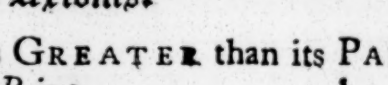
To thoſe *five* general *Axioms* already laid down in *Page* 146. *(which I here ſuppoſe the Reader to be very well acquainted with)* it will be convenient to underſtand theſe following, which begin their *Number* where the other ended.

Axioms.

Axioms.

6. *Every* whole Thing is GREATER than its PART.
That is, the whole Line *AB* is *greater* than its *Part Ac*, &c. 

The ſame is to be underſtood of *Superficies*'s and *Solids*.

7. *Every* Whole is EQUAL to all its PARTS taken together.
That is, the whole Line *AB* is *equal* to its *Parts* $AC + cd + de + eB$. 

The ſame is alſo true in *Superficies*'s and *Solids*.

8. Thoſe *Things* which being laid one upon another, do *agree* or *meet* in all their Parts, are equal one to the other.

But the Converſe of this *Axiom*, to wit, that *equal Things* being laid one upon the other will meet, is only *true* in *Lines* and *Angles*, but not in *Superficies*'s, unleſs they be alike, *viz.* of the ſame *Figure* or Form: As for Inſtance, a Circle may be *equal* in *Area* to a Square; but if they are *laid* one upon the other, 'tis plain they cannot *meet* in all their *Parts*, becauſe they are *unlike Figures*. Alſo, a *Parallelogram* and a *Triangle* may be equal in their *Area*'s one to another, and both of them may be equal to a *Square*; but if they are *laid* one upon the other, they will not *meet* in all their *Parts*, &c.

Note, *Beſides the* Characters *already explain'd in* Part I, *and in other Places of this Tract, theſe following are added.*

Viz. $\angle$ denotes an Angle in general, and $\angle\angle$ ſignifies Angles; $\triangle$ ſignifies a Triangle; $\square$ ſignifies a Square, and $\square\!\!-$ denotes a Parallelogram. And when an Angle is denoted by any *three Letters* (as, *A B C*) the middle *Letter* (as *B*) always denotes the Angular Point; and the other *two Letters* (*as A B* and *B C*) denote the *Lines* or *Sides* of the Triangle which includes that Angle.

Theſe Things being *premiſed*, the young *Geometer* may proceed to the *Demonſtrations* of the following *Theorems*; wherein he may perceive an abſolute Neceſſity of being well *verſed* in ſeveral Things that have been already deliver'd: And alſo it will be very *advantageous* to ſtore up ſeveral uſeful *Corollaries* and *Lemma*'s, as they become *diſcover'd Truths*: For it often happens, that a *Propoſition* cannot be clearly demonſtrated *à priori*, or of itſelf, without a great Deal of Trouble; therefore it will be uſeful to have *Recourſe* to thoſe *Truths* that may be aſſiſting in the *Demonſtrations* then in Hand.

THEO-

THEOREM I.

If a Right-line ſtand upon (or meet with) another Right-line, and make Angles with it, they will either be two Right-angles, or two Angles equal to two Right-angles. (13. *e.* 1.)

Demonſtration.

Suppoſe the Lines to be *AB* and *DC*, meeting in the *Point* at *C*: Upon *C* deſcribe any Circle at pleaſure; then will the Arch *AD* be the *Meaſure* of the ∠ *b*, and the Arch *DB* the *Meaſure* of ∠ *e*; but the Arches *AD* + *DB* = 180°, *viz.* they compleat the *Semicircle*. Conſequently the ∠ *b* + ∠ *e* = 180°. *Which was to be prov'd.*

Corollaries.

1. Hence it follows, that if the ∠ *b* = 90° then ∠ *e* = 90°; but if ∠ *b* be *obtuſe*, then the ∠ *e* will be *acute*, &c.

From hence it will be eaſy to conceive, that if ſeveral Right-lines ſtand upon, or meet with any Right-line at one and the ſame Point, and on the ſame Side, then all the Angles taken together will be = 180°, *viz.* Two Right-angles.

THEOREM II.

If two Angles interſect (i. e. *cut or croſs*) *each other, the two oppoſite Angles will be equal.* (15. *e.* 1.)

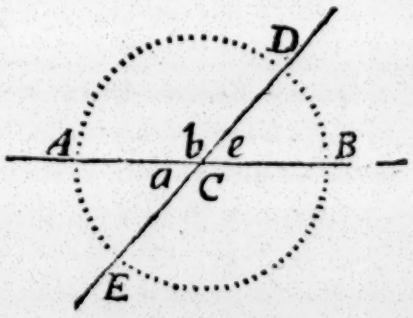
Demonſtration.

Let the *two Lines* be *AB* and *DE*, interſecting each other in the Center *C*.

Then ∠ *b* + ∠ *e* = 180°
And ∠ *b* + ∠ *a* = 180° } *per* laſt.

Conſequently ∠ *b* + ∠ *e* = ∠ *b* + ∠ *a*, *per Axiom* 5.

Subtract ∠ *b* on both Sides of the *Æquation*, and it will leave ∠ *e* = ∠ *a*.



Again, ∠ *b* + ∠ *e* = 180°, as before; and ∠ *e* + ∠ *C* = 180°, conſequently ∠ *e* + *C* = ∠ *b* + ∠ *e*. *Subtract* ∠ *e*, and then ∠ *C* = ∠ *b*. Q. E. D.

Corol-

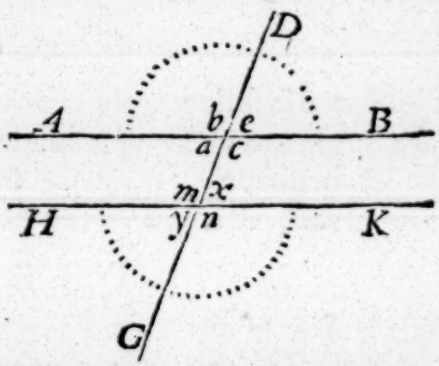
Corollary.

From hence it is evident, that if two Lines interſect each other, they will make four Angles; which, being taken together, will always be equal to Four Right-angles.

THEOREM III.

If a Right-line cut (or croſs) two parallel Lines, it will make the oppoſite Angles equal one to another. (29. *e.* 1.)

Suppoſe the two Lines *AB* and *HK* to be parallel, and the Right-line *DG* to cut them both at *C* and *n*: Upon the Point *C* (with *any Radius*) deſcribe a Semicircle; and with the ſame *Radius*, upon the Point at *n*, deſcribe another Semicircle oppoſite to the firſt, as in the Figure. Then 'tis plain, and I ſuppoſe very eaſy to conceive, that if the Center *C* were mov'd along upon the Line *DG*, until it came to the Center at *n*, the two Lines *AB* and *HK* would *meet* and concur, *viz.* become one Line (*for parallel Lines are as it were but one broad Line.*) Conſequently the two Semicircles would alſo meet, and become one entire Circle, like to that in the laſt *Demonſtration.*



And therefore the $\angle y = \angle x = \angle a = \angle e$ } { as before, *per*
And $\angle m = \angle n = \angle b = \angle c$ } { *laſt Theorem.*

Q. E. D.

Corollary.

Hence it follows, that if *three*, *four*, or ever ſo many Parallel-lines, are cut or croſs'd by one Right-line, all their oppoſite Angles will be equal.

THEOREM IV.

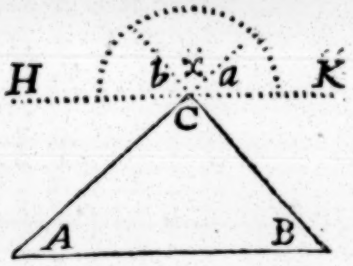
The three Angles of every plain Triangle are equal to two Right-angles. (32. *e.* 1.)

Conſequently, any two Angles of any plain Triangle muſt needs be leſs than two Right-angles. (17. *e.* 1.)

Demon-

Demonſtration.

Let the △ *ABC* be propos'd; draw the Right-line *HK* parallel to the Side *AB*, juſt touching the Vertical Angle *C*; and upon the ſame Angular Point *C* deſcribe any Semicircle, and produce the Sides *AC* and *BC* to its *Periphery*. Then will ≺ *b* = ≺ *B*, ≺ *a* = ≺ *A*, and ≺ *x* = ≺ *C*, *per* laſt *Theorem*. But ≺ *b* + ≺ *a* + ≺ *x* = 180°, or two Right-angles: Conſequently ≺ *B* + ≺ *A* + ≺ *C* = 180° per *Axiom* 5. Q. E. D.



Corollary.

Hence it follows, that the two *acute* Angles of every Right-angled Triangle are equal to a Right-angle, or 90°.

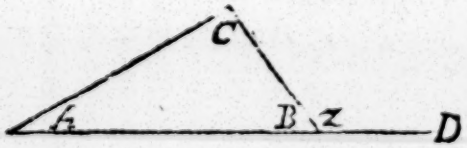
Conſequently, if one of the *acute* Angles be given, the other is alſo given, *viz.* 90°—the given ≺ leaves the other ≺.

THEOREM V.

If one Side of any plain Triangle be continued or produced beyond, or out of the Triangle, the outward Angle will always be equal to the two inward oppoſite Angles. (32. *e.* 1.)

Demonſtration.

Let the Side *AB* of the △ *ABC* be produced out of the △, ſuppoſe to *D*, &c. as in the Figure. Then ≺ *z* = ≺ *A* + ≺ *C*, for the ≺ *B* + ≺ *z* = 180° per *Theorem* 1. and the ≺ *B* + ≺ *A* + ≺ *C* = 180°, *per* laſt *Theorem*. Therefore ≺ *B* + ≺ *z* = ≺ *B* + ≺ *A* + ≺ *C*, per Axiom 5. *Subtract* ≺ *B* on both Sides the *Æquation*, and it will leave ≺ *z* = ≺ *A* + ≺ *C* (*per Axiom* 2.) Q. E. D.



Conſequently, the outward Angle (at z) of any plain Triangle, muſt needs be greater than either of the inward oppoſite Angles, viz. *greater than* ≺ *A, or* ≺ *C* (16. *e.* 1.)

Corollary.

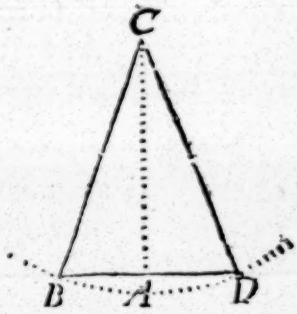
Hence it follows, that if one Angle of any plain Triangle be given, the *Sum* of the other two Angles is alſo given; for 180°— the given ≺ = the other two ≺ ≺.

THEO-

THEOREM VI.

In every plain Triangle, equal Sides ſubtend (viz. *are oppoſite to*) *equal Angles.* (5. e. 1.)
Conſequently, equal Angles are ſubtended by equal Sides (6. e. 1.)

Demonſtration.



Suppoſe the $\triangle BCD$ to be an *Iſoſceles* $\triangle$; that is, let $BC = CD$. Biſect the $\sphericalangle C$, or (*which is all one*) make CA *perpendicular* to BD; then will the $\sphericalangle\sphericalangle$ on each Side of it (*viz.* $\sphericalangle BAC$ and $\sphericalangle DAC$) be Right-angles.

Therefore $\left\{\begin{matrix}\frac{1}{2}\sphericalangle C + \sphericalangle B = 90^\circ \\ \frac{1}{2}\sphericalangle C + \sphericalangle D = 90^\circ\end{matrix}\right\}$ per *Corol.* to *Theorem* 4.
Conſequently, $\frac{1}{2}\sphericalangle C + \sphericalangle B = \frac{1}{2}\sphericalangle C + \sphericalangle D$, per *Axiom* 5.
Subtract $\frac{1}{2}\sphericalangle C$ from both Sides of the *Æquation*, and it will leave $\sphericalangle B = \sphericalangle D$, per *Axiom* 2. Q. E. D.

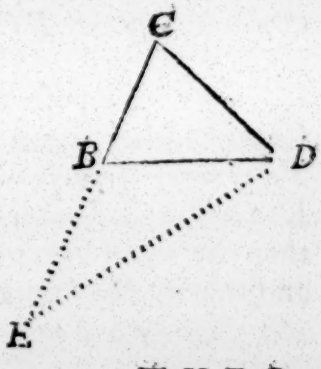
Corollary.

From hence it follows, that the *three Angles* of an *Equilateral Triangle* are equal one to another.

THEOREM VII.

In every plain Triangle, the longeſt Side ſubtends the greateſt Angle. (18. e. 1.)
Conſequently, the greateſt Angle of any plain Triangle is ſubtended by the longeſt Side.

This *Theorem* is evident by Inſpection only: For, let one of the Sides of any *plain Triangle* (as CB) be produced, ſuppoſe to E; join DE with a *Right-line*; then 'tis evident, that becauſe CE is now made longer than the Side BC, therefore the $\sphericalangle$ at D is become *larger* than it was before by the $\sphericalangle BDE$: And it's plain, the longer the Side CE had been made, the $\sphericalangle$ at D would have been the more enlarged.



THEOREM VIII.

If the Sides of two Triangles are equal, the Angles oppoſite to thoſe equal Sides will be equal. (8. *e.* 1.)

The Truth of this *Theorem* is evident by the two included *Triangles* in the 6th *Theorem*, for they have their reſpective Sides equal, *viz.* $BC = CD$, $BA = DA$, and CA common to both *Triangles*. And it is there prov'd, that the $\angle$ oppoſite to thoſe equal Sides are equal, &c. which needs no further *Proof*.

Note, *The Converſe of this* Theorem *holds not true; for the Angles of two Triangles may be equal, and their oppoſite or ſubtending Sides unequal; as will appear at* Theorem XII.

Corollary.

Hence it follows, that *Triangles mutually equilateral* are alſo *mutually equiangular*; and,

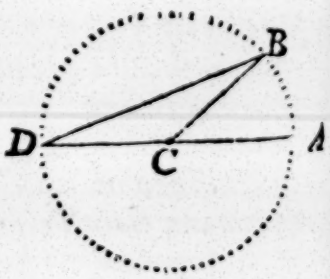
That *Triangles mutually equilateral* are equal one to another. (4. & 26. *e.* 1.)

THEOREM IX.

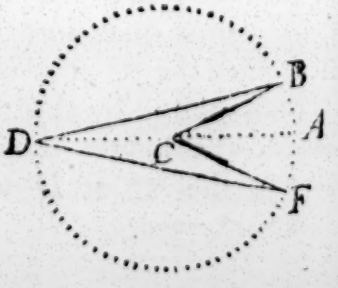
An Angle at the Center of any Circle is always double to the Angle at the Periphery, when both the Angles ſtand upon the ſame Arch. (20. *e.* 3.) This *Theorem* hath three Varieties or Caſes.

Demonſtration.

Caſe 1. Let the Diameter DA, and the Line DB, be the two Lines which form the $\angle D$ at the *Periphery*; draw the *Radius* BC, then $\angle BCA$ is the $\angle$ at the *Center*. But $\angle BCA = \angle D + \angle B$, *per Th* 5. and becauſe $DC = BC$, therefore $\angle D = \angle B$, per *Theorem* 6. conſequently $\angle BCA = 2 \angle D$.



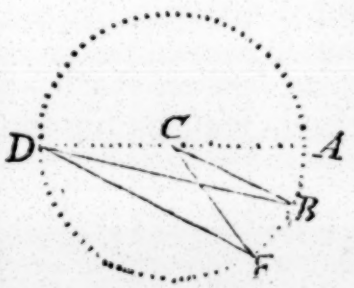
Caſe 2. Suppoſe the $\angle BCF$ at the Center to be within the $\angle BDF$ at the *Periphery*, *(as in the annexed Figure.)* Draw the Diameter DA;
then the $\angle BCA = 2 \angle BDA$
and the $\angle FCA = 2 \angle FDA$ } per *Caſe* 1.
add theſe two *Equations* together.



Then

Then will $\angle BCA + \angle FCA = 2 \angle BDA + 2 \angle FDA$, per *Ax.* 1. But $\angle BCA + \angle FCA = \angle BCF$, and $2 \angle BDA + 2 \angle FDA = 2 \angle BDF$. Consequently $\angle BCF = 2 \angle BDF$.

Case 3. Again, suppose the $\angle BCF$ at the Center to be out of the $\angle BDF$ at the Periphery. From the Angular Point D at the Periphery draw the Diameter DA.



Then $\angle FCA = 2 \angle FDA$ and $\angle BCA = 2 \angle BDA$ } per Case 1.

Subtract this last *Æquation* from the other, and it will leave $\angle FCA - \angle BCA = 2 \angle FDA - 2 \angle BDA$, per *Axiom* 2. But $\angle FCA - \angle BCA = \angle FCB$, and $2 \angle FDA - 2 \angle BDA = 2 \angle FDB$: Consequently $\angle FCB = 2 \angle FDE$. Q. E. D.

Corollary.

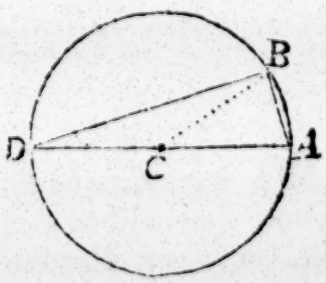
Hence 'tis evident, that all Angles at the Periphery, which stand on the same *Segment* or *Arch* of a Circle, or upon equal Arches, are equal one to another. (21. *e.* 3.)

THEOREM X.

An Angle in a Semicircle is a Right angle. (31. *e.* 3.) That is, *if the Diameter of any* Circle *be the Side of a Triangle, and the Angle opposite to that Side be any where in the* Circle's *Periphery, it will be a Right-angle.*

Demonstration.

Let DA be the Diameter, and DBA the Triangle, then, $\angle B = 90°$. Draw the Radius BC, then is the $\angle DBA = \angle D + \angle A$. For $\angle CBD = \angle D$, and $\angle CBA = \angle A$, per *Theorem* 6. Therefore $\angle DBA = \angle CBD + \angle CBA$, per *Axiom* 5. Again $\angle DBA + \angle D + \angle A = 180°$, per *Theorem* 4. Consequently, $\angle DBA = 90°$ or a Right-angle. Q. E. D.



Corollaries.

1. Hence it will be eaſy to conceive, that an Angle made in any Segment leſs than a Semicircle will be *obtuſe*, or greater than a Right-angle.

2. And an Angle, made in any Segment greater than a Semicircle, muſt conſequently be *acute*.

THEOREM XI.

In any Right-angled Triangle, the Square which is made of the Hypothenuſe, or Side ſubtending the Right angle, is equal to both the Squares which are made of the Sides including the Right-angle. (47. *e.* 1.)

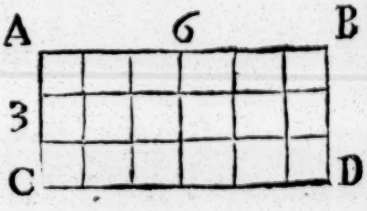
There are ſeveral Ways of demonſtrating this noble and uſeful *Theorem*, but, I preſume, none more eaſy to be underſtood by a *Learner* than that which I ſhall here propoſe: And, in order thereto, 'twill be neceſſary to premiſe the following *Lemma's*.

Lemma 1.

A *Right-line* is ſaid to be *multiply'd* with a *Right-line*, when either a *Square*, or other *Right-angled Parallelogram*, is made of the *two Lines*.

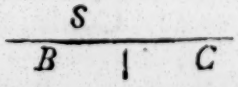
That is, the *Area* of any *Right-angled Parallelogram* is equal to the *Product* of thoſe *Numbers* which expreſs the *Meaſure* of its Sides.

Thus, if $AB = 6$ Inches and $AC = 3$ Inches: Then $AB \times AC = 6 \times 3 = 18$ ſquare Inches; which is the *Area* of the Parallelogram $ABCD$.



Lemma 2.

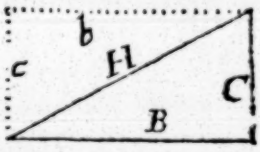
If a *Right-line* be any way cut into *two Parts*, the *Square* of the whole *Line* will be equal to the *Squares* of each Part, and a double *Rectangle* or *Parallelogram* made of both the Parts, (4. *e.* 2.) that is, if the *Line* S be cut into the two Parts B and C; then is $S = B + C$; But if both the Sides of the *Æquation* be involv'd, it will be $SS = BB + 2BC + CC$.



Lemma

Lemma 3.

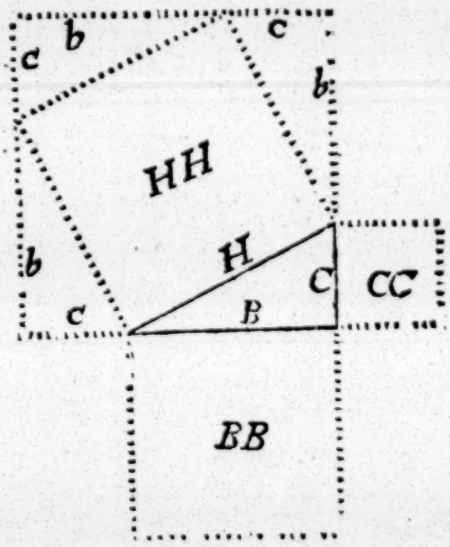
The *Area* of every *Right-angled Triangle* is half the *Parallelogram* made of its Baſe and Perpendicular.



For $B \times C =$ the *Area* of the whole *Parallelogram*, by the firſt *Lemma*. And $\triangle BCH + \triangle bcH =$ the *Parallelogram*; but $B = b$, and $C = c$. Therefore $\frac{1}{2} B \times C =$ the *Area* of each $\triangle$, *viz.* $\frac{1}{2} B \times C + \frac{1}{2} b \times c$ $B \times C$.

Theſe Things being premiſed, let us ſuppoſe the Triangle BCH to be a *Right-angled Triangle*, *viz.* the Side C perpendicular to the Side B; then will $BB + CC = HH$.

Demonſtration.



Make a *Square* whoſe Side is $= B + C$, and draw the included *Square* whoſe Side is $= H$, as in the Scheme: Then will the *Area* of the *great Square* be equal to the *Area* of the four *Triangles* $+ HH$; but the *Area* of each $\triangle = \frac{1}{2} BC$, or $B \times C$, per *Lemma* 3. Therefore the 4 $\triangle$'s $= \frac{1}{2} BC \times 4 = 2BC$, conſequently, the *Area* of the great *Square* is $HH + 2BC$. *Involve* $B + C$, and it will be $BB + 2BC + CC =$ the *Area* of the great *Square*; per *Lemma* 3. Conſequently, $HH + 2BC = BB + 2BC + CC$, per *Axiom* 5. Subtract $2BC$ from both Sides of the *Æquation*, and there will *remain* $HH = BB + CC$.

To illuſtrate this *Theorem* by *Numbers*, let us

Suppoſe $C = 3$. $B = 4$. and $H = 5$.
Then will $CC = 9$. $BB = 16$. and $HH = 25$.
Conſequently, $BB + CC = HH = 16 + 9 = 25$.

Conſectary.

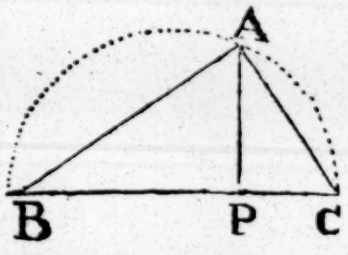
From this admirable *Theorem* (*ſaid to be firſt invented by* Pythagoras) is deduced the *Method* of adding and ſubtracting Squares, Parallelograms, Circles, *&c.*

THEO-

THEOREM XII.

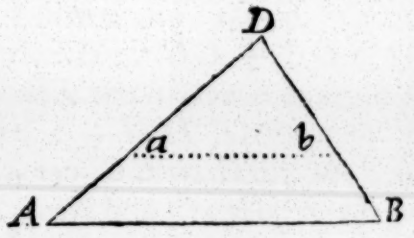
In any Right-angled Triangle, a Perpendicular being let fall from the Right-angle upon the Hypothenuſe will divide the Triangle into two Right angled Triangles, which will be both ſimilar (or alike) to the firſt Triangle, and to each other. (8. *e.* 6.)

Note, *All plain Triangles are ſaid to be ſimilar (viz. alike) when each ſingle Angle in one of the Triangles is equal to each ſingle Angle of the other; but if any two ſingle Angles of one Triangle are equal to two ſingle Angles of the other, the third Angle will be equal. Per* Theo. 4.



1. In the Right-angled $\triangle BAC$, let AP be ſuppoſed *perpendicular* to the *Hypothenuſe* BC; then $\prec BAP = \prec C$. For $\prec BAP + \prec B = 90°$, and $\prec B + \prec C = 90°$, per *Corollary* to *Theorem* 4. Therefore the $\prec BAP = \prec C$, per *Axiom* 5. again, $\prec PAC + \prec C = 90$, and $\prec B + \prec C = 90°$. Therefore $\prec PAC = \prec B$, &c. Conſequently the $\triangle BAP$ is alike to the $\triangle ACP$; and each is like to the whole $\triangle BAC$.

2. Or if a *Right line* be drawn parallel to one of the Sides of any *plain Triangle*, (*viz.* within it) it will cut off a Triangle ſimilar or alike to the whole Triangle. Thus:



In the $\triangle ABD$ draw the *Right-line* $a\,b$ parallel to the Side AB; then will the included $\triangle a\,D\,b$ be like the $\triangle ADB$: For $\prec a = \prec A$ and $\prec b = \prec B$, per *Theorem* 3; and $\prec D$ is common to both the Triangles; *Ergo*, &c.

THEOREM XIII.

If two Triangles are alike, their like Sides will be proportional.

That is, thoſe Sides which ſubtend the equal Angles, as alſo thoſe Sides which are *about* the equal Angles, will be *proportional* to each other; and conſequently, if any *two Triangles* have their Sides proportional, their Angles are equal. (4, 5, 6, 7. *e.* 6.)

Demon-

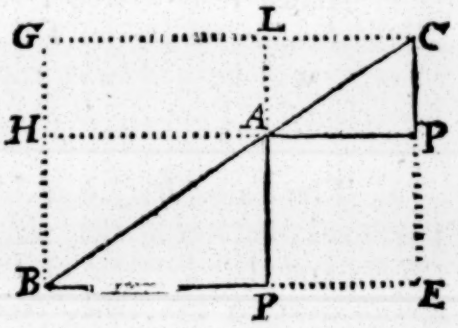
Demonſtration.

Let the *ſimilar Triangles* in the *Scheme* of the laſt *Theorem* be here propoſed again.

Then it will be $BP : AP :: AP : CP$, according to this *Theorem*. *Ergo* $BP \times CP = AP \times AP$.

Firſt.

Let us ſuppoſe the aforeſaid *Right-angled* $\triangle BAC$ cut through the Perpendicular AP, and there open'd until the Sides BA and CA become one *Right-line*. Let the Sides BP and CP be continued until they meet in E; then compleat the Parallelograms by drawing the parallel Lines GLC, HAP, GHB, and LAP, as in the *Figure*.

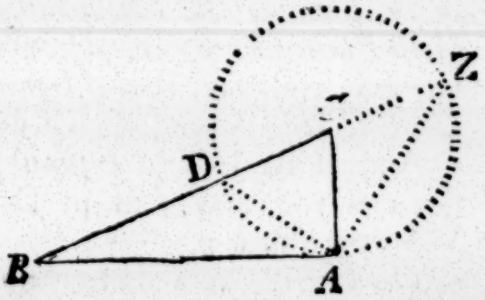


Then it is evident, that the $\triangle BHA = \triangle BPA$, and the $\triangle CPA = \triangle CLA$; alſo that the $\triangle BEC = \triangle BGC$, becauſe all their reſpective Sides are *equal*.

But the $\triangle BHA + \triangle CLA + \square HGLA = \triangle BPA, + \triangle CPA + \square APEP$. Now if from both *Sides* of this *Æquation* there be ſubtracted the equal *Triangles*, there will remain $\square HGLA = \square APEP$. But $\square HGLA = BP \times CP$, and $\square APEP = AP \times AP$. Conſequently $BP : AP :: AP : CP$. Which was to be prov'd.

Or otherwiſe, thus:

Suppoſe the $\triangle BAC$ to be *Right-angled* at A: Upon the $\prec$ Point C, with the *Radius* CA deſcribe a Circle, and continue the *Hypothenuſe* BC to Z; join ZA and AD with *Right-lines*; then will the $\triangle BAD$ be like to the $\triangle BZA$. For $\prec DAB + \prec DAC = 90°$, by *Conſtruction*. And $\prec ZAC + \prec DAC = 90°$, by *Theorem* X. Therefore $\prec DAB + \prec DAC = \prec ZAC + \prec DAC$. By *Axiom* 5. *ſubtract* $\prec DAC$ from both Sides of the *Æquation*, and there will remain $\prec DAB = \prec ZAC$. But $\prec ZAC = \prec CZA$, by *Theorem* 6. And $\prec B$ is common



to

to both Triangles. Therefore $\prec BDA = \prec BAZ$, by *Theorem* 6, conſequently $\triangle BAD$ is like to $\triangle BZA$.

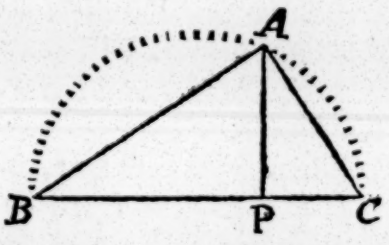
Let the *Sides* $\left\{\begin{matrix} BA = b \\ BC = h \\ CA = c \end{matrix}\right.$ Then $bb + cc = hh$, by *Theorem* 11. Conſequently $bb = hh = cc$, which gives the following Analogy, Viz. $b : h + c :: h - c : b$; that is, $BA : BZ :: BD : BA$.

Q. E. D.

Corollaries.

1. Hence it is evident, that, in any *Right-angled Triangle*, a *Perpendicular*, being let fall from the *Right-angle* upon the *Hypothenuſe*, will be a *Mean* proportional between the *Segments* of the *Hypothenuſe:* That is, $BP : PA :: PA : PC$.

2. The Baſe *(BA)* is a *Mean* proportional between the *Hypothenuſe (BC)* and that *Segment* of the *Hypothenuſe* next to the Baſe, *(viz. BP)* that is, $BC : BA :: BA : BP$.



3. The *Cathetus (AC)* is a *Mean* proportional between the *Hypothenuſe (BC)* and that *Segment* of the *Hypothenuſe* next to the *Cathetus*, (viz. *PC)*: That is, $BC : AC :: AC : PC$.

Scholium.

I have been more large upon this moſt excellent *Theorem*, in giving a double *Demonſtration* of it, becauſe it is ſo *univerſally uſeful* in all Parts of the *Mathematicks:* For the Buſineſs of *Trigonometry (both Plain and Spherical)* wholly depends upon it; and therefore one may truly ſay, that *Aſtronomy*, *Dialling*, *Navigation*, *Surveying*, *Opticks*, &c. depend upon a due Application of it.

And of its Uſe in *Geometry*, *Des Cartes* takes particular Notice; as you may find in Dr. *Pell*'s Algebra, *Page* 65, whoſe Words are theſe:

Des Cartes, in a Letter not yet printed, writes thus: "In "ſearching the *Solution* of *Geometrical Queſtions*, I always make "uſe of Lines parallel and perpendicular, as much as is poſſible, "[*he means as many Lines as are uſeful*] and I conſider no "other Theorems but theſe two, [*the Sides of like* Trian-"gles *have like Proportion*]. And [*in Rectangle Triangles*

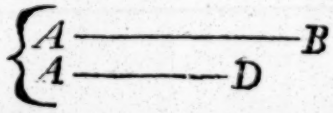
" *the Square of the greateſt Side is equal to the Squares of the two other* " *Sides.*] And I am not afraid to ſuppoſe many unknown Quanti- " ties, that I may reduce the propos'd Queſtion to ſuch Terms, as " to depend on no other Theorems but theſe Two."

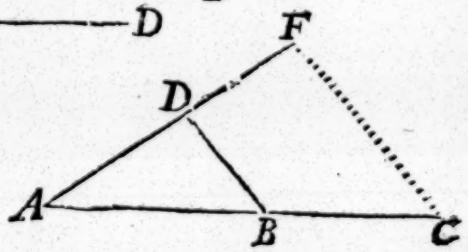
This I thought convenient to inſert, that the young *Learner* may ſee how the great *Des-Cartes* eſteem'd theſe two Theorems, *viz.* the laſt, and Theorem 11; for, in Truth, all the precedent Theorems are only (as it were) Preparatives to theſe Two.

This laſt Theorem demonſtrates the *Reaſon* of the *Method* uſed in finding out *Proportional Lines*; as in the Three following *Problems.*

PROBLEM I.

Two Right-lines being given, to find a Third in Proportion to them. (11. *e.* 6.)

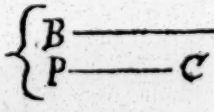
Let theſe *Lines* be 

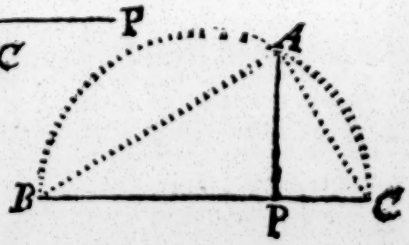


Set the Two given *Lines* at any Angle in the Point *A*, and produce the Line *AB* to *C*, making $BC = AD$; join the Points *B D* with a *Right-line*, and draw *CF* parallel to *BD*; then will the $\triangle ABD$ be like the $\triangle ACF$. Therefore $AB : BC (= AD :: AD : DF$, which is the third *Proportional* required.

PROBLEM II.

Two Right-lines being given, to find a Mean Proportional Line between them. (13. *e.* 6.)

Let the given *Lines* be 



Join the two given *Lines* into one, *viz.* make $BC = BP + PC$, and upon *BC*, as *Diameter*, deſcribe a *Semicircle*; then upon the Point *P*, where the two *Lines* meet, erect a Perpendicular to touch the Cir-

Circle's *Periphery*, as *P A*, and it will be the *Mean Proportional* requir'd, viz. *B P* : *A P* : : *A P* : *P C*.

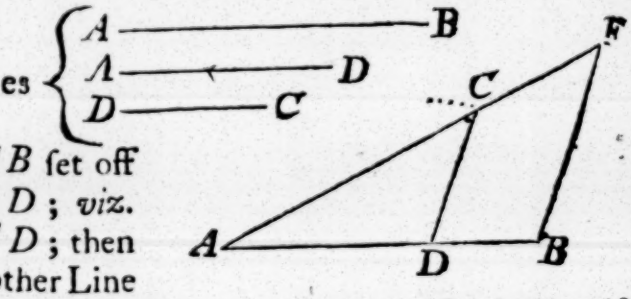
By this *Problem* 'tis eaſy to conceive how to make a *Square* equal to any given *Parallelogram*. (14. *e.* 6)

For if *B P* be the *Length*, and *P C* be the *Breadth* of the given *Parallelogram*, then will *A P* be the Side of the *Squa e*, equal in *Area* to that *Parallelogram*.

PROBLEM III.

Three Right-lines being given, to find a fourth Proportional Line (12. *e.* 6.)

Suppoſe the three Lines



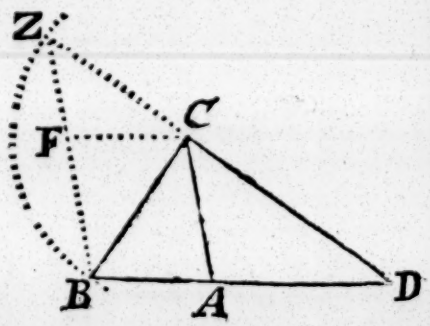
Upon the longeſt Line *A B* ſet off the next longeſt Line *A D*; *viz.* make *D B* = *A B* — *A D*; then upon the Point *D* ſet the other Line *D C* at an *Angle*, either *right* or *oblique*, and draw the *Right-line AC* continuing it a ſufficient Length; make *B F parallel* to *D C*, and it will be the fourth *Proportional* required; that is, *A D* : *DC* : : *AB* : *BF*.

THEOREM XIV.

If any Angle of a plain Triangle be biſected (viz. *divided into two equal Angles*) *with a Right-line*, (viz. *as C A is ſuppos'd to do the Angle B C D*) *it will cut the oppoſite Side* (viz. *B D*) *in Proportion to the other two Sides of the Triangle* (3. *e.* 6.) i. e. *B A* : *BC* : : *AD* : *C D*.

Demonſtration.

Produce the Side *D C*, until *C Z* = *C B*: join the Points *Z B* with a *Right line*, and draw the Line *FC* parallel to *B D*; whence the ∠ *Z* = ∠ *CBZ* per *Theorem* 6. and ∠ *Z* + ∠ *C B Z*, or 2 ∠ *C B Z* = ∠ *BCD*, per *Theorem* 5; or, dividing both Sides of the Æquation by 2, ∠ *C B Z* = ½ ∠ *BCD*. But ½ ∠ *B C D* = ∠ *ACB* = ∠ *ACD* by the *Hypotheſis*, therefore ∠ *ACB* = ∠ *C B Z* per *Axiom* 5: Whence *AC* is parallel to *B Z* per *Theorem* 3, and the Triangles *BDZ*, *ADC*, and *FCZ* are ſimilar by the ſecond Figure to *Theorem* 12. conſequently *B A* (= *FC*) : *BC* (= *ZC*) : : *AD* : *CD*. Q. E. D.

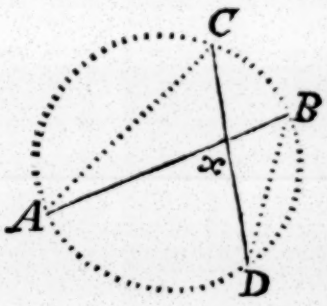


THEO-

THEOREM XV.

If two Right lines (howſoever drawn) within a Circle *do cut each other, the Rectangle made of the Segments (or Parts) of the one Line, will be equal to the Rectangle made of the Segments (or Parts) of the other Line.* (35. *e.* 3.)

That is, if two Lines (as AB and CD) do cut each other in any Point, as at x, then will $Ax \times Bx = Dx \times Cx$.



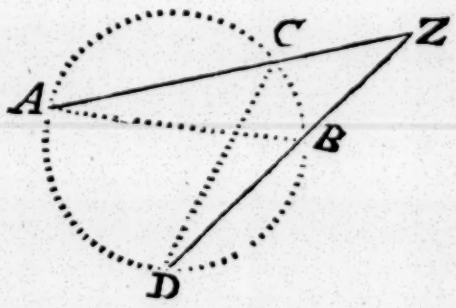
Demonſtration.

Join the Points of AC and BD with *Right-lines*, then will the $\triangle Cx A$ be like to $\triangle Bx D$: For $\prec B = \prec C$ and $\prec A = \prec D$. By Corollary to Theorem 9. and $\prec Ax C = \prec Bx D$. by *Theorem* 2. Therefore it will be $Ax : D :: Cx : Bx$. by *Theorem* 13. Conſequently $Ax \times Bx = Dx \times Cx$. Q. E. D.

THEOREM XVI.

If two Right lines are ſo drawn within a Circle, *as, being continued, they will meet in a Point out of the* Circle's *Periphery, the Rectangle made of the one whole Line, and its Part out of the* Circle, *will be equal to the Rectangle of the other whole Line, and its Part out of the* Circle. (36, 37. *e.* 3.)

That is, if the Lines AC and DB be continued unto the Point Z; then will $AZ \times CZ = DZ \times BZ$.



Demonſtration.

Draw the Lines AB and CD, then will $\triangle CZD$ be like to the $\triangle BZA$; for $\prec A = \prec D$, and $\prec Z$ is common to both *Triangles*, conſequently, $\prec ABZ = \prec DCZ$, by *Theorem* 4. therefore $AZ : BZ :: DZ :: CZ$. *Ergo*, $AZ \times CZ = DZ \times BZ$.

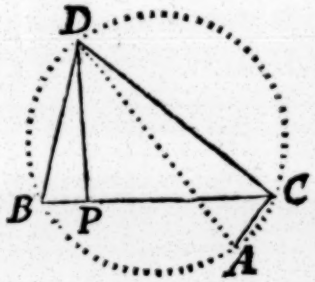
THEOREM XVII.

If from any Angle of a plain Triangle inſcribed in a Circle *there be let fall a Perpendicular upon the oppoſite Side, as DP; as that Perpen-*

dicular is in Proportion to one of the Sides including the Angle, ſo is the other Side including the Angle to the Diameter of the Circle.

Demonſtration.

Let BCD be the propoſed *Triangle*. From the $\prec$ at D draw the Diameter DA; then will $\prec A = \prec B$, becauſe they both ſtand upon the ſame Arch DC, and $\prec DCA = 90°$, by *Theorem* 10. conſequently the $\prec ADC = \prec BDP$, by *Theorem* 4. Therefore $\triangle DCA$ is like to the $\triangle DPB$; and therefore, $DP : DB :: DC : DA$; or $DP : DC :: DB : DA$. Q. E. D.



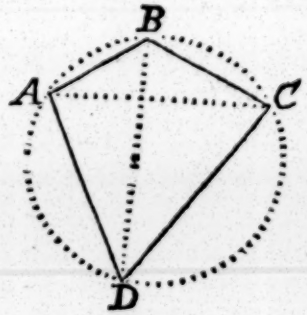
THEOREM XVIII.

If any Quadrangle (that is, a Trapezium*) be inſcrib'd within a* Circle, *the two oppoſite Angles, taken together, are equal to two Right-Angles,* viz. 180°. (22. *e.* 3.)

That is, in the *Quadrangle* $ABCD$ the $\prec A + \prec C = 180°$. And the $\prec B + \prec D = 180°$.

Demonſtration.

Draw the *two* Diagonals AC and BD; then will the $\prec BDA = \prec BCA$, and the $\prec BDC = \prec BAC$ by Corollary to *Theorem* 9. But $\prec ABC + \prec BCA + \prec BAC = 180°$. by *Theorem* 4. and the $\prec BDA + \prec BDC = \prec ADC$. Therefore the $\prec ABC + \prec ADC = 180°$. and by the ſame Way of arguing it may be prov'd, that the $\prec BAD + \prec BCD = 180°$. Q. E. D.



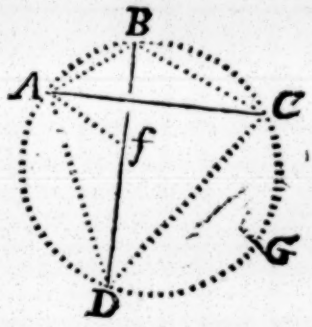
THEOREM XIX.

If in any Quadrangle inſcrib'd within a Circle *there be drawn two Diagonals, as AC and BD, the Rectangle made of the two Diagonals will be equal to both the Rectangles made of the oppoſite Side of the Quadrangle.*

That is, $AC \times BD = AB \times CD + AD \times BC$.

Demon-

Demonſtration.



Make the Arch DG = Arch BC, and from the Points AG draw the Line Af, and it will form the $\triangle Af D$, like to the $\triangle ABC$: For the $\prec fAD = \prec BAC$, becauſe the Arches DG and BC are equal.

Again, the $\prec fDA = \prec BCA$, becauſe they both ſtand upon the Arch AB: Conſequently, the $\prec AfD = \prec ABC$, by *Theorem* 4. Therefore it will be $AC : BC :: AD : Df$, by *Theorem* 13. *Ergo* $\frac{BC \times AD}{AC} = Df$.

Again, the $\triangle BAf$ and $\triangle ACD$ are alike: For $\prec ABf = \prec ACD$, and $\prec BAf = \prec CAD$, becauſe the $\prec fAD = \prec BAC$, and the $\prec CAf$ is common to both *Triangles*. Conſequently, the $\prec AfB = \prec ADC$. Therefore $AC : CD :: AB : Bf$, by *Theorem* 13. *Ergo* $\frac{CD \times AB}{AC} = Bf$. But $Df + Bf = BD$. Conſequently, $BC \times AD + CD \times AB = BD \times AC$. Q. E. D.

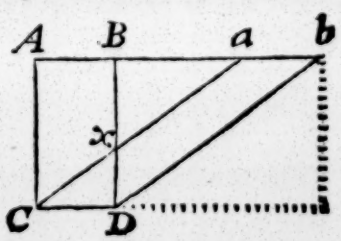
THEOREM XX.

All Parallelograms (whether Right or Oblique angled) that ſtand upon the ſame Baſe, *or upon equal* Baſes, *and betwixt the ſame Parallels, are equal to one another.* (35. & 36. *e.* 1.)

That is, $\square ABCD = \square abCD$.

Demonſtration.

Becauſe $AB = CD = ab$, by Suppoſition, therefore $Aa = Bb$; for Ba is common to both. And becauſe $AC = BD$, and the $\prec A = \prec B$, therefore the $\triangle ACa = \triangle BDb$: And if from both Triangles there be taken the $\triangle Bxa$ common to both, there will remain the *Trapezium* $ABxC = abxD$, per *Axiom* 5.



But

But the *Trapezium* $ABxC + \triangle CxD = \square ABCD$, and the *Trapezium* $abxD + \triangle CxD = \square abCD$, conſequently $\square ABCD = \square abCD$. Q. E. D.

Corollary.

Hence it will be eaſy to conceive, that all *Triangles* which ſtand upon the ſame Baſe, or upon equal Baſes, and between the ſame Parallels, (viz. *having the ſame Height)* are equal one to another. (37 & 38. *e.* 1.)

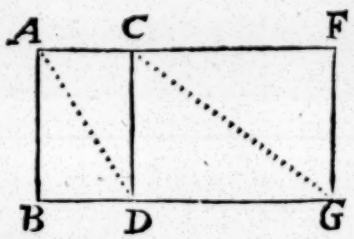
For all *Triangles* are the Halfs of their circumſcribing *Parallelograms*; and therefore, if the Wholes be equal, their Halfs will alſo be equal.

THEOREM XXI.

Parallelograms *(and conſequently* Triangles) *which have the ſame Heighth, have the ſame Proportion one to another as their* Baſes *have.* (1. *e.* 6.)

Demonſtration.

Draw AF parallel to BG, and draw AB, CD, FG Perpendiculars to them. Then will $BD \times AB = \square ABCD$. And becauſe $CD = AB$, therefore $DG \times AB = \square CDFG$, but $BD : DG :: BD \times AB : DG \times AB$. And conſequently $\triangle ABD : \triangle CDG :: BD : DG$, &c.



Q. E. D.

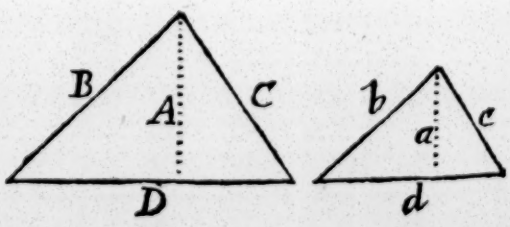
THEOREM XXII.

Like Triangles are in a duplicate Ratio *to that of their homologous Sides.* (19. *e.* 6.)

That is, the *Area*'s of like *Triangles* are in *Proportion* one to another as are the *Squares* of their like Sides.

Demonſtration.

Suppoſe the $\triangle BCD$ and $\triangle bcd$ to be alike, and their like Sides to be thoſe mark'd with the ſame Letters.



Let

Let A and a be Perpendiculars to the two Baſes D and d.

Then $\frac{1}{2} D A$ = the Area of $\triangle BCD$
And $\frac{1}{2} d a$ = the Area of $\triangle b c d$ } By *Lemma* 3, *Page* 303.

But	1	$B : b :: D : d$	} &c. By *Theorem* 13.
And	2	$B : b :: A : a$	
Conſeq.	3	$D : d :: A : a$	
3 ∴	4	$D a = d A$	
4 × $\frac{1}{2} D d$	5	$\frac{1}{2} D D d a = \frac{1}{2} D d d A$. By *Axiom* 3.	
5, Hence	6	$D D : d d :: \frac{1}{2} D A : \frac{1}{2} d a$. And ſo for other Sides.	

Q. E. D.

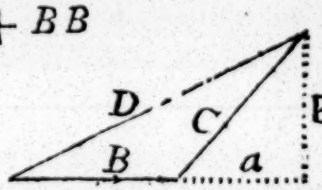
THEOREM XXIII.

In every Obtuſe-angled Triangle (as BCD) the Square of the Side ſubtending the obtuſe Angle (as D) is greater than the Squares of the other two Sides (B and C) by a double Rectangle made out of one of the Sides (as B) and the Segment or Part of that Side produced (as a) until it meet with the Perpendicular (P) let fall upon it. (12. *e.* 2.)

That is, $D D = B B + C C + 2 B a$.

Demonſtration.

Firſt	1	$D D = P P + a a + 2 B a + B B$
And	2	$C C = P P + a a$
1 — 2	3	$D D - C C = 2 B a + B B$
1 + $C C$	4	$D D = B B + C C + 2 B a$



Q. E. D.

Corollary.

Hence it is evident, that, if the Sides of any *Obtuſe-angled Triangle* are given, the *Segment (a)* of the Side produced (or the Perpendicular *P)* may be eaſily found.

THEOREM XXIV.

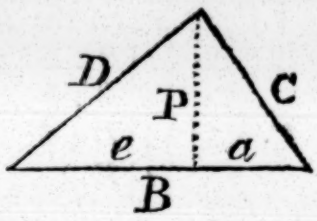
If a Perpendicular *(as P) be let fall into any Acute-angled Triangle (as BCD), the Square of either of the Two Sides (as D) is leſs than the Squares of the other Side, and that Side upon which the* Perpendicular *falls* (viz. *C and B*) *by a double Rectangle made of the Side B, and that Segment or Part of it* (viz. *a*) *which lies next to the Side C.* (13. *e.* 2.)

That is, $D D + 2 B a = B B + C C$.

Demon-

Demonſtration.

First	1	$DD = PP + ee$	} By *Theo.* 11.
And	2	$CC = PP + aa$	}
But	3	$B - a = e$, by Figure.	
3 & 2	4	$BB - 2Ba + aa = ee$	
4 — aa	5	$BB - 2Ba = ee - aa$.	
1 — 2	6	$DD - CC = ee - aa$.	
5, 6	7	$DD - CC = BB - 2Ba$.	
7 ±	8	$DD + 2Ba = BB + CC$.	



Q. E. D.

Corollary.

Hence it follows, that, if the Sides of any *Acute-angled Triangle* be known, the Perpendicular P, and the Segments of the Side whereon it falls (*viz.* a, e.) may be eaſily found.

CHAP. IV.

The Solution *of ſeveral Eaſy* Problems *in plain Geometry, whereby the Learner may (in Part) perceive the Application or Uſe of the foregoing Theorems.*

NOTE, *when a Line, or the Side of any plain Triangle, is any Way cut into two or more Parts, either by a Perpendicular Line let fall upon it, or otherwiſe, thoſe Parts are uſually call'd* Segments; *and ſo much as one of thoſe Parts is longer than the other, is call'd the* Difference of the Segments.

And when any Side of a Triangle, or any Segment of its Side is given, 'tis uſually mark'd with a ſmall Line croſs it, thus: ——|—— *and thoſe Sides or Parts of Sides, that are ſought, are marked with four Points, thus:* $\overset{\cdot\cdot}{\underset{\cdot\cdot}{\text{——}}}$

PROBLEM I.

To cut or divide a given Right-line (as S) into Extreme and Mean Proportion. (11. *e.* 2.)

That is, to *divide a Line* ſo, that the *Square* of the *greater Segment* (or *Part*) a, may be equal to the *Rectangle* made of the whole Line S, and the leſſer Segment e.

Viz.	1	$Se = aa$, by the *Problem*.	
And	2	$S - a = e$, for $S = a + e$.	

S

——|——

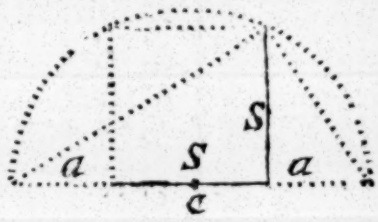
a | e

1 ÷ S

1 ÷ S	3	$\frac{aa}{S} = e$
2 and 3	4	$\frac{aa}{S} = S - a$. By *Axiom* 5.
4 × S	5	$aa = SS - Sa$
5 + Sa	6	$aa + Sa = SS$
6, ſolved	7	$a = \sqrt{SS + \frac{1}{4}SS} : \frac{1}{2}SS$. See *Pages* 195, 196.

Note, *The laſt* Problem *cannot be truly anſwered by Numbers, but Geometrically it may be performed, thus :*

1. Make a Square whoſe Side is $= S$ the given Line, and biſect one of its Sides in the Middle, as at C; upon the Point C deſcribe ſuch a Semicircle as will paſs through the remoteſt Points of the Square, and compleat its Diameter.



2. Then will either Part of the Diameter, on each End of the Side S, be $= a$, the *greater Segment* ſought.

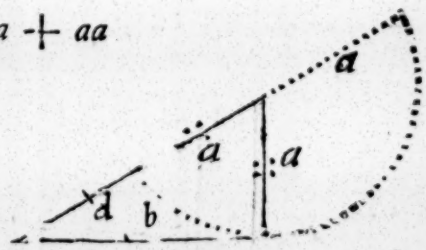
But $a + S : S :: S : a$. By *Theorem* 13.

Ergo, $aa + Sa = SS$. Which was to be done.

PROBLEM II.

The Baſe *of any Right-angled Triangle, and the Difference between the* Hypothenuſe *and* Cathetus *being given, to find the* Cathetus, *&c.*

Let	1	$b = 72$
	2	$d = 32$
And	3	$a =$ *Cathetus* ſought
Then	4	$bb + aa = dd + 2da + aa$ By *Theorem* 11.
4 — aa	5	$bb = dd + 2da$
5 — dd	6	$2da = bb - dd$
6 ÷ $2d$	7	$a = \frac{bb - dd}{2d} = 65$
Or,	8	$b : d + 2a :: d : b$. By *Theorem* 13.
8	9	$bb = dd + 2da$. As before at the 5th Step.

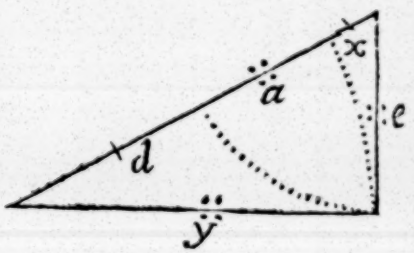


Here you ſee that either Way raiſes the ſame *Æquation*; neither is there any conſtant *Method* or *Road* to be obſerved in ſolving *Geometrical Problems*, but every one makes Uſe of ſuch Ways and Theorems as happen to come firſt into their Mind, the Reſult being every Way the ſame.

PROBLEM III.

The Difference between the Baſe *and* Hypothenuſe *of any Right-angled Triangle, and the Difference between the* Cathetus *and* Hypothenuſe *being both given, to find the Triangle.*

Let	1	$d = 32$
	2	$x = 25$
And	3	$d + x + a =$ the *Hypot.*
Then	4	$d + a = y$ } by the *Probl.*
	5	$x + a = e$ }
4 ⊙²	6	$dd + 2da + aa = yy$
5 ⊙²	7	$xx + 2xa + aa = ee$
3 ⊙²	8	$dd + 2dx + 2da + 2xa + xx + aa = \square$ *Hypothenuſe.*
6+7	9	$dd + 2da + 2xa + xx + 2aa = yy + ee.$



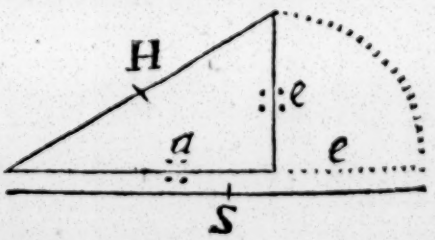
The two laſt Steps are equal, by *Theorem* 11. Conſequently, if thoſe Things that are equal in both be taken away, the Remainders will be equal. By *Axiom* 2.

That is	10	$aa = 2dx = 1600$
10 ww²	11	$a = \surd 2dx = 40$
1 + 11	12	$d + a = 72 = y$ *The* Baſe.
2 + 11	13	$x + a = 65 = e$ *The* Cathetus.
1+2+11	14	$d + x + a = 97$ *The* Hypothenuſe.

PROBLEM IV.

The Hypothenuſe, *and the Sum of the other two Sides, of any Right-angled Triangle, being given, thence to find the Sides.*

Let	1	$H = 97$
And	2	$a + e = S = 137$
By *Fig.* 2 ⊙²	3	$aa + ee = HH$
	4	$aa + 2ae + ee = SS$
4 — 3	5	$2ae = SS - HH$
3 — 5	6	$aa - 2ae + ee = 2HH - SS$
6 ww²	7	$a - e = \surd 2HH - SS$

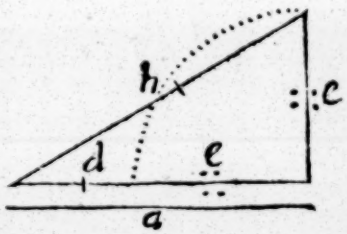


2 + 7

2	+	7	8	$2a = S + \sqrt{2HH - SS} = 144$
8	+	2	9	$a = \frac{S + \sqrt{2HH - SS}}{2} = 72$ *The* Base *required.*
2	—	9	10	$e = \frac{S - \sqrt{2HH - SS}}{2} = 65$ *The* Cathetus.

PROBLEM V.

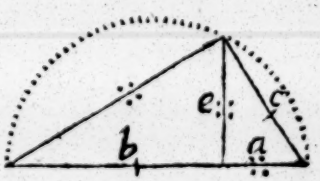
The Hypothenuse, *and the Difference of the other two Sides of any Right-angled Triangle being given, to find the Sides.*



	Let		1	$h = 97$ As before.
	And		2	$a - e = d = 7$ Quere a
	By *Fig.*		3	$aa + ee = hh$
2	⊙²		4	$aa - 2ae + ee = dd$
3	—	4	5	$2ae = hh - dd$
3	+	5	6	$aa + 2ae + ee = 2hh - dd$
6	ww²		7	$a + e = \sqrt{2hh - dd}$
2	+	7	8	$2a = d + \sqrt{2hh - dd} = 144$
8	÷	2	9	$a = 72$
7	—	2	10	$2e = \sqrt{2hh - dd} : - d = 130$
1	÷	2	11	$e = 65$

PROBLEM VI.

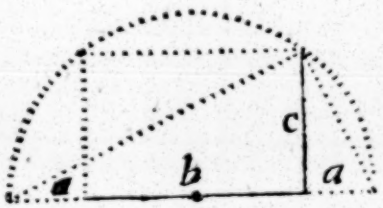
In any Right-angled Triangle, either the Base, *or* Cathetus, *and the alternate Segment of the* Hypothenuse *made by a Perpendicular let fall from the Right-angle, being given, to find the other Segment.*



	Let		1	$e = 45$ *The* Cathetus
	And		2	$b = 48$ *The* alternate *Segm.*
	Then		3	$b : e :: e : a$ Quere a
3	∴		4	$ba = ee$
	Again,		5	$cc - aa = ee$. By *Theor.* 11.
4,		5	6	$ba = cc - aa$
6	+	*aa*	7	$aa + ba = cc$
7,	*C*	□	8	$aa + ba + \frac{1}{4}bb = cc + \frac{1}{4}bb$
8	ww²		9	$a + \frac{1}{2}b = \sqrt{cc + bb}$
9	—	$\frac{1}{2}b$	10	$a = \sqrt{cc + \frac{1}{4}bb} : - \frac{1}{2}b = 27$ And so on for e, &c.

I ſhall now ſhew the *Geometrical Conſtruction* (or *Solution*) of the three Caſes of *Quadratick Æquations* promiſed in 202. Let the firſt Example be that above, *viz.* $aa + ba = cc$. Caſe 1.

Make the Co-efficient b, and the *Root* of the *Reſolvend* (which is here) c, into a *Right-angled Parallelogram*. And upon the middle Point of the Side $= b$, deſcribe ſuch a Semicircle, as will paſs through the remoteſt Points or Angles of the *Parallelogram*, compleating its Diameter, as in the annexed *Scheme*.

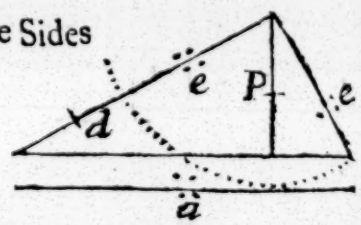


Then will either Part of the Diameter, on each End, be equal to a; the other Part will be $a + b$, and the Side c will be a *mean Proportional* between them: That is, $a + b : c :: c : a$. By *Theorem* 13, conſequently $aa + ba = cc$. Which was to be done.

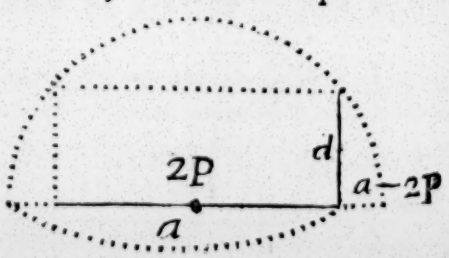
PROBLEM VII.

The Difference between the Baſe *and* Cathetus *of any Right-angled Triangle, and the* Perpendicular *let fall from the Right-angle upon the* Hypothenuſe, *being given; thence to find the* Hypothenuſe, *&c.*

Let	1	$d = 15$ The Difference of the Sides
And	2	$p = 36$
Quere a	3	$a =$ *The* Hypothenuſe.
By *Fig.* 4	4	$d + e : p :: a : e$
∴	5	$de + ee = pa$
Again,	6	$dd + 2de + 2ee = aa$. By *Theorem* 11.
5 × 2	7	$dde + 2ee = 2pa$
6 — 7	8	$dd = aa - 2pa$. Caſe 2.
8 *C* □	9	$aa - 2pa + pp = dd + pp = 1521$.
9 ww²	10	$a - p = \sqrt{dd + pp} = 39$
10 + p	11	$a = p + \sqrt{dd + pp} = 75$, *&c.* for e. per *Step* 5.



The Geometrical Conſtruction of this Caſe 2, viz. $aa - 2pa = dd$, may be performed in the very ſame Manner as the *laſt Caſe* was; that is, by making a Right-angled Parallelogram of the Co-efficient $2p$ and the $\sqrt{dd}$, viz. d, &c. As in the annexed Figure.

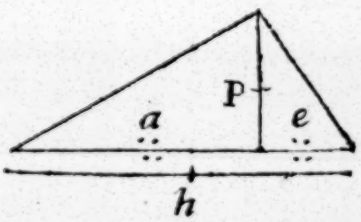


Then

Then will the *greater* Part of the Diameter to one End of the *Parallelogram* be $= a$, and the *leſſer* Part will be $a - 2p$. For $a : d :: d : a - 2p$ by *Theorem* 13. Conſequently, $aa - 2pa = dd$. Which was to be done.

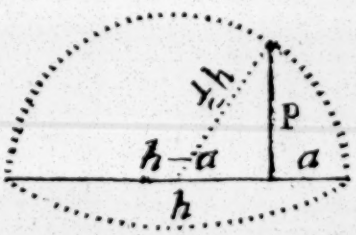
PROBLEM VIII.

The Hypothenuſe *of any Right-angled Triangle, and the* Perpendicular *let fall from the Right-angle upon the* Hypothenuſe, *being given, to find the greater Segment of the* Hypothenuſe, *&c.*



Let	1	$h = 75$ *The* Hypothenuſe
And	2	$p = 36$
Then	3	$a + e = h$ Quere a
per *Fig.*	4	$a : p :: p : e$
4 ∴	5	$\frac{pp}{a} = e$
3 − a	6	$h - a = e$
5, 6	7	$h - a = \frac{pp}{a}$
7 × a	8	$ha - aa = pp$ Caſe 3.
8 +	9	$aa - ha = -pp$
9 C □	10	$aa - ha + \frac{1}{4}hh = \frac{1}{4}hh - pp = 110,25$
10 ω^2	11	$a - \frac{1}{2}h = \sqrt{\frac{1}{4}hh - pp} = 10,5$
11 $+\frac{1}{2}h$	12	$a = \frac{1}{4}h \pm \sqrt{\frac{1}{2}hh - pp} = 48$. Or, $a = 27$.

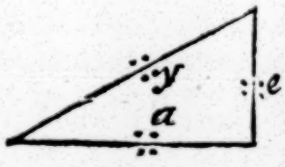
The Geometrical Conſtruction of *Caſe* 3, viz. $ha - aa = pp$, may be thus performed: Draw a *Right-line* (of any convenient Length at Pleaſure) and near its Middle erect a Perpendicular $= p$, *viz.* of the ſame Length with the *Root* of the *Reſolvend.* From the top Point or upper End of that Perpendicular, ſet off Half the Length of the Co-efficient, *viz.* $\frac{h}{2}$ and upon the Point where $\frac{h}{2}$ juſt touches the firſt Line (with the ſame Diſtance) deſcribe a Semicircle; then will its Diameter h be cut by the Perpendicular p into two Segments, which are the two Values of the Root a, *viz.* the *greater* and *leſſer Roots*, both taken together, being always equal to the Co-efficient: (*vide* Page 201.) For $h - a : p :: p : a$ by *Theorem* 13. Ergo, $ha - aa = pp$. Which was to be done.



PRO-

PROBLEM IX.

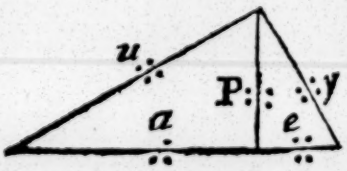
The Perimeter, *i. e. the Sum of all the three Sides of any Right-angled Triangle, and its* Area, *being given, thence to find each Side.*



Viz. Let	1	$a+e+y=s=234$	The *Sum* of the *Sides.*
And	2	$\frac{1}{2}ae=A$	The *Area* $=2340$
Again,	3	$aa+ee=yy$	By Figure
2 × 4	4	$2ae=4A$	
3 + 4	5	$aa+2ae+ee=yy+4A$	
1 — 3	6	$a+e=s-y$	
6 $\odot^2$	7	$aa+2ae+ee=ss-2sy+yy$	
5, 7	8	$yy+4A=ss-2sy+yy$	
8 ±	9	$2sy=ss-4A=45396$	
9 ÷ 2s	10	$y=\frac{ss-4A}{2s}=\frac{1}{2}s-\frac{2A}{s}=97$	*The* Hypothenuse.
6, 10	11	$a+e=s-y=137$	
3 — 4	12	$aa-2ae+ee=yy-4A=49$	
12 au^2	13	$a-e=\sqrt{49}=7$	
11 + 13	14	$2a=137+7=144$	
13 ÷ 2	15	$a=72$	*The* Base.
11 — 15	16	$e=137-72=65$	*The* Cathetus.

PROBLEM X.

In any Right-angled Triangle a Perpendicular *being let fall from the Right angle upon the* Hypothenuse; *if the Sum of each Segment, when added to its adjacent or next Side, be given, thence to find each Side, and the Segments.*



Viz. If	1	$a+u=s=108$	
And	2	$e+y=z=72$	
To find		$a, e, u, y,$ and p	
1 — a	3	$u=s-a$	
3 ^{2}	4	$uu=ss-2sa+aa$	
4 — aa	5	$uu-aa=ss-2sa=pp$	
2 — e	6	$z-e=y$	
6 $\odot^2$	7	$zz-2ze+ee=yy$	
7 — ee	8	$zz-2ze=yy-ee=pp$	
5, 8	9	$zz-2ze=ss-2sa$	
By *Fig.*	10	$a:p::p:e$	
10 ∴	11	$ae=pp$	
5, 11	12	$ae=ss-2sa$	

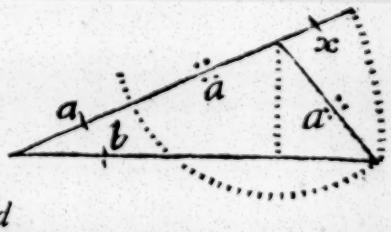
12 ÷ a

12 ÷ a	13	$e = \frac{ss - 2sa}{a}$
13 × $2z$	14	$2ze = \frac{2zss - 4zsa}{a}$
9 + 14	15	$zz = ss - 2sa + \frac{2zss + 4zsa}{a}$
15 × a	16	$zza = ssa - 2saa + 2zss - 4zsa$
16 ±	17	$2saa + zza + 4zsa - ssa = 2zss$
17 ÷ $2s$	18	$aa + \frac{zza}{2s} + 2z - \frac{1}{2}sa = zs$
Substitute	19	$2x = \frac{zz}{2s} + 2z - \frac{1}{2}s = 114$
Then	20	$aa + 2xa = zs = 7776$
20 C □	21	$aa + 2xa + xx = zs + xx = 11025$
21 $\sqrt{\ }$	22	$a + x = \sqrt{zs + xx} = 105$
22 — x	23	$a = \sqrt{zs + xx} : - x = 48$
1 — 23	24	$u = 60 =$ *The* Base.
per 13	25	$e = \frac{ss}{a} - 2s = 27$
2 — 25	26	$y = 45 =$ *the* Cathetus.
23 + 25	27	$a + e = 75 =$ *the* Hypothenuse.

PROBLEM XI.

The Difference of the Sides of any Oblique-angled plain Triangle, the Difference of the Segments of the Base, *and the Difference between the greater Side and the* Base, *being given, to find the* Base, *&c.*

Let	1	$d =$ the Difference of the Sides $= 405$
	2	$b =$ the Difference of the Segments $= 495$
	3	$x = 165$ the Differ. of the greater Side and Base
And	4	$a =$ the least Side
Then	5	$d + a + x =$ the Base
And	6	$d + a + x : d + 2a :: d : b$ By *Theorem* 16.
6 ∴	7	$db + ba + bx = dd + 2da$
7 ±	8	$2da - ba = db + bx - dd$
8 ÷ $2d - b$	9	$a = \frac{db + bx - dd}{2d - b} = \frac{118125}{315} = 375$
1 + 9	10	$d + a = 780 =$ *the* greatest Side.
3 + 10	11	$d + a + x = 945 =$ *the* Base.



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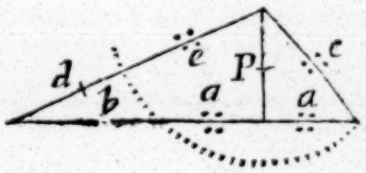
PROBLEM XII.

The Difference of the Sides of any plain Triangle, the Difference of the Segments of the Base, *and the* Perpendicular *let fall from the Vertical Angle, being given, thence to find all the Sides.*



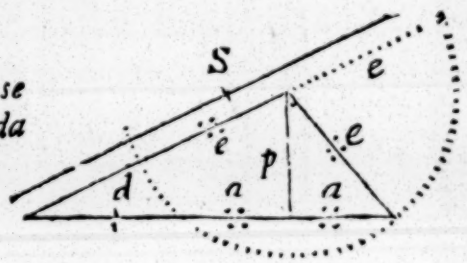
Let {	1	$d = 405$ } as before.
	2	$b = 495$ }
And	3	$p = 300$
Quere	4	$a =$ the leſſer *Segment.*
Then	5	$b + 2a : d + 2e :: d : b.$
5	6	$bb + 2ba = dd + 2de$
6 — dd	7	$bb - dd + 2ba = 2de$
Subſtitute	8	$2x = bb - dd = 81000$
7, 8	9	$2x + 2ba = 2de$
9 ÷ $2d$	10	$\frac{x + ba}{d} = e$
But	11	$pp + aa = ee$ By *Theorem* 11.
10 $\odot^2$	12	$\frac{xx + 2xba + bbaa}{dd} = ee$
11, 12	13	$\frac{xx + 2xba + bbaa}{dd} = pp + aa$
13 × dd	14	$xx + 2xba + bbaa = ppdd + ddaa$
14 ±	15	$bbaa - ddaa + 2xba = ppdd - xx$
8, 15	16	$2xaa + 2xba = ppdd - xx$
16 ÷ $2x$	17	$aa + ba = \frac{ppdd}{2x} - \frac{1}{2}x$
17 C □	18	$aa + ba + \frac{1}{4}bb = \frac{1}{4}bb + \frac{ppdd}{2x} - \frac{1}{2}x$
18 $\sqrt{}$	19	$a + \frac{1}{2}b + \sqrt{\frac{1}{4}bb + \frac{ppdd}{2x} - \frac{1}{2}x}$
19 — $\frac{1}{2}b$	20	$a = \sqrt{\frac{1}{4}bb + \frac{ppdd}{2x} - \frac{1}{2}x} : -\frac{1}{2}b = 225$
20 × 2	21	$2a = 450$
2 + 21	22	$b + 2a = 945$ *the* Baſe
10, *Num.*	23	$e = 375 =$ *the* leſſer Side.
1 + 23	24	$d + e = 780 =$ *the* greater Side.

PROBLEM XIII.

The Sum of the two Sides of any plain Triangle, the Difference of the Segments of the Baſe, *and the* Perpendicular *let fall from the Vertical*

Vertical Angle upon the Base, *being given, thence to find the* Base *and the Sides.*

Let	1	$s = 1155$ the *Sum* of the *Sides*.
	2	$d = 495$ the *Difference* of the *Segments*.
	3	$p = 300$ the *Perpendicular*.
Put	4	$a =$ the least *Segment*.
	5	$e =$ the least *Side*.
Then	6	$d + 2a =$ the *Base*.
And	7	$s - 2e =$ the *Difference* of the *Sides*.
Per *Fig.*	8	$d + 2a : s :: s - 2e : d$
	9	$aa - + pp = ee$
9 ww	10	$\sqrt{aa + pp} = e$
8 ∴	11	$dd + 2da = ss - 2se$
11 ±	12	$2se = ss - dd - 2da$
Suppose	13	$2x = ss - dd$
Then	14	$2se = 2x - 2da$
14 ÷ 2s	15	$e = \frac{x - da}{s}$
10, 15	16	$\frac{x - da}{s} = \sqrt{aa + pp}$
16 ⊙²	17	$\frac{xx - 2xda + ddaa}{ss} = aa + pp$
17 × ss	18	$xx - 2xda + ddaa = ssaa + sspp$
18 ±	19	$ssaa - ddaa + 2xda = xx - sspp$
13 19	20	$2xaa + 2xda = xx - sspp$
20 ÷ 2x	21	$aa + da = \frac{1}{2}x - \frac{sspp}{2x}$, &c. as before.
21, hence	22	$a = 225$
22 × 2	23	$2a = 450$
2 + 23	24	$d + 2a = 945$ the *Base*.
10, *Num.*	25	$e = 375$ the lesser Side.
1 − 25	26	$s - e = 780$ the greater Side.

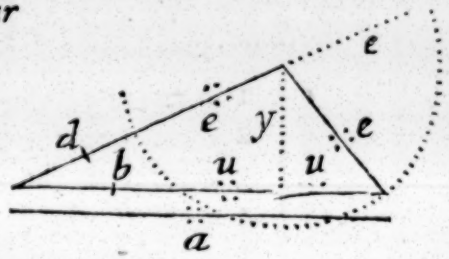


PROBLEM XIV.

The Area of any Oblique-angled plain Triangle, *the Difference of the Sides, and the Difference of the Segments of the* Base, *being given, thence to find the* Base, &c.

Let	1	$A = 141750 =$ the *Area*
	2	$d = 405$
	3	$b = 495$

Put {	4	$y =$ the *Perpendicular*
	5	$a =$ the *Baſe*.
Then	6	$\frac{1}{2}\,ya = A$
Per *Fig.*	7	$a : d + 2e :: d : b$
7	8	$ba = dd + 2de$
8 — dd	9	$ba - dd = 2de$
9 $\circledcirc^2$	10	$bbaa - 2ddba + dddd = 4ddee$
Per *Fig.*	11	$\frac{a-b}{2} = u$ the leſſer Segment of the *Baſe*.
11 $\circledcirc^2$	12	$\frac{aa - 2ba + bb}{4} = uu$
6 × 2	13	$ya = 2A$
13 ÷ a	14	$y = \frac{2A}{a}$
14 $\circledcirc^2$	15	$yy = \frac{4AA}{aa}$
Per *Fig.*	16	$yy + uu = ee = \frac{4AA}{aa} + \frac{aa - 2ba + bb}{4}$
10 ÷ $4dd$	17	$\frac{bbaa - 2ddba + dddd}{4dd} = ee$
16, 17	18	$\frac{bbaa - 2ddba + d^4}{4dd} = \frac{4AA}{aa} + \frac{aa - 2ba + bb}{4}$
18 × aa	19	$\frac{bba^4 - 2ddba^3 + d^4aa}{4dd} = 4AA + \frac{a^4 - 2ba^3 + bba^2}{4}$
19 × $4dd$	20	$\begin{cases} bba^4 - 2ddba^3 + d^4aa = 16\,AAdd + dda^4 \\ -2ddba^3 + ddbba^2 \end{cases}$
20 ±	21	$bba^4 - dda^4 + d^4a^2 - ddbba^2 = 16\,AAdd$
21 ÷	22	$aaaa - ddaa = \frac{16\,AA\,dd}{bb - dd}$
22 C □	23	$aaaa - ddaa + \frac{1}{4}dddd = \frac{16\,AA\,DD}{bb - dd} + \frac{1}{4}d^4$
23 $\sqrt{\ }^2$	24	$aa - \frac{1}{2}dd = \sqrt{\frac{16\,AA\,dd}{bb-dd} + \frac{1}{4}d^4}$
24 + $\frac{1}{2}dd$	25	$aa = \frac{1}{2}dd + \sqrt{\frac{16\,AAdd}{bb-dd} + \frac{1}{4}d^4}$
25 $\sqrt{\ }^2$	26	$a = \surd : \frac{1}{2}dd + \sqrt{\frac{16\,AAdd}{bb-dd} + \frac{1}{4}d^4} = 945$



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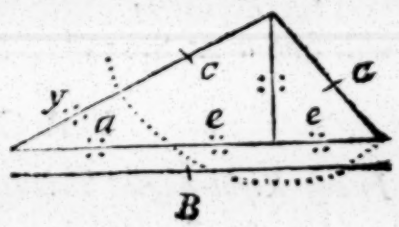
PROBLEM XV.

There is an Oblique-angled plain Triangle, wherein a Perpendicular *is let fall from the Vertical Angle upon the* Base; *the least Side and the* Base *are given; and the Rectangle of the Difference of the Sides into the least Side is equal to the Square of the Difference of the Segments of the* Base: *'Tis requir'd to find the Segments of the* Base, *&c.*

Let	1	$c = 56 =$ the least Side
	2	$B = 92 =$ the *Base*.
And	3	$a + 2e = B$
Put	4	$y =$ the Difference of the *Sides*.
Then	5	$cy = aa$ by the Question.
By *Figure* 6	6	$B : 2c + y :: y : a$, for $B = a + 2c$
$\therefore$	7	$Ba = 2cy + yy$
5 × 2	8	$2cy = aa$
7 — 8	9	$B - 2aa = yy$
5 $\odot^2$	10	$ccyy = aaaa$
10 ÷ cc	11	$yy = \frac{aaaa}{cc}$
9, 11	12	$Ba - 2aa = \frac{aaaa}{cc}$
12 × cc	13	$cc\,Ba - 2ccaa = aaaa$
13 ÷ a	14	$cc\,B - 2cca = aaa$
14 + $2cca$	15	$aaa + 2cca = cc\,B$
15 in *Num.*	16	$aaa + 6272a = 288512$



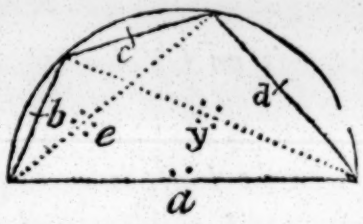
The *Value* of a, in this *Æquation*, may be found as in the Examples *Page* 238, *viz.* by putting $r + e = a$, &c. as in those Examples you will find $a = 37{,}55502$, &c.

PROBLEM XVI.

The three Chords or Subtenses of three Arches compleating a Semicircle *being each given, thence to find the Diameter of that* Circle. That is,

Any Trapezium *being inscrib'd in a* Semicircle, *if one of its Sides be the Diameter, and the other three Sides be given, thence to find the Diameter or fourth Side.*

Let	1	$b = 3$	the 3 Sides.
	2	$c = 4$	
	3	$d = 5$	
Quære	4	$a =$ the *Diam.* ſought	



Draw the two *Diagonals* e and y

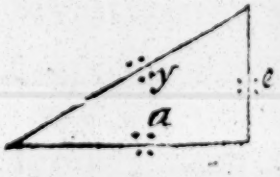
Then	5	$ca + bd = ey$.	By *Theorem* 19.
And	6	$aa - bb = yy$	By *Theorem* 10 and 11.
	7	$aa - dd = ee$	
5 $\odot^2$	8	$ccaa + 2bdca + bbdd = eeyy$	
6 × 7	9	$aaaa - bbaa - ddaa + bbdd = eeyy$	
8, = 9	10	$aaaa - bbaa - ddaa = ccaa + 2bdca$	
10 — a	11	$aaa - bba - dda - cca = 2bdc$	
11 — cca	12	$aaa - bba - dda - cca = 2bdc$	
12, *Numb.*	13	$aaa - 50a = 120$	

This *Æquation* being ſolv'd as in *Example* 2, *Page* 240, you will find $a = 8{,}05581$, &c.

PROBLEM XVII.

In any Right-angled Triangle, the Area *and the Sum of the* Hypothenuſe, *when added to either Side, being given, thence to find the Sides*, &c.

Suppoſe	1	$\frac{ae}{2} = A = 1350$	the *Area*
	2	$y + e = s = 120$	the *Sum*, &c.
	3	Quære a, e, and y	
1 × 2	4	$ae = 2A$	
4 ÷ a	5	$e = \frac{2A}{a}$	
Per *Fig.*	6	$aa + ee = yy$	
2 — e	7	$y = s - e$	
5, 7	8	$y = s - \frac{2A}{a}$	
8 $\odot^2$	9	$y = s - \frac{4sA}{a} + \frac{4AA}{aa}$	
5 $\odot^2$	10	$ee = \frac{4AA}{aa}$	
10 + aa	11	$aa + ee = \frac{4AA}{aa} + aa$	



6, 9, 11

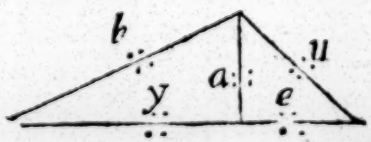
6, 9, 11	12	$\frac{4AA}{aa} + aa = yy = ss - \frac{4sA}{a} + \frac{4AA}{aa}$
12, That is	13	$aa = ss - \frac{4sA}{a}$
13 × a	14	$aaa = ssa - 4sA$
14 +	15	$ssa - aaa = 4sA$
15, in *Num.*	16	$14400a - aaa = 648000$

The *Value* of a, in this *Æquation*, may be found as in the third Example, *Page* 241; that is, by making $r + e = a$, &c. it will be found that $a = 60$.

PROBLEM XVIII.

There is an Oblique-angled plain Triangle, *wherein a* Perpendicular *is let fall from the* Vertical Angle *upon the* Base; *the Sum of each Segment of the* Base, *when added to its adjacent or next Side, and the* Area *of the* Triangle, *are given, to find the* Perpendicular *and each Side.*

Let	1	$y + b = z = 1500$	Quære y, b, e, and u
	2	$e + u = s = 600$	
	3	$A =$ the *Area* $= 141750$	
And	4	$a =$ the *Perpendicular* sought.	
Then	5	$y + e \times \frac{1}{2} a = A$	
5 × 2 ÷ a	6	$y + e = \frac{2A}{a}$	
Per *Fig.*	7	$yy + aa = bb$	
	8	$ee + aa = uu$	
1 — y	9	$b = z - y$	
2 — e	10	$u = s - e$	
9 $\square^2$	11	$bb = zz - 2zy + yy$	
10 $\square^2$	12	$uu = ss - 2se - ee$	
7, 11	13	$zz - 2zy = aa$	
8, 12	14	$ss - 2se = aa$	
13 +	15	$zz - aa = 2zy$	
14 +	16	$ss - aa = 2se$	
15 ÷ $2z$	17	$\frac{zz - aa}{2z} = y$	Having found the Value of a from the 24th Step, e and y will be easily found by these two Steps, and b, u, by the 9th and 10th Step.
16 ÷ $2s$	18	$\frac{ss - aa}{2s} = e$	
17 ÷ 18	19	$\frac{zz - aa}{2z} + \frac{ss - aa}{2s} = y + e$	



6, 19

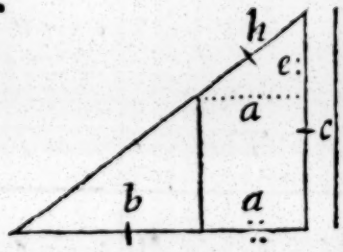
6, 19	20	$\frac{zz-aa}{2z}+\frac{ss-aa}{2s}=\frac{2A}{a}$
20 × 2z	21	$zz-aa+\frac{zzs-zaa}{s}=\frac{4za}{a}$
21 × s	22	$zzs-saa+zss-zaa=\frac{4zA}{a}$
22 × a	23	$zzsa-saaa+zssa-zaaa=4zAs$
23, *Numb.*	24	$900000a-aaa=243000000$

Here $a=300$ found as in the laſt *Problem*.

PROBLEM XIX.

There is a Right-angled Triangle, wherein a Right-line is drawn parallel to the Cathetus; *there is given the* Cathetus, *that Segment of the* Hypothenuſe *next to the* Cathetus, *and the alternate Segment of the* Baſe; *thence to find the* Baſe, *&c.*

viz. Let	1	$b=20\,.\,c=24\,.$ and $h=15$
Then	2	$b+a=$ the *Baſe*. Quære a
Here	3	$b+a:c::a:e$ per *Figure*.
And	4	$aa+ee=hh$ per *Figure*.
3 ∴	5	$\frac{ca}{b+a}=e$
5 ⊙²	6	$\frac{ccaa}{bb+2ba+aa}=ee$
4 − aa	7	$hh-aa=ee$
6, 7	8	$\frac{ccaa}{bb+2ba+aa}=hh-aa$
8 ×	9	$ccaa=hhbb-bbaa+2hhba-2ba^3+hhaa-a^4$
9 +	10	$a^4+2baaa+ccaa+bbaa-hhaa-2hhba=hhbb$
That is,	11	$aaaa+40aaa+751aa-9000a=90000$



For a *Solution* of this *Æquation*, let it be made

$aaaa+baaa-caa-da=G$ *Viz.* $\begin{cases} b=40 & .\ c=751 \\ d=9000 & .\ G=90000 \end{cases}$

Put $r+e=a$

Then $\left.\begin{cases} r+4rrre+6rree=a^4 \\ brrr+3brre+3bree=baaa \\ crr+2cre+cee=caa \\ -dr-de=-da \end{cases}\right\}=G=90000$

Let $r=10$

Then

Then $\left\{\begin{array}{l} +\ 10000 + 4000e + 600ee \\ +\ 40000 + 12000e + 1200ee \\ +\ 75100 + 15020e + 751ee \\ -\ 90000 - 9000e \end{array}\right\} = G = 90000$

That is, $35100 + 22020e + 2551ee = 90000$

Hence it will be $22020e + 2551ee = 54900$

Consequently, $8,63e + ee = 21,52 = D$

And $\frac{D}{8,63 + e} = e$

Operation, 8,63) 21,52 (2,1 = *e*

$+ e = 2,1$ 20

1. *Divisor* = 10 1,52 First $r = 10$

2. *Divisor* = 10,7 1,07 $+ e = 2,1$

45 &c. $r + e = 12,1 = r$ for a second Operation, which being involv'd, and multiply'd into the *Co-efficients*, as before, will produce these *Numbers*:

$\left\{\begin{array}{l} +\ 21435,8881 + 7086,24e + 878,46ee \\ +\ 70862,4400 + 17569,20e + 1452,00ee \\ +\ 109953,9100 + 18174,20e + 751,00ee \\ -\ 108900,0000 - 9000,00e \end{array}\right\} = C.$

Viz. $93352,2381 - 33829,64e + 3081,46ee = 90000$

Here, because $93352,2381 \not> 90000$ therefore $12,1 \not> a$, and therefore it must be made $r - e = a$, which will produce the same *Numbers*, only all the second Signs must be changed.

Thus, $93352,2381 - 33829,64e + 3081,46ee = 90000$ from whence will arise this *Æquation*:

$+ 33829,64e - 3081,46ee = 3352,2381$

Consequently, $10,9784e - ee = 1,08787332 = D$

Operation, 10,9784) 1,08787332 (0,0999 = *e*

$- e = ,0999$ 9792

1. *Divisor* 10,88 108673 Last $r = 12,1$

2. *Divisor* 10,879 97911 $- e = 0,0999$

3. *Divisor* 10,8785 1076232 $r - e = 12,0001 = e$

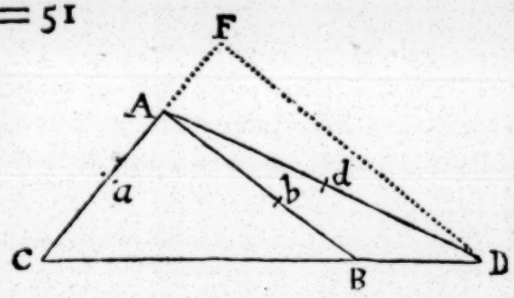
979065

&c.

PROBLEM XX.

In the Oblique-angled Triangle CAD, there is given the Side AD, and the Sum of the Sides AC + CD; also within the Triangle is given

the Line AB perpendicular to the Side CA; thence to find the Side CA, &c.



Let	1	$CA + CD = s = 51$
	2	$AD = d = 32$
	3	$AB = b = 21$
And	4	$CA = a$ fought.
Then	5	$s - a = CD$
Suppoſe		the *Line D F* *parallel* to *AB*; *CA* being produced to *F*
Then		$\triangle CAB$ and $\triangle CFD$ will be alike.
And	6	$BC : CA :: DC : CF$
But	7	$BC = \sqrt{bb + aa}$. Let $AF = e$, and $FD = y$
6, 7	8	$\sqrt{bb + aa} : a :: s - a : a + e$
8	9	$\frac{sa - aa}{\sqrt{bb + aa}} = a + e$
5 $\circledcirc^2$	10	$ss - 2sa + aa = \square CD$
Per *Fig.*	11	$ss - 2sa + aa = aa + 2ae + ee + yy = \square CF + \square FD$
11 $-aa$	12	$ss - 2sa = 2ae + ee + yy$
But	13	$dd = ee + yy = \square AF + \square FD$
12 $-$ 13	14	$ss - 2sa - dd = 2ae$
Let	15	$2x = ss - dd$
14, 15	16	$x - sa = ae$
16 $\div a$	17	$\frac{x - sa}{a} = e$
17 $+a$	18	$\frac{x - sa + aa}{a} = a + e$
9 $\circledcirc^2$	19	$\frac{ssaa - 2saaa + a^4}{bb + aa} = \square\, a + e$
18 $\circledcirc^2$	20	$\frac{xx - 2xsa + 2xaa + ssaa - 2sa^3 + a^4}{aa} = \square\, a + e$
19, 20	21	$\frac{ssaa - 2saaa + a^4}{bb + aa} = \frac{xx - 2xsa + 2xaa + ssaa - 2sa^3 + a^4}{aa}$

This *Æquation* being brought out of the *Fractions*, and into *Numbers*, will become $-2018a^4 + 125409a^3 - 2464230{,}25a^2 + 35468307a = 274183922{,}25$; which being divided by 2018, the *Co-efficient* of the higheſt *Power* of *a*, will be $-a^4 + 62{,}145\,6a^3 - 1221{,}125a^2 + 17575{,}9697a = 135869{,}138875$, &c.

And

And from hence the *Value* of *a* may be found, as in the *laſt Problem*, due Regard being had to the *Signs* of every *Term*.

This Work of *reducing*, or preparing *Æquations* for a *Solution* by *Diviſion*, hath always been taught both by *ancient* and *modern* Writers of *Algebra*, as a Work ſo neceſſary to be done, that they do not ſo much as give a Hint at the *Solution* of any *adfected Æquation* without it.

Now it very often happens, that, in *dividing* all the *Terms* of an *Æquation*, ſome of their *Quotients* will not only run into a long *Series*, but alſo into imperfect *Fractions* (as in this *Equation* above) which renders the *Solution* both tedious and *imperfect*.

To remedy that *Imperfection*, I ſhall here ſhew how this *Æquation* (and conſequently any other) may be *reſolv'd* without ſuch *Diviſion* or *Reduction*.

Let $b = 2018$. $c = 125409$. $d = 2464230{,}25$
$f = 35468307$. And $G = 274183922{,}25$
Then the *precedent Equation* will ſtand thus:
$$-baaaa + caaa - daa + fa = G$$
Put $r + e = a$ as before.

Then will
$$\left\{\begin{array}{llll} -br^4 & -4brrre & -6brree & = -ba^4 \\ +cr^3 & +3crré & +3créé & = +ca^3 \\ -drr & -2dré & -dèè & = -daa \\ +fr & +fé & \ldots\ldots\ldots & = +fa \end{array}\right\} = G$$

This is plain and eaſily conceived. The next Thing will be, how to eſtimate the firſt *Value* of *r*; and, to perform that, let *G* be *divided* by *b*, only ſo far as to determine how many *Places* of whole *Numbers* there will be in the *Quotient*; conſequently, how many *Points* there muſt be *(according to the Height of the Æquation.)*

Thus $b = 2018)\ G = 274183922{,}25\ (130000$
2018

7238, &c.

Now from hence one may as eaſily gueſs at the *Value* of *r*, as if all the *Terms* had been *divided*. That is, I ſuppoſe $r = 10$, which being *involved*, &c. as the *Letters above direct*, will be

 -20880000

$$\left.\begin{array}{l} -\ 20880000 - 8072000e - 1210800ee \\ +\ 125409000 + 37622700e + 3762270ee \\ -\ 246423025 - 49284605e - 2464230,25ee \\ +\ 354683070 + 35468307e \end{array}\right\} = G$$

Viz. $213489045 + 15734402e + 87239,75ee = 2741839$ &c.

Hence $15734402e + 87239,75ee = 60694877,25$

Conſequently, $180,3e + ee = 695,72 = D$

And $\frac{D}{180,3 + e} = e$

Operation.

	$180,3)$	$695,72$	$(3,7 = e$
	$+e = 3,7)$	549	
1. *Diviſor*	$= 183$	$146,72$	First $r = 10$
2. *Diviſor*	$= 184,0$	$128,80$	$+e = 3,7$
		&c.	$r + e = 13,7 = r$ for a

ſecond *Operation*, with which you may proceed, as in the laſt *Problem*, and ſo on to a third *Operation*, if Occaſion require ſuch Exactneſs. But this may be ſufficient to ſhew the Method of reſolving any *adfected Æquation*, without reducing it; which is not only very exact, but alſo very ready in Practice, as will fully appear in the laſt *Chapter* of this *Part*, concerning the *Periphery* and *Area* of the *Circle*, &c. wherein you will find a farther Improvement in the *Numerical Solution* of *High Equations* than hath hitherto been publiſh'd.

CHAP. V.

Practical Problems, *and* Rules *for finding the* Superficial Contents, *or* Area's *of Right-lin'd Figures.*

BEfore I proceed to the following *Problems*, it may be convenient to acquaint the *Learner*, that the *Superficies* or *Area* of any *Figure*, whether it be *Right-lin'd* or *Circular*, is *compos'd* or made up of *Squares*, either *greater* or *leſs*, according to the different *Meaſures* by which the *Dimenſions* of the *Figures* are taken or *meaſur'd*.

That is, if the *Dimenſions* are taken in *Inches*, the *Area* will be *compos'd* of *ſquare Inches*; if the *Dimenſions* are taken in *Feet*, the *Area* will be *compos'd* of *ſquare Feet*; if in *Yards*, the *Area* will be *ſquare Yards*; and if the *Dimenſions* are taken by *Poles or Perches* (*as in ſurveying of Land*, &c.) then the *Area* will be *ſquare Perches*, &c.

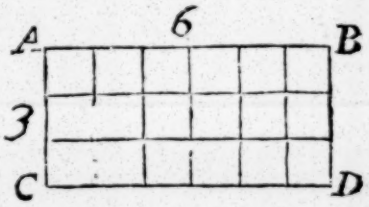
&c. These Things being understood, and the *Definitions* in the 283d and 284th *Pages* well consider'd, will help to render the following *Rules* very easy.

PROBLEM I.

To find the Superficial Content, or Area of a Square; *or of any Right-angled* Parallelogram.

RULE. { *Multiply the Length into its Breadth, and the Product will be the Area requir'd.* (*See* Lemma 1. Page 302.)

Example. Suppose the *Line* $AB = 6$ *Yards*, and the *Breadth* AC or $BD = 3$ *Yards*, then $AB \times AC = 6 \times 3 = 18$ will be the Number of *square Yards* contain'd in the *Area* of the *Parallelogram* $ABCD$. This is so evident by the *Figure* only, that it needs no *Demonstration*.

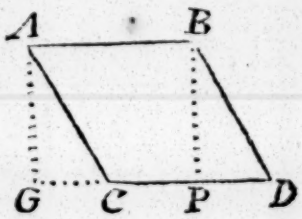


PROBLEM II.

To find the Area of any Oblique-angled Parallelogram, viz. *either of a* Rhombus *or* Rhomboides.

RULE. { *Multiply the Length into its perpendicular Height (or Breadth) and the Product will be the Area requir'd.*

That is, the *Side* $AB \times BP =$ the *Area* of the *Rhombus* $ABCD$. For if BP be drawn *perpendicular* to CD, and AG be made *parallel* to BP, then will $GC = PD$ and $GP = CD$. Consequently $\triangle AGC = \triangle BPD$, and $\square ABGP =$ *Rhombus* $ABCD$. But $AB \times BP = \square ABGP$. Therefore $AB \times BP$, or $CD \times BP =$ the *Area* of the *Rhombus* $ABCD$.



Example. Suppose the Side $AB = 23$ *Inches*, and the *Perpendicular* $BP = 17{,}5$ *Inches (being the shortest or nearest Distance between the two Sides, AB and CD.)* then $AB \times BP = 23 \times 17{,}5 = 402{,}5$ *square Inches*, being the *Area* of the *Rhombus* required.

The like may be done for any *Rhomboides* whose *Length* and *perpendicular Breadth* is given.

PROBLEM III.

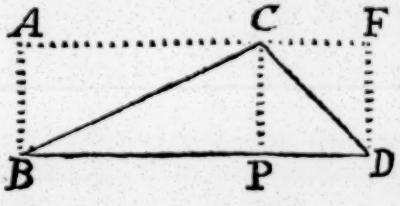
To find the Superficial Content, or Area of any plain Triangle.

Every plain Triangle is equal to half its circumſcribing Parallelogram, (41. *e.* 1.) *which affords the following Rule:*

RULE. { *Multiply the* Baſe *of the given Triangle into half its perpendicular Height, or half the* Baſe *into the whole Perpendicular, and the Product will be the Area.*

That is, $BD \times \frac{1}{2} CP$, or $\frac{1}{2} BD \times CP =$ *Area of* $\triangle BCD$.

For $AC = BP$, $AB = CP$, *and* BC is common to both $\triangle\triangle$; therefore $\triangle ABC = \triangle BCP$, and for the like Reaſons $\triangle CFD = \triangle CPD$. Therefore $\triangle BCP + \triangle CPD = \frac{1}{2} \square ABCD$. Conſequently $BD \times CP$, or $BD \times \frac{1}{2} CP$ will be the *Area* of $\triangle BCD$.



Example. Suppoſe the *Baſe* $BD = 32$ *Inches*, and the *perpendicular Height* $CP = 14$ *Inches*.

Then $\frac{1}{2} BD \times CP = 16 \times 14 = 224$. Or $BD \times \frac{1}{2} CP = 32 \times 7 = 224$. Or thus, $32 \times 14 = 448$. Then 2) 448 (224 = the *Area* of the *Triangle BCD* in *ſquare Inches*.

PROBLEM IV.

To find the Superficies, or Area of any Trapezium.

Firſt, *divide* the given *Trapezium* into *two Triangles*, by drawing a *Diagonal* from one of its acute *Angles* to the oppoſite *Angle*; and let fall two *Perpendiculars (from the other two Angles)* upon the *Diagonal*, as in the following *Figure*. Then

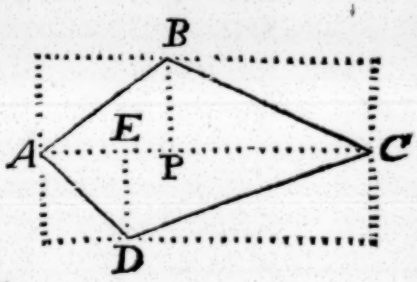
RULE. { *Multiply half the Diagonal into the Sum of the two Perpendiculars, or half the Sum of the Perpendiculars into the Diagonal, and the Product will be the Area.*

That is, $\frac{1}{2} AC \times \overline{BP + ED}$. Or $AC \times \frac{1}{2} \overline{BP + \frac{1}{2} ED} =$ *Area* of the *Trapezium ABCD*.

For the $\triangle ABC$ is *Half* its *circumſcribing Parallelogram*; and the $\triangle ACD$ is alſo *Half* of its *circumſcribing Parallelogram*, as hath been prov'd at the laſt *Problem*.

Conſequently,

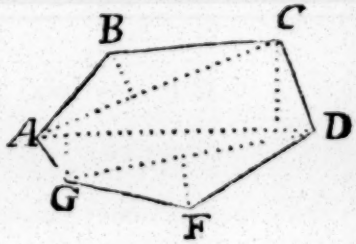
Consequently, $\overline{BP + ED} \times \frac{1}{2} AC$, or $\frac{1}{2} \overline{BP + \frac{1}{2} ED} \times AC$ will be the *Area* of the *Trapezium*, as above.



Example. Suppose the *Diagonal* $AC = 33$ *Feet*, and the *Perpendicular* $BP = 15$ *Feet*, and the *Perpendicular* $ED = 14$ *Feet*. Then $BP + ED = 29$ *Feet*, and $\overline{BP + ED} \times \frac{1}{2} AC = 29 \times 16{,}5 = 478{,}5$. Or $AC \times \overline{\frac{1}{2} BP + \frac{1}{2} ED} = 33 \times \frac{29}{2} = 478{,}5$. Or thus, $29 \times 33 = 957$. Then 2) 957 (478,5 any of these *Products* are the *Area* of the *Trapezium* $ABCD$.

PROBLEM V.

To find the Superficial Content or Area of any irregular Polygon, or *many sided Figure, which by some Authors is call'd a Triangulate, because (as I suppose) it must be divided into Triangles*, as in the *annexed Figure* $ABCDFG$; by which it is evident, that the *Sum* of the *Area*'s of all those *Triangles*, found as in the last *Problem*, &c. will be the *Area* of their *circumscribing Polygon*.



PROBLEM VI.

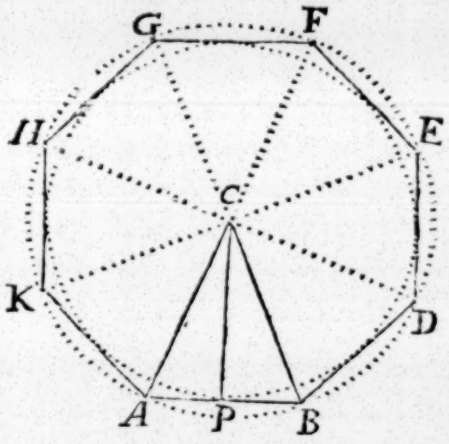
To find the Superficies, or Area of any regular Polygon, viz. *of any regular* **Pentagon, Hexagon, Heptagon, Octagon.** &c.

General RULE. { *Multiply half the Sum of its Sides into the Radius of the inscrib'd Circle, or half the said Radius into the Sum of the Sides, and the Product will be the Area required.*

That is, $\frac{AB + BD + DE + EF + FG + GH + HK + KA}{2} : \times CP$ = the *Area* of the *annexed Octagon*; wherein it is evident, that its *Area* is compos'd of so many *equal Isosceles Triangles* as there are *Numbers* of *Sides in the Polygon*, viz. of *eight Isosceles Triangles*, whose *Bases* are the *Sides* of the *Octagon*, viz. $AB = BD = DE$ &c. And the *Sides* of those *Triangles* CA, CB, CD, &c. are the *Radius's* of the *circumscribing Circle*; and their *perpendicular Heights*, viz. PP, is the *Radius* of the *inscrib'd Circle*.

But

But the *Area* of any one of thoſe *Triangles* is $\frac{1}{2}\ AB \times CP$ by *Problem* 3. Conſequently the *Sum* of all their *Area*'s will be *CP* into *half* the *Sum* of all their *Baſes*, as above.



This, being *equally* evident in all *regular Polygons* whatſoever, makes the *Rule general* for finding their *Area's*

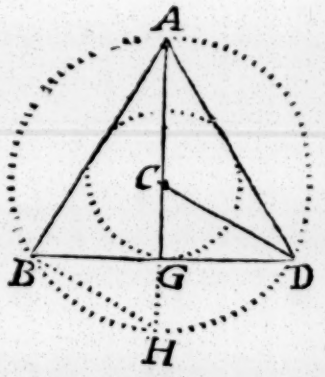
Now, becauſe it is requir'd to have the *Radius* of the propos'd *Polygon's inſcrib'd Circle*, I ſhall here inſert *(and demonſtrate)* the *Proportions* that are between the *Sides* of ſeveral *regular Polygons* and the *Radius's* both of their *inſcrib'd* and *circumſcribing Circles*; the one will help to *delineate* or *project* the *Polygon (if Occaſion require it)* and the other will help to find its *Area*.

And *Firſt*, Of an Equilateral Triangle.

The Side of any *Equilateral plain Triangle* is in *Proportion* to the *Radius* of

its { *Circumſcribing Circle*, / *Inſcrib'd Circle* / *Perpendicular Height*, } As 1 : To { 0,57735027 &c. / 0,28867513 &c. / 0,86602540 &c. }

i. e. { *AB* : *CD* : : 1 : 0,57735027 / *AB* : *CG* : : 1 : 0,28867513 / *AB* : *AG* : : 1 : 0,86602540 }



Demonſtration.

Let $AB = BD = 1$, then will $BG = GD = 0{,}5$; but $\square\ AB - \square\ BG = \square\ AG$ by *Theorem* 11. That is, $1 - 0{,}25 = 0{,}75 = \square\ AG$, conſequently, $\surd\ 0{,}75 = 0{,}86602540 = AG$: Then $AG : AB :: AB : AH$, by *Theorem* 13, that is, 0,8660254 : 1 : : 1 : 1,15470054 &c. $= AH$, then $\frac{1}{2}\ AH = 0{,}57735027 = AC$. Again, $AG : DG :: DG : CG$, that is, 0,8660254 : 0,5 : : 0,5 : 0,28867513 $= CG$. Q. E. D.

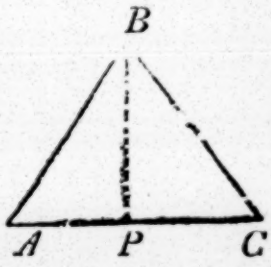
Now, by the Help of the *Firſt* of theſe *Proportions*, it will be eaſy to *reſolve* the following *Problem*.

PRO-

PROBLEM VII.

The Side of any Equilateral plain Triangle being given, to find its Area.

Example. Suppoſe the Side of the propoſed *Triangle ABC* to be 25 *Inches*, viz. $AB = BC = CA = 25$ Firſt $1 : 0{,}866254 :: AB = 25 : 21{,}650635 = BP$ by *Theorem* 13. Then $AP (= \frac{1}{2} CA) \times BP =$ the *Area* of $\triangle ABC$ by *Rule* to *Problem* 3, that is, $12{,}5 \times 21{,}650635 = 270{,}6329$ the *Area* in *ſquare Inches*.



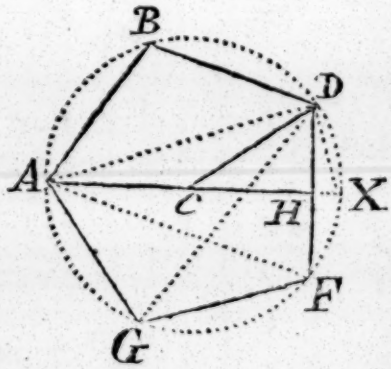
Or this *Problem* may be otherwiſe *reſolv'd* thus: Let $b = AP = AC$. Then $2b = AB$. But $\square AB - \square AP = \square BP$. By *Theorem* 11. That is, $4bb - bb = 3bb = \square BP$. Conſequently, $\surd\, 3bb = BP$. Then $b \surd\, 3bb = BP \times \frac{1}{2} AC$. viz. $\surd\, 3bbbb \times \surd\, 3 =$ the *Area* of the *Triangle*.

Secondly, For a Pentagon.

The Side of any Pentagon is in Proportion to the Radius of

its { *Circumſcribing Circle*, *Inſcrib'd Circle*, *Perpendicular Height*, } As 1 : To { 0,85065080 &c. 0,68819096 &c. 1,53884176 &c.

Viz. { $AB : AC :: 1 : 0{,}85065080$; $AB : CH :: 1 : 0{,}68819096$; $AB : AH :: 1 : 1{,}53884176$



Demonſtration.

Let $AB = 1$. And draw the *Diagonals* AD, AF, and DG, which will be *equal* to one another. Then will $AG \times DF + AD \times GF = AF \times DG$ by *Theorem* 19. Conſequently, $AG \times DF = AF \times DG : - AD \times GF$, that is, $\square AB = \square AD : - AD \times GF = 1$ (becauſe $AB = AG = DF$, and $AD = AF = DG$) hence it will be $AD = 1{,}61803398$, then $\square AD - \square DH = \square AH$ by *Theor.* 11. But $DH = \frac{1}{2} AB$, therefore $\surd\, \square AD - \frac{1}{4} \square AB = AH = 1{,}53884176$. Again, $AH : AD :: AD : AX = 2AC$. For $\triangle AHD$ and $\triangle ADX$ are alike.

Ergo

Ergo $\frac{\square AD}{AH} = 2AC = 1{,}70130161$. Hence $AC = 0{,}85065080$
But $AH - AC = CH = 0{,}68819096$, &c. Q. E. D.

From hence it will be *eaſy* to *reſolve* the following *Problem*.

PROBLEM VIII.

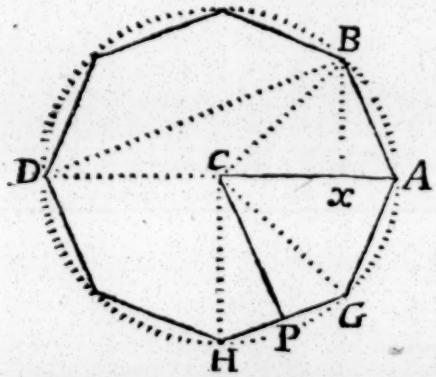
The Side of any regular Pentagon being given, to find its Area.

Example. Suppoſe the given *Side* to be 15 *Inches* long, then it will be, as 1 : 1,53884176 : : 15 : 22,0826264 the *perpendicular Height*; and by the general *Rule* $22{,}0826264 \times \frac{15}{2} = 165{,}619698$ the *Area* requir'd.

Thirdly, For an Octagon.

The Side of any regular Octagon is in Proportion to the Radius of its { *Circumſcribing Circle*, As 1 : to 1,30656296, &c.
{ *Inſcrib'd Circle*, As 1 : to 1,20710678, &c.

Viz. { $BA : CA$: : 1 : 1,30656296
{ $BA : CP$: : 1 : 1,20710678



Demonſtration.

Draw the *Right Line DB*, and from the *Point B* let fall the *Perpendicular Bx* upon the *Diameter DA*.

Then will $\triangle DBA$ and $\triangle DxB$ be alike, by *Theorem* 10 and 12.

Let { $b = BA = 1$. $a = CA$
{ $c = BD$, and $y = Bx$

	Then	1	$2a : b :: e : y$. viz. $DA : BA :: DB : Bx$
1	∴	2	$\frac{2ay}{b} = e = DB$
2	⊙.²	3	$\frac{4aayy}{bb} = ee = \square DB$
	But	4	$4aa - \frac{4aayy}{bb} = bb$
	That is	5	$\square DA - \square DB = \square BA$. By *Theorem* 11.
4	× bb		$4bbaa - 4aayy = bbbb$
	Again	6	{ $\frac{1}{2}aa = yy$. For $Cx = Bx$ and $\square Cx + \square Bx = \square CB = aa$

5, 6

5, 6	7	$4bbaa - 2a^4 = b^4$. Or $2a^4 - 4bbaa = -b^4$
7 ÷ 2	8	$aaaa - 2bbaa = -\frac{1}{2}b^4$
8 *C* □	9	$a^4 - 2bbaa + b^4 = b^4 - \frac{1}{2}b^4 = \frac{1}{2}b^4$
9 ww	10	$aa - bb = \surd \frac{1}{2}b^4$
10 + *bb*	11	$aa = bb + \surd \frac{1}{2}b^4$
11 ww	12	$a = \surd : bb + \surd \frac{1}{2}b^4 = 1,30656296$, &c. $= CA$
Then	13	$aa - \frac{1}{4}bb = \square CP$, viz. $\square CH - \square HP = \square CP$
13 ww	14	$\surd\, aa - \frac{1}{4}bb = 1,20710678$, &c. $= CP$.

From hence 'twill be eaſy to find the *Area* of any *Octagon*.

PROBLEM IX.

The Side of any regular Octagon *being given, to find its* Area.

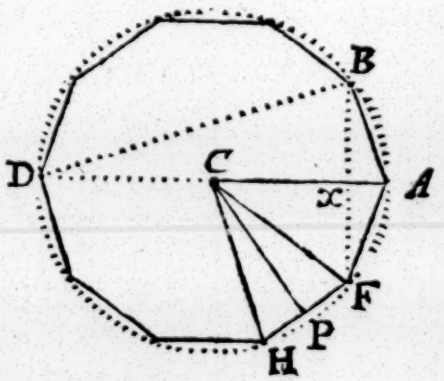
Example. Suppoſe the Side given to be 12 Inches long; *Firſt*, as 1 : 1,20710678 :: 12 : 14,48528136 = the *Radius* of its inſcrib'd Circle; then $12 \times 4 = 48$ is half the Sum of its Sides, and $48 \times 14,48528136 = 695,2935$ the *Area* required.

Fourthly, For a Decagon.

The Side of any regular Decagon *(viz. a* Polygon *of ten equal Sides) is in Proportion to the* Radius *of*

Its { *Circumſcribing Circle*, as 1 : to 1,61803398, &c.
Its { *Inſcrib'd Circle* as 1 : to 1,53884176, &c.

Viz. { $BA : CA :: 1 : 1,61803398$
Viz. { $BA : CP :: 1 : 1,53884176$



Demonſtration.

Let { $b = BA = 1$. $a = CA$
Let { $c = DB$, and $y = Bx$

Then	1	$2a : b :: e : y$
That is,		$DA : BA :: DB : Bx$
1 ∴	2	{ $2ay = be$ and $2y = \frac{be}{a}$
But	3	$2y : e :: 1 : 1,61803398$. See *Pentagon*.
3 ∴	4	$\frac{1e}{1,61803398} = 2y = \frac{be}{a} = \frac{1e}{a}$
4 ÷ 1*e*	5	$1,61803398 = a = CA$
Again	6	{ $aa - \frac{1}{4}bb = \square CP$. viz. $\square CF = \square PF = \square CP$. By *Theorem* 11.
That is,	7	$\surd\, 2,61803398 - 0,25 - 1,53884176 = CP$

PROBLEM X.

The Side of any regular Decagon *being given, to find its* Area.

Example. Let the given Side be 14 Inches long; then, as 1 : 1,53884176 :: 14 : 21,543784 = the *Radius* of the inſcrib'd Circle; and 14 × 5 = 70 is half the Sum of its Sides. Laſtly, 21,543784 × 70 = 1508,06488 the *Area* requir'd.

Fifthly, For a Dodecagon.

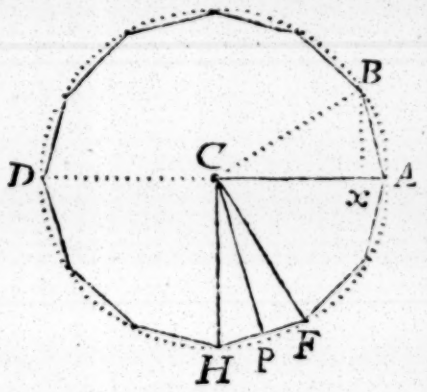
The Side of any regular Dodecagon (viz. *a* Polygon *of twelve equal Sides*) *is in Proportion to the* Radius *of*

Its { *Circumſcribing Circle*, as 1 : to 1,93185165, &c.
Inſcrib'd Circle as 1 : to 1,86632012, &c.

Viz. { $BA : CA :: 1{,}03185165$
$BA : CP :: 1{,}86632012$

Demonſtration.

Let $b = BA = 1$. $a = CA$ as before
And $e = xA$; then $a - e = Cx$



	Firſt,	1	$\left\{ \begin{array}{l} bb - \square Bx = ee \\ \text{By Figure.} \end{array} \right.$
	But	2	$Bx = \frac{1}{2} CA = \frac{1}{2} a$
2	$\odot^2$	3	$\square Bx = \frac{1}{4} aa$
1,	3	4	$bb - \frac{1}{4} aa = ee$
4	m^2	5	$\surd\, bb - \frac{1}{4} aa = ee$
	Again	6	$aa - \frac{1}{4} aa = aa - 2ae + ee$
	Viz.		$\square CB - \square Bx = \square Cx$
5	× $2a$	7	$2a \surd\, bb - \frac{1}{4} aa = 2ae$
4	— 7	8	$bb - \frac{1}{4} aa - 2a \surd\, bb - \frac{1}{4} aa = ee - 2ae$
7,	8	9	$aa - \frac{1}{4} aa = aa + bb - \frac{1}{4} aa - 2a \surd\, bb - \frac{1}{4} aa$
9	+	10	$2a \surd\, bb - \frac{1}{4} aa = bb$
10	$\odot^2$	11	$4bbaa - aaaa = b^4$
11	+	12	$aaaa - 4bbaa = -b^4$
13,	C $\square$	13	$aaaa - 4bbaa + 4b^4 = 3b^4 = 3$
1	ω^2	14	$aa - 2bb = \surd\, 3 = 1{,}7320508075$
14	+ $2bb$	15	$aa = 2bb + \surd\, 3 = 3{,}7320508075$
15	ω^2	16	$a = \surd\, 3{,}7320508075 = 1{,}93185165 = CA$
	Again	17	$aa - \frac{1}{4} bb = \square CP$. viz. $\square CF - \square PF = \square CP$
17,	Hence	18	$CP = \surd\, aa - \frac{1}{4} bb = 1{,}86632012$. Q. E. D.

Con-

Conſectary.

Hence if the Side of any *regular Dodecagon* be given, the *Radius* of its *inſcrib'd Circle* may be eaſily obtain'd, and thence the *Area* found; as in the laſt Problem.

The Work of the 'foregoing *Polygons*, being well conſider'd, will help the young *Geometer* to raiſe the like Proportions for others, if his Curioſity requires them: And not only ſo, but they will alſo help to form a true Idea of a Circle's *Periphery* and *Area*, according to the Method which I ſhall lay down in the next Chapter for finding them both.

CHAP. VI.

A new and eaſy Method of finding the Circle's Periphery *and* Area *to any aſſign'd Exactneſs (or Number of Figures) by one Æquation only. Alſo a new and facile Way of making Natural* Sines *and* Tangents.

LET us ſuppoſe *(what is very eaſy to conceive)* the Circle's *Area* to be compos'd or made up of a vaſt Number of *plain Iſoſceles Triangles*, having their *acuteſt Angles* all meeting in the Circle's Center. And let us imagine the *Baſes* of thoſe *Triangles* ſo very ſmall, that their Sides and their Perpendicular Heights, *viz.* the *Radius*'s of their circumſcrib'd and inſcrib'd Circles *(vide* Problem 6.) may become ſo very near in *Length* to each other, as that they may be taken one for another without any ſenſible Error: Then will the *Peripheries* of their circumſcribing and inſcrib'd Circles become *(altho' not co-incident, yet)* ſo very near to each other, as that either of them may be indifferently taken for one and the ſame Circle.

But how to find out the Sides of a *Polygon* (viz. *the Baſes of thoſe Iſoſceles Triangles)* to ſuch a convenient Smallneſs as may be neceſſary to determine and ſettle the Proportion betwixt a Circle's *Diameter* and its *Periphery (to any aſſign'd Exactneſs)* hath hitherto been a Work which requir'd great *Care* and much *Time* in its Performance; as may eaſily be conceived from the *Nature* of the *Method* us'd by all thoſe who have made any conſiderable Progreſs in it, viz. *Archimedes*, *Snellius*, *Hugenius*, *Mœtius*, *Van Culen*, &c. Theſe proceeded with the biſecting of an *Arch*, and found the *Value* of its *Chord* to a convenient Number of Figures

at every ſingle Biſection, repeating their *Operations* until they had approach'd to the *Chord* deſign'd.

And this *Method* is made Choice of by the learned Dr. *Wallis* in his Treatiſe of *Algebra*; wherein, after he hath given us a large Account of the different Enquiries made by ſeveral *(very eminent in Mathematical Sciences)* in order to find out ſome eaſier and more expeditious Way of approaching to the Circle's *Periphery*, as in *Chap.* 82, 84, 85, 86, and ſeveral other Places, he comes to this Reſult, (*Page* 321.)

" 'Tis true, *ſaith he*, we might in like Manner proceed by con-
" tinual Triſection, Quinquiſection, or other Section, if we had
" for theſe as convenient Methods of Operation as we have for
" Biſection: But becauſe *Euclid* ſhews how to biſect an Arch
" Geometrically, but not to triſect, *&c.* and the one may be done
" *(Algebraically)* by reſolving a Quadratick Æquation, but not
" thoſe other, without Æquations of a higher Compoſition, I
" therefore make Choice of a continual Biſection, *&c.*"

And then he lays down theſe following Canons:

The Subtenſe of $\frac{1}{6}$	1 into 6
of $\frac{1}{2}$	$\sqrt{:2-\sqrt{3}}$ into 12
of $\frac{1}{3\,4}$	$\sqrt{:2+\sqrt{:2+\sqrt{3}}}$ &c. 24
of $\frac{1}{4\,8}$	$\sqrt{:2-\sqrt{:2+\sqrt{:2+\sqrt{3}}}}$ 48
of $\frac{1}{9\,6}$	$\sqrt{:2-\sqrt{:2+\sqrt{:2+\sqrt{:2+\sqrt{3}}}}}$ 96
&c.	$\sqrt{:2-\sqrt{:2+\sqrt{:2+\sqrt{:2+\sqrt{:2+\sqrt{3}}}}}}$ 192
	$\sqrt{:2-\sqrt{:2+\sqrt{:2+\sqrt{:2+\sqrt{:2+\sqrt{:2+\sqrt{3}}}}}}}$ 384
	$\sqrt{:2-\sqrt{:2+\sqrt{:2+\sqrt{:2+\sqrt{:2+\sqrt{:2+\sqrt{:2+\sqrt{3}}}}}}}}$ 768
	&c. &c.

How tedious and troubleſome the Work of theſe complicated Extractions *is, I leave to the Conſideration of thoſe, who either have had Experience therein, or out of Curioſity will give themſelves the Trouble of making Trial.*

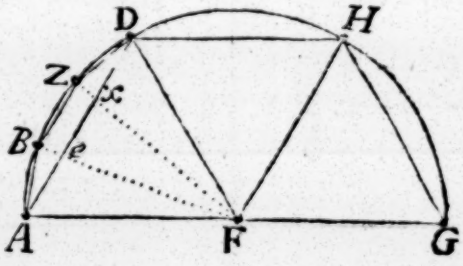
Again, in *Page* 347, the *Doctor* inſerts a particular Method propoſed by *Libnitius*, publiſh'd in the *Acta Eruditorum* at *Leipſick*, for the Month of *February* 1682, in order to find the Circle's *Area*, and conſequently its *Periphery*, which is this:

As 1 : to $\frac{1}{2}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\frac{1}{17}-\frac{1}{19}$, &c. infinitely : : ſo is the Square of the *Diameter* to the Circle's *Area*. But this convergeth ſo very ſlowly, that it is not worth the Time to purſue it.

I ſhall here propoſe a new Method of my own, whereby the Circle's *Periphery*, and conſequently its *Area*, may be obtain'd

infinitely near the Truth, with much greater Eaſe and Expedition than either that of *Biſection*, or that of *Libnitius*, as above, or any other Method that I have yet ſeen; it being perform'd by *reſolving* only one *Æquation*, deduced by an eaſy Proceſs from the Property of a Circle, *(known to every Cooper)* which is this:

The Radius *of every Circle is equal to the* Chord *of one ſixth Part of its* Periphery. That is, $AD = DH = HG$, the *Chords* of one third Part of the *Semicircle*, are each equal to AF its *Radius*. Then if the Arch AD be triſected, it will be $AB = BZ = ZD$.



Let $\begin{cases} R = AF = 1 \\ c = AD = 1 \\ a = AB. \end{cases}$ Quære a.

Then	1	$R : a :: a : \frac{aa}{R} = Be$
And	2	$R : a :: R - \frac{aa}{R} : c - 2a$
That is, For	3	$FB : BZ :: Fe : ex = AD - 2a$ $\triangle AFB$, and $\triangle BAe$, are alike. And $AB = Ae = Dx$, &c.
2 ∴	4	$Rc - 2Ra = Ra - \frac{aaa}{R}$
4 × &c.	5	$3R^2a - aaa = RRc$. That is, $3a - aaa = 1$ Here a = the Chord of $\frac{1}{18}$ Part of the *Circle*. For $\frac{1}{3}$ of $\frac{1}{6} = \frac{1}{18}$

Next, *To triſect the* Arch AB.

Let	1	$3y - y^3 = a$ the laſt Chord.
1 ⊙ 5	2	$27y^3 - 27y^5 + 9y^7 - y^9 = a^3$
1 × 3	3	$9y - 3y^3 = 3a$
3 − 2	4	$9y - 30y^3 + 27y^5 - 9y^7 + y^9 = 3a - a^3 = 1$ Here y = the Chord of $\frac{1}{54}$ Part of the Circle.

Again, *To triſect the* Arch *whereof* y *is the* Chord.

Let	1	$3a - a^3 = y$
1 ⊙ 3	2	$27a^3 - 27a^5 + 9a^7 - a^9 = y^3$
1 ⊙ 5	3	$243a^5 - 405a^7 + 270a^9 - 90a^{11} + 15a^{13} - a^{15} = y^5$

1 ⊙ 7

1 ⊙.7	4	$\left\{\begin{array}{l} 2187a^7 - 5103a^9 + 5103a^{11} - 2835a^{13} + \\ 945a^{15} = y^7 \end{array}\right.$
1 ⊙.9 −	5	$\left\{\begin{array}{l} 19683a^9 - 59049a^{11} + 78732a^{13} - \\ 61236a^{15} = y^9 \end{array}\right.$
1 × 9 −	6	$27a - 9a^3 = 9y$
2 × 30	7	$810a^3 - 810a^5 + 270a^7 - 30a9 = 30y^3$
3 × 27 −	8	$\left\{\begin{array}{l} 6561a^5 - 10935a^7 + 7290a^9 - 2430a^{11} + \\ 405a^{13} + 27a^{15} = 27y^5 \end{array}\right.$
4 × 9	9	$\left\{\begin{array}{l} 19683a^7 - 45927a^9 + 45927a^{11} - \\ 25515a^{13} + 8505a^{15} = 9y^7 \end{array}\right.$
6 − 7 +8−9 + 5	10	$\left\{\begin{array}{l} 27a - 819a^3 + 7371a^5 - 30888a^7 + \\ 72930a^9 - 107406a^{11} + \\ 104652a^{13} - 69768a^{13} \end{array}\right\} = 1$

Here $a =$ the *Chord* of $\frac{1}{102}$ Part of the Circle.

Proceeding on in this Method of continually triſecting the Arch of every new Chord, and ſtill connecting the produced *Æquations* into one, as in the *two laſt Triſections*, 'twill not be difficult to obtain the *Chord* of any aſſign'd Arch, how ſmall ſoever it be.

Now, in order to facilitate the Work of raiſing theſe *Æquations* to any conſiderable Height, 'twill be convenient to add a few uſeful Obſervations concerning their Nature, and of ſuch Contractions as may be ſafely made in them; which, being well underſtood, will render the Work very *eaſy*.

1. *I have obſerv'd, that every* Triſection *will gain or advance one Figure in the Circle's* Periphery, *but no more. Therefore ſo many Places of Figures as are at firſt deſign'd to be perfect in the* Periphery, *ſo many* Triſections *muſt be repeated to raiſe an Æquation that will produce a* Chord *anſwerable to that Deſign.*

2. *I have alſo found, that all the ſuperior Powers (of* a) *whoſe Indices are greater than the Number of* Triſections, *(viz. whoſe Indices are greater than the Number of deſign'd Figures) may be wholly rejected as inſignificant.*

3. *When once the Number of* Triſections, *and thence the higheſt Power (of* a) *is determined, the third* Proceſs *(viz. the third* Triſection) *may be made a fix'd or conſtant Canon; for by it, and Multiplication only, all the ſucceeding* Triſections *(how many ſoever they are) may be compleated without repeating the ſeveral Involutions.*

4. *In*

4. *In raiſing and collecting the Co-efficients of the ſeveral Powers (of* a*) 'twill be ſufficient to retain only ſo many ſignificant Figures (at* a^3*) as there is deſigned to be Places of Figures in the* Periphery *(or at moſt but two more) and every ſucceeding ſuperior Power may be allow'd to decreaſe two Places of ſignificant Figures: But herein great Care muſt be taken to ſupply the Places, of thoſe Figures that are omitted, with Cyphers, that ſo the whole and exact Number of Places may be truly adjuſted; otherwiſe all the Work will be erroneous.*

Now the Number of thoſe ſupplying Cyphers may be very conveniently denoted by Figures placed within a Parentheſis, *thus:* 576 (8) a^3, *may ſignify* 576000000a^3, *as in the following Æquations. The like may be done with* Decimal Parts, thus: (,7)658 *may ſignify* ,0000000658, &c. *which will be found very uſeful in the Solution of theſe and the like Æquations.*

The aforeſaid Contractions may be ſafely made, becauſe both the ſuperior Powers of *a*, which are rejected; as alſo thoſe Numbers that are omitted in the *Co-efficients* (and ſupply'd with Cyphers) would produce Figures ſo very remote from Unity, as that they would not affect the *Chord* deſign'd; that is, they would not affect the *Chord* in that Place wherein the deſign'd *Periphery* is concerned; as will in Part appear in the following Example.

If theſe Directions be carefully minded, 'twill be eaſy to raiſe an *Æquation* that will produce the Side of a *regular Polygon*, whoſe Number of Sides ſhall be vaſtly numerous, conſequently infinitely ſmall: But, I preſume, 'twill be ſufficient for an *Example* to find the Side of a *Polygon* conſiſting of 258280326 equal Sides; that is, if I find the *Chord* of $\frac{1}{258280326}$ Part of the *Circle*'s Periphery, and that requires but *ſixteen Triſections*, which being ordered, as before directed, will produce this *Æquation*.

$$\left\{\begin{array}{l} 43046721a - 33236017948696812(4)a^3 \\ +769837653199714(20)a^5 - 8491218532841(35a^7 \\ +546333311435(50)a^9 - 230083348(66)a^{11} \\ +6830988(79)a^{13} - 15072(94)a^{15} \end{array}\right\} = 1$$

Here the Value of *a* will have 23 Places of *Figures* true; that is, the Sides of the *inſcrib'd* and *circumſcrib'd Polygons* will be exactly the ſame to 23 Places of *Decimal Parts*, but no farther; all which may be eaſily obtain'd at *two Operations*. And for the firſt, 'twill be ſufficient to take only *three Terms* of the *Æquation*, which will admit of being yet farther contracted, thus:

Let

$$\text{Let } \left\{ \begin{matrix} 43046721a - 3323601794(12)a^3 \\ +76983765(27)a^5 \end{matrix} \right\} = 1$$

And let $r + e = a$; then rejecting all the *Powers* of e, that arise by Involution above eee,

It will be $r^3 + 3rre + 3ree + eee = aaa$

And $r^5 + 5r^4e + 10r^3ee + 10rreee = a^5$

Then the first single Value of r may be thus found:

43046721) 1,00000000 (,00000002 $= r$

This ,00000002 $= r$ being duly involv'd, and its Powers multiply'd into their respective *Co-efficients*, will produce

$$\left. \begin{matrix} +,86093441 + 43046721e \\ -,02658881 - 3988322e - 199416(9)ee - 3324(18)eee \\ +,00024635 + 61587e + 6159(9)ee + 308(18)eee \end{matrix} \right\} = 1$$

viz. ,83459196 + 39119986e — 193257(9)ee — 3016(18)eee = 1

Hence 39119986e — 193257(9)ee — 3016(18)ee = 0,16540804

All the *Terms* of this last *Æquation* being divided by 193257(9) the *Co-efficient* of ee, it will then become

,0000002024 e — ee —, 156 (5) eee =, 00000000000008558968 = D

$$\textit{Consequently, } \left\{ \frac{D + 156(5)eee}{,0000002024 - e} = e \right.$$

Operation.

	,0000002024)	,00000000000008558968	(,00000004 = e
— e	,0000000043 :	+ ,00000000000000009984	= 156 (5) eee
1 *Di.*	,0000001981)	,00000000000008568952	(,00000004327
2 *Di.*	,0000001981	792	
		6489	
		5943	
		5465	
		3962	
		&c.	

First r = ,00000002

$+ e$ = ,000000004327

$r + e$ = ,000000024327 = a.

Or rather new r for a second Operation.

Now, if this first Value of a = ,000000024327 *were not continued to more Places of Figures by a second Operation, but only multiply'd into the Number of* Chords, *viz.* ,000000024327 × 258280326 = 6,28318539, &c. *the Periphery of that* Circle *whose Diameter is* 2, *nearer than either* Archimedes, *or* Mœtius's *Proportion*: *For* Ar-

Archimedes *makes it* 6,285714, &c. *viz. As* 7 *to* 22. *And* Mœtius *makes it* 6,28318584, *viz. As* 113 *to* 355.

But if the whole *Æquation* before propos'd be now taken, and we proceed to a second *Operation*, the Value of a may be increas'd with twelve Places of *Figures* more, and those may be obtain'd by plain *Division* only.

Thus, let $r + e = a$, as before, and let all the Powers of e be now rejected as insignificant;

$$\text{Then will}\left\{\begin{array}{l} r + e = a \\ r^3 + 3r^2e = a^3 \\ r^5 + 5r^4e = a^5 \\ r^7 + 7r^6e = a^7 \end{array}\right\} \text{and} \left\{\begin{array}{l} r^9 + 9r^8e = a^9 \\ r^{11} + 11r^{10}e = a^{11} \\ r^{13} + 13r^{12}e = a^{13} \\ r^{15} + 15r^{14}e = a^{15} \end{array}\right.$$

The several Powers of $r = ,000000024327$ being rais'd, and multiply'd into their respective *Co-efficients*, will produce these following Numbers:

$$\left.\begin{array}{lr} +1,047197581767 & +\ 43046721e \\ -\ ,047849196598394865 & -\ 5900751e \\ +\ ,000655906484595355 & +\ 134810e \\ -\ ,000004281440413375 & -\ 1232e \\ +\ ,000000016302517803 & +\ 6e \\ -\ ,000000000040631167 & -\ 0e \\ +\ ,000000000000071388 & +\ 0e \\ -\ ,000000000000000093 & -\ 0e \end{array}\right\} = 1$$

Viz. $1,000000026474745106 + 37279554e = 1$

Hence $37279554e = -\ ,000000026474745106 = D$: Or rather $-\ 37279554e = \ ,000000026474745106 = D$

Consequently, $\left\{\begin{array}{c} D \\ 37279554 \end{array}\right. = -e$

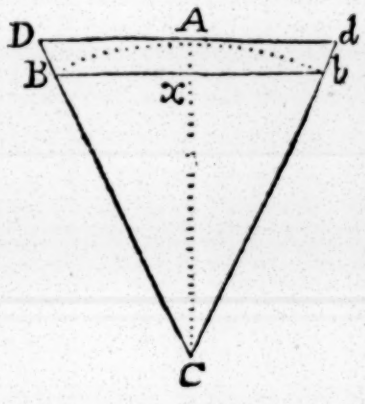
Operation.

$$\begin{array}{l} 37279554)\ ,000000026474745106)\ (,15)710167967 = -e \\ \qquad\qquad\qquad\quad 260956878 \\ \qquad\qquad\qquad\quad \overline{37905730} \\ \qquad\qquad\qquad\qquad 37279554 \\ \qquad\qquad\qquad\qquad \overline{62617660} \\ \qquad\qquad\qquad\qquad\quad 37279554 \\ \qquad\qquad\qquad\qquad\quad \overline{\&c.} \end{array}$$

Laſt $r = ,000000024327$
$-e = ,000000000000000710167967$

$r - e = ,000000024326999289832033 = a$ the *Chord* or *Side* of the *Polygon* required.

The next Work will be to examine how many Places of theſe *Figures* will hold true to the *Circle's Periphery*: In order to that, let a be repreſented by the *Chord* $B\,b$, in the *annexed Scheme*; and let $Bx = xb$. Then will $Bx = \frac{1}{2}a = (,7)121634996449160165$ and $\square\,BC - \square\,Bx = \square\,Cx$. Let the *Radius* $BC = 1$ as before. Then will the $\surd\,\square BC - \square Bx = Cx = ,9999999999999999$, &c.



But $Cx : xB :: CA : AD$ or $Cx : Bb :: CA : Dd$ } per *Fig.*
Ergo $Dd = (,7)243269992898320354$ the Side of the *Circumſcribing Polygon*. Then will $a \times 258280326$ be the *Perimeter* of the *Inſcrib'd Polygon*. And $Dd \times 258280326$ will be the *Perimeter* of the *Circumſcribing Polygon*. That is, $6,283185307179585\,9 =$ the *Perimeter* of the *Inſcrib'd Polygon*. And, $6,283185307179586\,5 =$ the *Perimeter* of the *Circumſcrib'd Polygon*.

Hence 'tis evident, that the *Circle's Periphery*, whoſe *Diameter* is 2, may be concluded $6,283185307179586\,4$ true, becauſe the *Perimeters* of the inſcrib'd and circumſcrib'd *Polygons* are ſo far very near being *Co-incident*, or the ſame.

'Tis poſſible there may be ſome who will think this is tedious and troubleſome Work; but if thoſe pleaſe to conſider, that, if this *Periphery* were to be found by the aforeſaid *Method* of *Biſection*, it would require theſe following *Extractions*.

Viz. $\left\{\begin{array}{l} \surd:2-\surd:2+\surd:2+\surd:2+\surd:2+\surd:2+\surd:2 \\ +\surd:2+\surd:2+\surd:2+\surd:2+\surd:2+\surd:2+\surd:2 \\ +\surd:2+\surd:2+\surd:2+\surd:2+\surd:2+\surd:2+\surd:2 \\ +\surd:2+\surd:2+\surd:2+\surd:2+\surd:2+\surd:2+\surd:3 \text{ } \textit{mutiplied} \text{ into } 402809984. \end{array}\right.$

Here the firſt Root (*viz.* $\surd\,3$) muſt be *extracted* at leaſt to one hundred and two Places of *Figures*. The ſecond Root

(*viz.* $\sqrt{:2+\sqrt{3}}$) muſt have 99 Places of *Figures* in it. The third Root (*viz.* $\sqrt{:2+\sqrt{:2+\sqrt{3}}}$) muſt have 96 Places in it, &c. every *Extraction* being allow'd to *decreaſe three Places*, that ſo the laſt *Root* (viz. *the Chord ſought*) may conſiſt of 24 Places of *Figures*, as above.

I ſay, whoever duly conſiders the Trouble of theſe ſo often repeated *Extractions* will, I preſume, be pleas'd with what I have done. For truly, when I conſider the great Time and Care required in them, I cannot but admire at the Patience of the laborious *Van Culen*, who proceeded that Way until he had found the *Circle's Periphery* to *Thirty-ſix Places of Figures*, to wit, 6,283185307179586476925286766559005 76. *Theſe Numbers are ſaid to be engraven upon his Tomb-Stone in St.* Peter's *Church in* Leyden, *for a Memorial of ſo great a Work.*

Having thus obtain'd the *Circle's Periphery*, its *Arch* may eaſily be found (*to the ſame Number of Figures*) by *Problem* 6. That is, if *Half the Periphery* of any *Circle* be *multiply'd* into *Half* its *Diameter*, the *Product* will be that *Circle's Area*, as will appear farther on. Therefore 3,14159265358979 3 will be the *Area* of the *Circle* whoſe *Diameter* is 2.

Thus I have ſhew'd the young *Geometer* how to find the *Circle's Periphery* and *Area* to what Exactneſs he pleaſes to approach; for preciſely true cannot be found, notwithſtanding the late *Pretenſions* of a certain *Frenchman* who hath publiſhed to the World (in the *Works of the Learned*) that after twenty-five Years *Study* he had found the *Quadrature* of the *Circle*: But if he had perus'd the 83d *Chapter* of Dr. *Wallis's Algebra*, he might there have ſeen his *Error*, *viz.* the *Impoſſibility* of what be pretended to; for it is as impoſſible to ſquare the *Circle* (*that is*, *to find its true Area*) as it is to find the *Root* of a *Surd Number*.

Note, What I have here propoſed and done by the Triſection of an Arch, may as eaſily and much more ſpeedily be perform'd by Quinqueſection or Septiſection, *&c.* But becauſe the Scheme for Triſection is more ſimple, and may be eaſier underſtood by a Learner than thoſe of the other Sections (of which ſee my *Compendium of Algebra*, Pages 76 and 79) I have for that Reaſon made Choice of Triſection.

As to the *Proportion* of one *Circle* to another, and of the *Circle* to the *Ellipſis*, &c. thoſe ſhall be fully ſhew'd when we come to the Fifth Part.

Before I conclude this Part, I ſhall make ſome *Uſe* or *Application* of the above-found *Periphery*, in finding the *Quantity* of *Angles*, which is done by the Help of *Right-lines*, call'd *Sines* and *Tangents*, the Lengths whereof are calculated to every Degree and Minute of a *Quadrant*, by much Labour. But I ſhall here ſhew how to find the natural *Sine* (and conſequently the natural *Tangent*) of any propos'd *Arch* or *Angle*, by two *Æquations*, without the Help of any *precedent Sine*, as uſual; which I did ſome Years ago communicate to the ingenious Mr. *Joſeph Ralphſon*, and he ſo well approv'd of them as to make them the 20th and 21ſt *Problems* in the *ſecond Edition* of his *Analyſis Æquationum Univerſalis*.

And becauſe, in finding the Quantity of *Angles*, every *Circle* is ſuppos'd to be *divided* into 360 *equal Parts*, call'd *Degrees*; every *Degree* is *ſubdivided* into 60 *Parts*, call'd *Minutes*; and every *Minute* into 60 *Seconds*, &c. (See *Page* 294.)

Therefore 360) 6,2831853, &c. (0,0174532925, &c. is an *Arch* of the above-found *Periphery*, equal to the *Arch* of one *Degree*.

And 60) 0,0174532925, &c. (0,0002908882, &c. = the *Arch* of one *Minute*.

Then if the given *Arch (or Angle)* be leſs than 45 *Degrees*, reduce it into *Minutes*, and *multiply* thoſe *Minutes* into this conſtant *Multiplicator*, *viz.* 0,0002908882 calling the *Product p*. And for the *Sine* ſought put *a*. Then it will be $-aaaa + 12paaa - 195aa - 36ppaa + 240pa = 45pp$.

Example.

Let it be required to find the Sine of 19°. 13′. = 1153′. Here 0,0002908882 × 1153 = 0,3353940946 = *p*. And $-a^4 + 4{,}024729a^3 - 199{,}049611aa + 80{,}49583a = 5{,}06201394$.

Let $r + e = a$

Then $\begin{cases} rr + 2re + ee = aa \\ rrr + 3rre + ee = aaa \\ rrrr + 4rrre + 6rree = aaaa \end{cases}$

Note, *In this Caſe the firſt* r *may always be taken equal to the firſt Figure in the Product* = p. Viz. *here* r = 0,3 *which being involved as its Powers direct, and thoſe Powers multiply'd into the reſpective Co-efficients of the Æquation; it will be*

$$\left\{\begin{array}{rrrr} + & 24{,}1483 & + 80{,}49e & \\ - & 17{,}9144 & - 119{,}43e & - 199{,}05ee \\ + & 0{,}1086 & + 1{,}08e & + 3{,}62ee \\ - & 0{,}0081 & - 0{,}11e & - 0{,}54ee \end{array}\right\} = 5{,}06201394$$

Viz. $6{,}3344 - 37{,}97e - 135{,}97ee = 5{,}06201$

Hence

Hence $37,97ee + 195,97ee = 1,27239$
And $0,123e + ee = 0,006492 = D$

THEOREM $\left\{ \frac{D}{,193+e} = e \right.$

Operation. 0,193) 0,006492 (0,029 = *e*

$+ e =$	,029	42
1. *Diviſor*	,21	2292
		1998
2. *Diviſor*	,222	

First $r = 0,3$
$+ e = 0,029$

$r + e = 0,329 = r$ for a ſecond *Operation.*

Which being *involv'd* and *multiply'd*, &c. as before, will produce theſe *Numbers:*

$+26,48271781 + 80,49458e$
$-21,54532894 - 130,97464e - 199,0496ee$
$+ 0,14332578 + 1,30692e + 3,9724ee$
$- 0,01171611 - 0,14244e - 0,6494ee$

Viz. $5,06899854 - 49,31558e - 195,7266ee = 5,06201394$

Hence $49,31558e + 195,7266ee = ,0069846$; which being divided by 195,7266 the *Co-efficient* of *ee*, will become $,25196e + ee = ,0000356854 = D$.

Then $\left\{ \frac{D}{,25196+e} = e \right.$

. . . . .

Operation. 0,25196) ,0000356854 (0,0001415 = *e*

$+ e =$	0,00014	2520
1. *Diviſor*	0,2520	104854
2. *Diviſor*	0,25210	100840
		40140
		25210
		&c.

Laſt $r = 0,329$
$+ e = 0,0001415$

$r + e = a = 0,3291415$ being the natural *Sine* of 90°. 13′. As was required.

Thus you may find the *Right Sine* of any *Arch* or *Angle* leſs than 45 *Degrees.*

But,

But, if the given *Arch* be greater than 45 *Degrees*, you muſt take its Complement to 90°. *viz.* ſubtract it from 90 *Degrees*, and reduce the Remainder into *Minutes*, as before. Then *multiply* the *Square* of theſe *Minutes* into this conſtant *Multiplicator*, 0,000000084616 calling their *Product* p, and putting $a =$ the *Sine* ſought, as before. Then will $a^4 + 28a^3 + 195aa + 36paa + 108pa - 28a = 196 - 81p$.

Example.

Suppoſe it were required to find the *Sine* of 75°. 32′. or (which is the ſame Thing) to find the *Co-ſine* of 14°. 28′. = 868′, whoſe *Square* 753424 × 0,000000084616 = 0,063751725 18 = p. Hence the *Æquation* in *Numbers* will be $aaaa + 28aaa + 197{,}295062aa - 21{,}114814a = 190{,}8361102588$.

Let $r - e = a$ And $r = 1$.

Then $\begin{cases} rr - 2re + ee = aa \\ rrr - 3rre + 3ree = aaa \\ rrrr - 4rrre + 6rree = aaaa \end{cases}$

Note, *I here take* $r = 1$ *becauſe the Arch is ſo near to* 90°. *and therefore I make it* $r - e = a$.

Then $\left\{\begin{array}{rrrrr} - & 21{,}1148 & + & 21{,}11e & + \\ + & 197{,}2956 & - & 394{,}59e & + & 197{,}29ee \\ + & 28{,}0000 & - & 84{,}00e & + & 84{,}00ee \\ + & 1{,}0000 & - & 4{,}00e & + & 6{,}00ee \end{array}\right\} = 190{,}8361$

Viz. $205{,}1808 - 461{,}48e + 287{,}29ee = 190{,}8361$

Hence $461{,}48e - 287{,}29ee = 14{,}3447$

And $1{,}606e - ee = {,}049930 = D$

THEOREM $\left\{ \frac{D}{1{,}606 - e} = e \right.$

Operation. 1,606) ,049930 (0,031 = e

$-e = 0{,}031$ 471

1. *Diviſor*	1,57	2830	First $r = 1{,}000$
		1575	$-e = 0{,}031$
2. *Diviſor*	1,575		$r - e = 0{,}969 = r$
		&c.	

for a ſecond *Operation*; which, being *involv'd* as before, will produce theſe following Numbers:

— 20,

$$\begin{array}{rrrr}
- & 20{,}460254766 & + \; 21{,}11481e & \\
+ & 185{,}252368710 & - \; 382{,}35783e & + \; 197{,}2951ee \\
+ & 25{,}475889852 & - \; 78{,}87272e & + \; 81{,}5960ee \\
+ & 0{,}881647759 & - \; 3{,}63941e & + \; 5.6337ee \\
\hline
\textit{Viz.} & 191{,}149651515 & - \; 443{,}75515e & + \; 284{,}5248ee \\
= & 190{,}836110259 & &
\end{array}$$

Hence it will be $443{,}75515e - 284{,}5248ee = 0{,}313541256$

And $1{,}55963e - ee = {,}0011019821 = D$

$$\text{Then} \left\{ \frac{D}{1{,}55963 - e} = e \right.$$

$$\begin{array}{lll}
\textit{Operation.} & 1{,}55963) & 0{,}0011019821 \; (0{,}0007068 = e \\
- e = & 0{,}00070 & \quad 109123 \\
\hline
1. \textit{Divisor} & 1{,}5589 & \quad 1075210 \\
\hline
2. \textit{Divisor} & 1{,}55893 & \quad 935358 \\
\hline
 & & \quad 1398520 \\
 & & \quad 1247144, \text{ \&c.} \\
\hline
\end{array}$$

Last $r = 0{,}969$

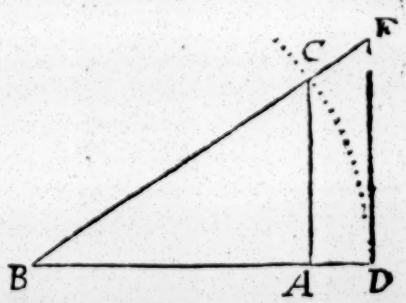
$- e = 0{,}0007068$

$r - e = a = 0{,}9682932$ the *Sine* of $75^\circ.\ 32'.$ as was required.

Having found the *Sine* and *Co-sine* of any *Arch*, the *Tangent* is usually found by this *Proportion :*

Viz. { *As the Co-sine of any Arch : is to the Sine* of that *Arch : : so is the Radius : to the Tangent of the same Arch.*

For supposing $BC = BD$ *Radius*, AC the *Sine* of the *Arch* CD. Then BA is the *Co-sine*, and FD the *Tangent* of the same *Arch*. But $BA : CA :: BD : FD$, &c. Now by this *Proportion* there is *required* to be given both the *Sine* and *Co-sine* of the *Arch*, to find the *Tangent*. 'Tis true, if the *Radius*, and either the *Sine* or the *Co-sine* be given,



the other may be found, thus, $\sqrt{\square BC - \square CA} = BA$. Or $\sqrt{\square BC - \square BA} = CA$. But, if either the *Sine* or *Co-sine* be given, the *Tangent* may (I presume) be more easily found by the following *Theorems :*

Let

Let $BC = 1$. $CA = S$. $BA = x$ and $FD = T$. Then, if S be given, T may be found by this

$$\text{Theorem.} \left\{ \surd \frac{SS}{1 - SS} = T \right.$$

Or if x be given, T may be found by this

$$\text{Theorem} \left\{ \surd \frac{xx}{1 - xx} = T \right.$$

Let the *Sine* of 90°. 13′. *(before found)* be given, *viz.* 0,3291415 $= S$, to find T the *Tangent* of the ſame *Arch.* Firſt 0,3291415 × ,3291415 = 0,108334127 $= SS$. Again 1 — 0,108334127 = 0,891665873 = 1 — SS. Then 0,891665873) 0,108334127 (0,1214963253 and $\surd$ 0,1214963253 = 0,3485632 = T, the *Tangent* of 19°. 13′. As was required. And ſo you may proceed to find T = the *Tangent*, when x = the *Co-ſine* is given.

Perhaps it may here be *expected*, that I ſhould have ſhew'd and *demonſtrated (or at leaſt have inſerted)* the *Proportions* from whence the foregoing *Æquations* for making *Sines* were *produced*; but I have omitted that, as alſo their *Uſe* in computing the *Sides* and *Angles* of *plain Triangles* by the *Pen* only *(viz. without the Help of Tables)* for the *Subject* of my *Diſcourſe* hereafter, if *Health* and *Time* permit.

In the mean *Time*, what is here done may ſuffice to ſhew, that the making of *Sines* by ſuch a *laborious* and *operoſe* Way, as was formerly uſed, is in a great *Meaſure* overcome; which, I think, I may juſtly claim as my own.

AN INTRODUCTION TO THE Mathematicks.

PART IV.

CHAP. I.

Definitions of a Cone, *and its* Sections.

THERE are ſeveral Definitions given of a *Cone:* The Learned Dr. *Barrow*, upon *Euclid*, hath it thus:

" A Cone *(ſaith he)* is a Figure made when one Side of " a Rectangle Triangle, *(viz.* one of thoſe Sides that contain the " Right Angle) remaining fix'd, the Triangle is turn'd round " about, 'till it return to the Place from whence it firſt moved: " And if the fix'd Right Line be equal to the other which con- " taineth the Right Angle, then the Cone is a Rectangled Cone: " but if it be leſs, 'tis an Obtuſe-angled Cone; if greater, an Acute- " angled Cone. The *Axis* of a Cone is that fix'd Line about which " the Triangle is mov'd: The *Baſe* of a Cone is the Circle, which " is deſcrib'd by the Right-Line mov'd about."

(Defin. 18, 19, 20. *Euclid.* 11.)

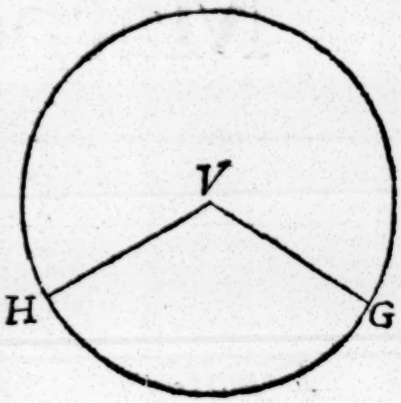
Sir *Jonas Moor*, in his Treatiſe of *Conical Sections* (taken out of the Works of *Mydorgius)* defines it thus:

" If a Line of ſuch a Length as ſhall be needful ſhall, upon a " Point fix'd above the Plain of a Circle, ſo move about the Cir- " cle, until it return to the Point from whence the Motion began, " the Superficies that is made by ſuch a Line is call'd a *Conical* " *Superficies*; and the ſolid Figure contain'd within that Superficies " and the Circle is call'd a *Cone*. The Point remaining ſtill is the " Vertex of the Cone, *&c.*"

Altho' both theſe *Definitions* are equally true, and, with a little Conſideration, may be pretty eaſily underſtood; yet I ſhall here propoſe one very different from either of them; and, as I preſume, more plain and intelligible, eſpecially to a *Learner*.

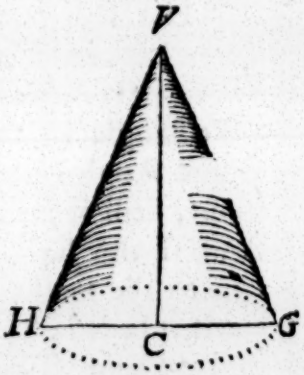
If a Circle deſcribed upon ſtiff Paper *(or any other pliable Matter)* of what Bigneſs you pleaſe, be cut into *two*, *three*, or more *Sectors*, either equal or unequal, and one of thoſe *Sectors* be ſo roll'd up, as that the *Radii* may exactly meet each other, it will form a *Conical Superficies*.

That is, if the *Sector HVG* be cut out of the Circle, and ſo roll'd up as that the *Radii VH* and *VG* may juſt meet each other in all their Parts, it will form a *Cone*, and the Center *V* will become a *Solid Point*, call'd the *VERTEX* of the *Cone*; the *Radius VH*, being every-where equal, will be the Side of the Cone, and the Arch *HG* will become a Circle, whoſe *Area* is call'd the *Cone's Baſe*.

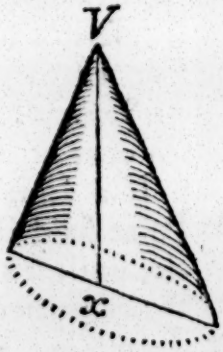


A *Right Line* being ſuppos'd to paſs from the *Vertex*, or Point *V*, to the *Center* of the Cone's *Baſe*, as at *C*, that Line (viz. *VC*) will be the *AXIS*, or *perpendicular Height* of the Cone.

If a *Solid* be actually made in ſuch a Form, it will be a compleat or perfect Cone; which I ſhall all along call a Right Cone, becauſe its *Axis VC* ſtands at *Right Angles* with the Plain of its *Baſe HG*, and its Sides are every-where equal.



Any Cone, whoſe *Axis* is not at *Right Angles* with the Plain of its *Baſe*, may be properly call'd an *imperfect Cone*, becauſe its Sides are not every-where equal *(as in the annexed Figure.)* Now, ſuch an imperfect Cone is uſually call'd a *Scalene*, or *Oblique Cone*.



Any ſolid Cone may be cut by Plains *(which I ſhall all along hereafter call Right Lines)* into *five Sections*.

Sect.

Sect. 1.

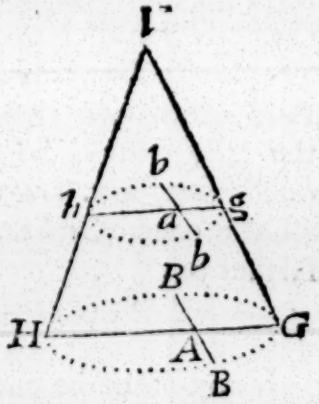
If a Right Cone be cut directly thro' its *Axis*, the *Plain* or *Superficies* of that *Section* will be a plain *Iſoſceles Triangle*, as *HVG Fig.* 2, viz. the Sides (*HV* and *VG*) of the Cone will be the Sides of the Triangle, the Diameter (*HG*) of the Cone's *Baſe* will be the *Baſe* of the Triangle, and (*VC*) its *Axis* will be the *perpendicular Height* of the Triangle.

Sect. 2.

If a Right Cone be cut *(any where)* off by a Right line parallel to its *Baſe*, as *h g (it will be eaſy to conceive, that)* the Plain of that *Section* will be a Circle, becauſe the Cone's *Baſe* is ſuch: wherein one Thing ought to be clearly underſtood, which may be laid down as a *Lemma*, to demonſtrate the Properties of the following *Sections*.

LEMMA. *If any two Right Lines, inſcrib'd within a Circle, do cut or croſs each other* (as *h g* doth *b b* in the annexed Figure) *the Rectangle made of the Segments of one of the Lines will be equal to the Rectangle made of the Segments of the other Line.* (See *Theorem* 15. *Page* 315.)

That is, $ha \times ga = ba \times ab$ } &c.
And $HA \times GA = BA \times AB$ }
conſequently if $baa = ab$, and if $BA = AB$, then it will be $ha \times ga = \square ba$, and in the *Cone's* Baſe $HA \times GA = \square BA$.

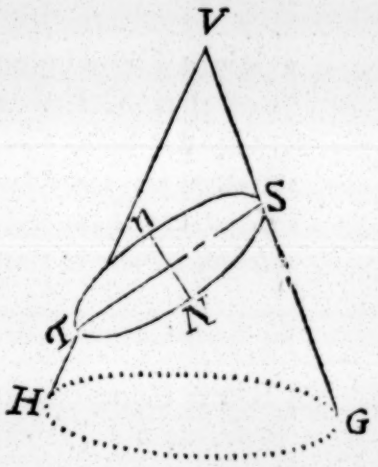


Sect. 3.

If a *Right Cone* be *(any where)* cut off by a *Right Line* that cuts both its *Sides*, but not *parallel* to its *Baſe (as TS in the following Figure)* the Plain of that Section will be an *Ellipſis (vulgarly called an* Oval) *viz.* an oblong or imperfect Circle, which hath ſeveral *Diameters)* and *two particular Centers*. That is,

1. Any *Right Line* that divides an *Ellipſis* into *two* equal Parts is call'd a *Diameter*; amongſt which the *longeſt* and the *ſhorteſt* are particularly diſtinguiſh'd from the reſt, as being of moſt general Uſe; the other are only applicable to particular Caſes.

2. The longeſt *Diameter* (as *TS*) is called the *Tranſverſe Diameter*, or *Tranverſe Axis*, being that *Right Line* which is drawn thro' the Middle of the *Ellipſis*, and doth ſhew or limit its Length.



3. The ſhorteſt *Diameter*, call'd the *Conjugate Diameter*, is a *Right-Line* that doth interſect or croſs the *Tranſverſe Diameter* at *Right Angles*, in the Middle or common Center of the *Ellipſis* (as *Nn*) and doth limit the *Ellipſis*'s *Breadth*.

4. The *two Points*, which I call particular *Centers* of an *Ellipſis* (*for a Reaſon which ſhall be ſhew'd farther on*) are two Points in the *Tranverſe Diameter*, at an equal Diſtance each Way from the *Conjugate Diameter*, and are uſually call'd NODES, FOCI, or *burning Points*.

5. All *Right Lines* within the *Ellipſis* that are paralled to one another, and can be divided into two equal Parts, are called ORDINATES with Reſpect to that *Diameter* which divides them: And if they are parallel to the *Conjugate*, viz. at *Right Angles* with the *Tranſverſe Diameter*, then they are call'd *Ordinates rightly apply'd*. And thoſe two that paſs through the *Foci* are remarkable above the reſt, which, being equal and ſituated alike, are call'd both by one Name, *viz.* LATUS RECTUM, or *Right Parameter*, by which all the other *Ordinates* are regulated and valued; as will appear farther on.

Sect. 4.

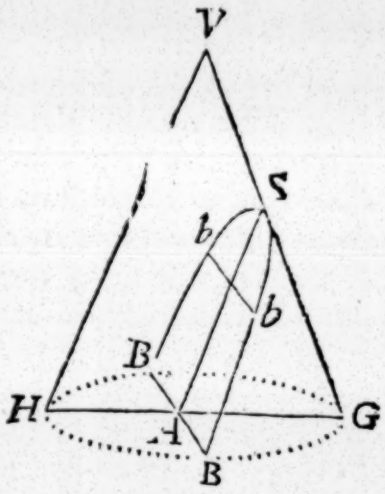
If any Cone be cut into *two Parts* by a *Right-line* parallel to one of its Sides (*as S A in the following Scheme*) the Plain of that Section (*viz. SbBABbS*) is call'd a PARABOLA.

1. A *Right Line* being drawn thro' the Middle of any *Parabola* (as *SA*) is call'd its *Axis*, or intercepted Diameter.

2. All *Right Lines* that interſect or cut the *Axis* at *Right Angles* (*as BB and bb are ſuppos'd to cut or croſs SA*) are call'd *Ordinates rightly apply'd* (*as in the Ellipſis*) and the greateſt *Ordinate*, as *BB*, which limits the *Length* of the *Parabola's Axis* (*SA*) is uſually call'd the Baſe of the *Parabola*.

3. That

3. That *Ordinate* which passes thro' the *Focus*, or burning Point of the *Parabola*, is called the *Latus Rectum*, or *Right Parameter (as in the Ellipsis)* because by it all the other *Ordinates* are *proportion'd*, and may be found.

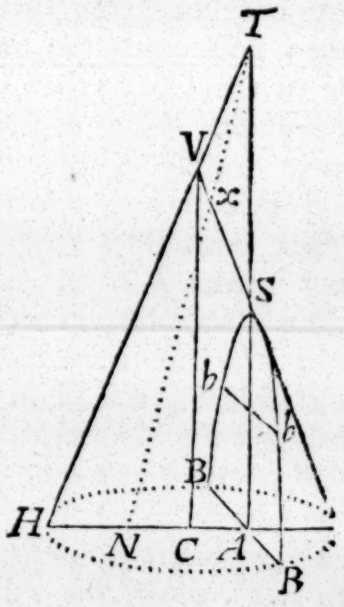


4. The *Node*, *Focus*, or *burning Point* of the *Parabola*, is a Point in its Axis *(but not a Center, as in the Ellipsis)* distant from the *Vertex*, or Top of the Section, (viz. *from S)* just $\frac{1}{4}$ Part of the *Latus Rectum*; as shall be shewn farther on.

5. All *Right Lines* drawn within a *Parabola* parallel to its *Axis* are call'd *Diameters*; and every *Right Line*, that any of those *Diameters* doth bisect or cut into two equal Parts, is said to be an *Ordinate* to the *Diameter* which bisects it.

Sect. 5.

If a Cone be any where cut by a *Right Line*, either parallel to its *Axis (as S A, or otherwise as x N)* so as the cutting Line being continued thro' one Side of the Cone (as at *S* or *x*) will meet with the other Side of the Cone if it be continued or produced beyond the *Vertex V*, as at *T*; then the Plain of that Section (viz. *the Figure S b B B b S)* is call'd an HYPERBOLA.



1. A *Right Line* being drawn thro' the Middle of any *Hyperbola*, viz. within the Section *(as S A, or x N)* is call'd the *Axis* or intercepted *Diameter (as in the Parabola)* and that Part of it which is continued or produced out of the Section, until it meet with the other Side of the Cone continued, viz. *T S* or *T x*, &c. is call'd the *Transverse Diameter*, or *Transverse Axis* of the *Hyperbola*.

2. All *Right Lines* that are drawn within an *Hyperbola*, at *Right Angles* to its *Axis*, are call'd *Ordinates right'y apply'd*; as in the *Ellipsis* and *Parabola*.

3. That

3. That *Ordinate* which passes thro' the *Focus* of the *Hyperbola* is call'd *Latus Rectum*, or *Right Parameter*, for the same Reason as in the other Sections.

4. The middle Point of the *Transverse Diameter* is call'd the *Center* of the *Hyperbola:* from whence may be drawn *two Right Lines* (out of the Section) call'd ASYMPTOTES, because they will always incline *(that is, come nearer and nearer)* to both Sides of the *Hyperbola*, but never meet with (or touch) them, altho' both they and the Sides of the *Hyperbola* were infinitely extended; as will plainly appear in its proper Place.

These five Sections, *viz.* the *Triangle*, *Circle*, *Ellipsis*, *Parabola*, and *Hyperbola*, are all the Plains that can possibly be produced from a Cone; but of them, the *three last* are only called *Conick Sections*, both by the ancient and modern *Geometers*.

Scholium.

Besides the 'foregoing *Definitions*, it may not be amiss to add, by Way of Observation, how one Section may *(or rather doth)* change or degenerate into another.

An *Ellipsis* being that Plain of any Section of the Cone which is between the *Circle* and *Parabola*, 'twill be easy to conceive that there may be great Variety of *Ellipses* produced from the same Cone; and when the Section comes to be exactly parallel to one Side of the Cone, then doth the *Ellipsis* change or degenerate into a *Parabola*. Now a *Parabola*, being that Section whose Plain is always exactly parallel to the Side of the Cone, cannot vary, as the *Ellipsis* may; for so soon as ever it begins to move out of that *Position* (viz. *from being parallel to the Cone's Side)* it degenerates either into an *Ellipsis*, or into an *Hyperbola:* That is, if the Section incline towards the Plain of the Cone's Base, it becomes an *Ellipsis*; but if it incline towards the Cone's Vertex, it becomes an *Hyperbola*, which is the Plain of any Section that falls between the *Parabola* and the *Triangle*. And therefore there may be as many *Varieties* of *Hyperbola's* produced from one and the same Cone, as there may be *Ellipses*.

To be brief, a *Circle* may change into an *Ellipsis*, the *Ellipsis* into a *Parabola*, the *Parabola* into an *Hyperbola*, and the *Hyperbola* into a plain *Isosceles Triangle:* And the Center of the Circle, which is its *Focus* or burning Point, doth, as it were, part or divide itself into two *Foci* so soon as ever the Circle begins to degenerate into an *Ellipsis*; but when the *Ellipsis* changes into a *Parabola*, one End of it flies open, and one of its *Foci* vanishes, and the

the remaining *Focus* goes along with the *Parabola* when it degenerates into an *Hyperbola:* And when the *Hyperbola* degenerates into a plain *Iſoſceles Triangle*, this *Focus* becomes the *vertical* Point of the *Triangle* (*viz.* the *Vertex* of the Cone); ſo that the Center of the Cone's Baſe may be truly ſaid to paſs gradually through all the Sections, until it arrives at the *Vertex* of the Cone, ſtill carrying its *Latus Rectum* along with it: For the *Diameter* of a Circle being that *Right Line* which paſſes through its *Center* or *Focus*, and by which all other *Right Lines* drawn within the Circle are regulated and valued, may *(I preſume)* be properly called the Circle's *Latus Rectum:* and although it loſes the Name of *Diameter* when the Circle degenerates into an *Ellipſis*, yet it retains the Name of *Latus Rectum*, with its firſt Properties, in all the Sections, gradually ſhortening as the *Focus* carries it along from one Section to another, until at laſt *it* and the *Focus* become co-incident, and terminate in the *Vertex* of the Cone.

I have been more particular and fuller in theſe *Definitions* than is uſual in Books of this Subject, which I hope is no Fault, but will prove of Uſe, eſpecially to a *Learner:* And altho' they may perhaps ſeem a little ſtrange, and at firſt hard to be underſtood, yet, when they are well conſidered, and compar'd with a Cone cut into ſuch Sections as have been defined, they will not only be found true, but will alſo help to form a true and clear *Idea* of each Section.

CHAP. II.

Concerning the Chief Properties of an Ellipſis.

NOTE, *If the tranſverſe Diameter of an* Ellipſis, *as TS in the following* Figure, *be interſected or divided into any two Parts by an* Ordinate rightly apply'd, *as at the Points A, C, a,* &c. *then are thoſe Parts TA, TC, Ta, and SA, SC, Sa,* &c. *uſually called* Abſciſſæ *(which ſignifies Lines or Parts cut off) and by the Rectangle of any two* Abſciſſæ *is meant the Rectangle of ſuch two Parts as, being added together, will be equal to the Tranſverſe Diameter.*

As $TA + SA = TS$. And $TC + SC = TS$.
Or $TA + SA = TS$, &c.

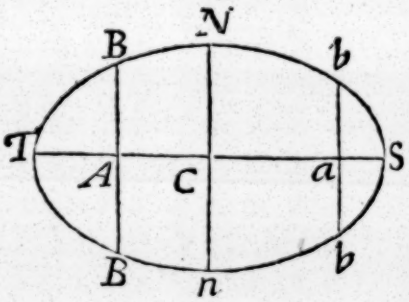
Section 1.

Every *Ellipſis* is proportion'd, and all ſuch Lines as relate to it are regulated, by the Help of one general *Theorem*.

THEOREM. { *As the Rectangle of any two* Abſciſſæ : *is to the Square of Half the* Ordinate *which divides them* : : *ſo is the Rectangle of any other two* Abſciſſæ : *to the Square of Half that* Ordinate *which divides them.*

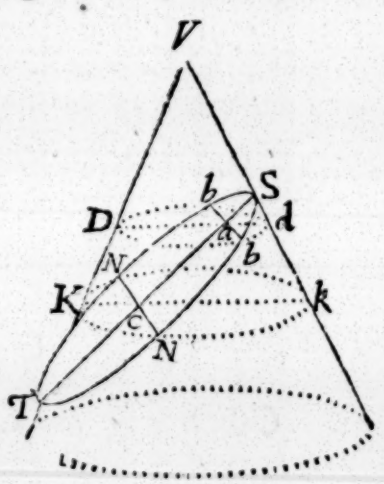
That is,

$TA \times SA : \square BA :: Ta \times Sa : \square ba$
$TA \times SA : \square BA :: TC \times SC : \square NC$
$TC \times SC : \square NC :: Ta \times Sa : \square ba$
&c.



Demonſtration.

Let the annexed Figure repreſent a Right Cone, cut thro' both Sides by the Right Line TS; then will the Plain of that Section be an *Ellipſis* (by *Sect.* 3. *Chap.* 1.) TS will be the *Tranſverſe Diameter*, NCN and bab will be *Ordinates rightly apply'd*; as before. Again, if the Lines Dd and Kk be parallel to the Cone's Baſe, they will be Diameters of Circles (by *Sect.* 2. *Chap.* 1.) Then will $\triangle TCK$ and TaD be alike. Alſo $\triangle Sad$ and $\triangle SCk$ will be alike.

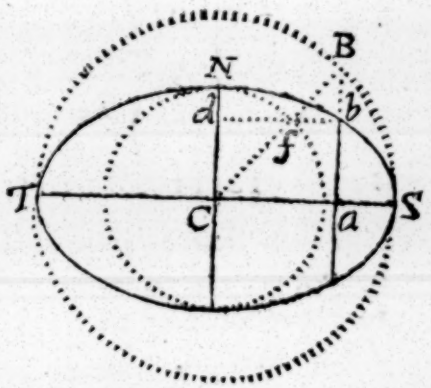


Ergo	1	$Sa : ad :: SC : Ck$ }
And	2	$TC : CK :: Ta : aD$ } per *Theorem* 13.
1 ∴	3	$Sa \times Ck = ad \times SC$
2 ∴	4	$Ta \times CK = TC \times aD$
2 × 3	5	$Sa \times Ck \times Ta \times CK = ad \times SC \times TC \times aD$. Per *Axiom* 3.
But	6	$CK \times Ck = \square NC$ }
And	7	$aD \times ad = \square ba$ } per *Lemma Sect.* 2.
Then		for $CK \times Ck$, and $aD \times ad$, take $\square NC$ and $\square ba$
5, 6, 7	8	$Sa \times Ta \times \square NC = TC \times SC \times \square ba$. Per *Axiom* 5.
Hence	9	$Sa \times Ta : \square ba :: TC \times SC : \square NC$. See *Page* 194.

Q. E. D.

Or

Or, the Truth of these Proportions may be otherwise prov'd by a Circle, without the Help of the Cone; thus: Let any *Ellipsis* be circumscrib'd and inscrib'd with Circles, as in the following Figure; then from any Point in the circumscrib'd Circle's Periphery, as at *B*, draw the Right Line *Ba*, parallel to the semi conjugate Diameter N *c*, then will *ba* be a *Semi-ordinate* rightly apply'd to the transverse Diameter *TS*, as before. Again, from the Point *b* (in the *Ellipsis Periphery*) draw the Right Line *bd* parallel to the Transverse *TS*; and draw the *Radius BC*. Then will $\triangle BCa$ and $\triangle Cfd$ be alike.



Therefore	1	$BC : Ba :: Cf : dC$ per *Theorem* 13.
But	2	$TC = BC, NC = Cf$ and $ba = dC$
Conseq.	3	$TC : Ba :: NC : ba$
Or	4	$TC : NC :: Ba : ba$
4 in □'s	5	$\square TC : \square NC :: \square Ba : \square ba$
But	6	$Ta \times Sa = \square Ba$ per Lem. *Sect.* 2. *Ch.* 1.
Therefore	7	$Ta \times Sa : \square ba :: TC \times SC = \square TC : \square NC$, as before.

And so for any other *Abscissæ* and their *Semi-ordinates*.

These Proportions being found to be the true and common Properties of every *Ellipsis*, all that is farther requir'd in (or about) that *Section* may be easily deduced from them.

Sect. 2. *To find the* Latus Rectum, *or* Right Parameter *of any Ellipsis.*

There are several Ways of finding the *Latus Rectum*, but I think none so easy, and shews it so plainly to be the Third Principal Line in the *Ellipsis*, as the following.

THEOREM. *As the* Transverse Diameter : *is in Proportion to the* Conjugate :: *so is the* Conjugate : *to the* Latus Rectum.

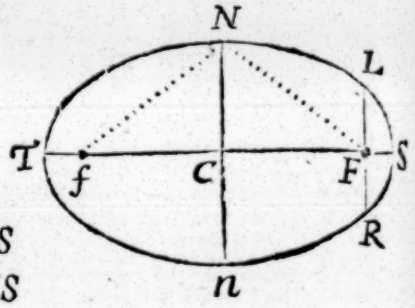
Viz. *(in the following Fig.)* $TS : Nn :: Nn : LR$ the *Latus Rectum*.

Demonstration.

From the last *Proportions* take either of the Antecedents, and its Consequent, *viz.* either $TC \times SC : \square NC$, or $Ta \times Sa : \square ba$,

and make TS the third Term, to which find a fourth Proportional, and it will be $= LR$:

Thus	1	$TC \times SC : \square NC :: TS : LR$
But	2	$\begin{cases} TC = SC \\ \text{and } NC = Cn \end{cases}$
Therefore	3	$TC \times SC = \frac{1}{4} \square TS$
And	4	$\square NC = \frac{1}{4} \square Nn$
1, 3, 4	5	$\frac{1}{4} \square TS : \frac{1}{4} \square Nn :: TS : LR$
5 ∴	6	$\frac{1}{4} \square TS \times LR = \frac{1}{4} \square Nn \times TS$
6 × 4	7	$\square TS \times LR = \square Nn \times TS$
7 ÷ TS	8	$\begin{cases} TS \times LR = \square Nn \\ \text{which gives the following } \textit{Analogy}. \end{cases}$
viz.	9	$TS : Nn :: Nn : LR$
Again	10	$\begin{cases} TC \times SC : \square NC :: Ta \times Sa : \square ba. \\ \text{by common } \textit{Properties}. \end{cases}$
1, 10	11	$TS : LR :: Ta \times Sa : \square ba.$



From hence 'tis evident that LR, thus found, is that *Ordinate* by which the other *Ordinates* may be regulated and found. Therefore (according to its Definition *Sect.* 3, *Chap.* 1.) it is the true *Latus Rectum*. Q. E. D.

Conſectary:

Hence it follows, that if the tranſverſe and conjugate Diameters of any *Ellipſis* are given *(either in Lines or Numbers)* the *Latus Rectum* may be eaſily found; and then any *Ordinate*, whoſe Diſtance from the Conjugate is given, may be found, as above.

Sect. 3. *To find the* Focus *of any* Ellipſis.

The *Focus* is the Diſtance of the *Latus Rectum* from the Conjugate or Middle of the *Ellipſis (vide* Definition 4, Page 364.) and that Diſtance is always a *Mean Proportional* between the half Sum and half Difference of the tranſverſe and conjugate Diameters, which gives this *Theorem.*

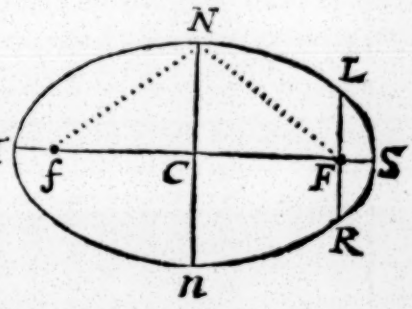
THEOREM. *From the* Square *of half the* Tranverſe *ſubtract the* Square *of half the* Conjugate, *the ſquare Root of their Difference will be the Diſtance of each* Focus *from the Middle or common Center of the Ellipſis.*

That is, ſuppoſing the Points f and F to be the two *Foci*, viz. $fC = Cf$, and $TC = \frac{1}{2} TS$. $NC = \frac{1}{2} Nn$. Then, $TC + NC : fC :: FC : TC - NC$ Ergo $\square FC = TC - \square NC$. Conſequently, $FC = \sqrt{\square TC - \square NC}$.

Demon-

Demonstration.

First,	1	$TS \times LR = \square Nn$, by 8th Step of the last Process.
And	2	$TS : LR :: TF \times SF : \square LF$, common Properties.
That is	3	$TS : LR :: \overline{TC + CF} \times \overline{TC - CF} : \frac{1}{4} \square LR = \square LF$
3	4	$\left\{ \begin{array}{l} \frac{1}{4} \square LR \times TS = \\ \overline{\square TC - \square CF} \times LR \end{array} \right.$
$4 \div LR$	5	$\frac{1}{4} LR \times TS = \square TC - \square CF$
$1 \div 4$	6	$\frac{1}{4} TS \times LR = \frac{1}{4} \square Nn \square = NC$
5, 6	7	$\square NC = \square TC - \square CF$
7, $\pm$	8	$\square CF = \square TC - \square NC$
8 ww	9	$CF = \surd \overline{\square TC - \square NC}$



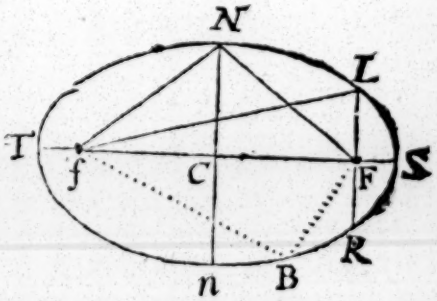
Now from hence is deduced that notable Proposition, upon which is grounded the usual Method of describing an *Ellipsis*, and drawing of *Tangents*, &c.

PROPOSITION. *If from the two* Foci *of any* Ellipsis *there be drawn two* Right Lines, *so as to meet each other in any Point of the* Ellipsis's Periphery, *the* Sum *of those* Lines *will be equal to the* Transverse.

Viz. $fN \times NF = TS$. $fL \times LF = TS$. Or $fB + BF = TS$, &c.

Demonstration.

First,	1	$\left\{ \begin{array}{l} \square CF \times \square NC = \square TC \\ \textit{by 8th of the last.} \end{array} \right.$
But	2	$\left\{ \begin{array}{l} \square CF + \square NC = \square NF \\ \textit{by} \text{ Theorem 11.} \end{array} \right.$
1, 2	3	$\left\{ \begin{array}{l} \square NF = \square TC \\ \textit{by} \text{ Axiom 5.} \end{array} \right.$
3 ww	4	$NF = TC$
Hence		$2\,NF = 2\,TC = TS$
Again.	5	$TS : LR :: TF \times FS : \square LF$, by common Properties.
Consequ.	6	$\frac{1}{2} TS : \frac{1}{2} LR :: TF \times FS : \square LF$
But,		$\frac{1}{2} TS = TC$. And $\frac{1}{2} LR = LF$
Ergo	7	$TC : LF :: \overline{TC + CF} \times \overline{TC - CF} : \square LF$
7	8	$TC \times LF = \square TC - \square CF$
But,	9	$\square fF + \square LF = \square fL$, *by* Theorem 11.
That is,	10	$4 \square CF + \square LF = \square fL$, for $2\,CF = fF$
8×4	11	$4 \square TC - 4 \square CF = 4\,TC \times LF$
$10 + 11$	12	$4 \square TC + \square LF = 4\,TC \times LF + \square fL$
12 —	13	$4 \square TC - 4\,TC \times LF + \square LF = \square fL$



13 ww^2	14	$2TC - LF = fL$
14 + LF	15	$2TC = fL + LF$. But $2TC = TS$
Ergo		$TS = fL + LF$. Q. E. D.

And this *Propoſition* muſt needs hold true to every Point in the *Ellipſis*'s *Periphery*, *viz.* at *B*, &c. As will evidently appear to any one who rightly conſiders, That, as a Thread juſt the Length of the Diameter of any Circle having its two Ends ty'd together, and then mov'd about a Point in the Center (viz. *by making it a double Radius*) will, by drawing another Point in its Extremity, deſcribe the *Periphery* of a Circle; [*vide* Definition *Page* 280] even ſo, if a Thread juſt the Length of the tranſverſe Diameter (*TS*) having its two Ends ſo fix'd upon the two *Foci* (*f* and *F*) that it may be mov'd about them, by drawing a Point in its Extremity (viz. *at its full Stretch*) it will deſcribe the true *Periphery* of an *Ellipſis*.

Now, altho' this eaſy Way of deſcribing, or, as uſually phras'd, drawing an *Ellipſis*, be mechanical, and known even to moſt *Joiners*, *Carpenters*, &c. yet it gives as compleat and clear an Idea of that Figure as any other Way whatſoever; and by deſcribing it thus about its two *Foci*, as a Circle is about its Center, doth plainly ſhew that thoſe two Points are not improperly call'd particular Centers in *Definition* 4, *Sect.* 3, *Chap.* 1. for each of them bears much the ſame Reſpect to the *Ellipſis*'s Periphery, as the Circle's Center doth to its Periphery.

Sect. 4. *To deſcribe or delineate an* Ellipſis *ſeveral Ways.*

There are ſeveral (*other*) Ways of deſcribing an *Ellipſis*, both Geometrically and Numerically, according to peculiar Occaſions, but I ſhall only mention two or three of them, leaving the reſt to the *Learner's Genius*. Now, in order to that Work, it will be convenient to conſider what *Lines* are requiſite to limit or bound its Form, which I take to be chiefly theſe following.

1. If the Tranſverſe and Conjugate are given, the *Ellipſis* is perfectly limited; (vide *Conſectary Page* 363.) for if *TS* and *Nn* be ſet at *Right Angles* in their Middle at *C*, and *TC* or *CS* be ſet *off* from *N*, or *n*, both Ways upon the Tranſverſe to *f* and *F*, (viz. *make* $fN = TC = NF$) then will thoſe Points *f* and *F* be the two *Foci* (by *4th Step of the laſt Proceſs*) and then the *Ellipſis* may be deſcrib'd as above.

2. If

2. If the *Tranſverſe Diameter* and *Latus Rectum* are given, the *Ellipſis* is truly limited, becauſe by them the Conjugate may be found, by Sect. 2.

3. Or if only the *Tranſverſe*, and the Proportion it hath either to the *Conjugate* or *Latus Rectum*, be given, the *Ellipſis* is thereby limited. As for Inſtance; ſuppoſe the given *Ratio* between the Tranſverſe and Conjugate to be, as a : to d :

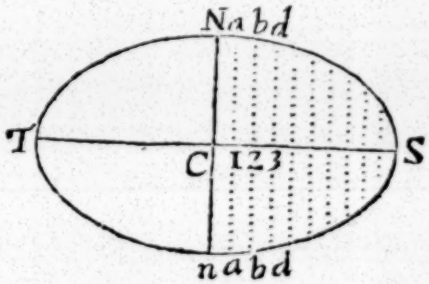
Viz. $a : d :: TS : Nn$, then $\frac{TS \times d}{a} = Nn$, &c.

4. If either the Tranſverſe or Conjugate, and the Diſtance of the *Focus* from the *Conjugate* be given, the *Ellipſis* is limited, becauſe by them the Conjugate or Tranſverſe may be found.

Theſe being premis'd, and the *precedent Work* a little conſider'd, it muſt be eaſy to deſcribe or delineate any *Ellipſis* in *Plano*, either Geometrically or Numerically.

1. *To deſcribe an* Ellipſis *Numerically* by *Points*.

Suppoſe the *Tranſverſe Diameter* $TS = 20$, and the *Conjugate* $Nn = 12$, *(either Inches, or any other equal Parts)* and let them croſs each other at *Right Angles* in their Middles, as in the Point C; then will $TC = CS = 10$, and $NC = Cn = 6$, and it will be $20 : 12 :: 12 : 7,2 =$ the *Latus Rectum*.



Again $20 : 7,2$. Or rather take their *Ratio*.

Thus
$1 : 0,36 :: \sqrt{10+1 \times 10-1} : \square a. \parallel 1.$
$1 : 0,36 :: \sqrt{10+2 \times 10-2} : \square b. \parallel 2.$
$1 : 0,36 :: \sqrt{10+3 \times 10-3} : \square d \parallel 3.$ &c.

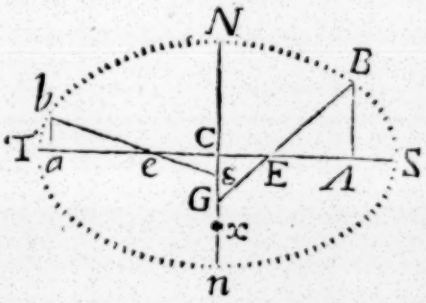
Viz.
$\sqrt{100-1 \times 0,36} = \square a$ 1. Hence $\sqrt{99 \times 0} = 5,97$ &c. $= a$. 1
$\sqrt{100-4 \times 0,36} = \square b$ 2. $\sqrt{96 \times} = 5,88$ &c. $= b$. 2
$\sqrt{100-9 \times 0,36} = \square d$ 3. $\sqrt{91 \times 0,36} = 5,72$ &c. $= d$. 3

If ſo many *Semi ordinates* as may be thought convenient *(the more the better)* be found in this Manner, and every one of them be ſet of at *Right Angles* from its reſpective Point in the *Tranſverſe Diameter* each Way, *viz*, from 1 to a, from 2 to b, from 3 to d, &c. Then if a Curve Line be carefully drawn with an even Hand thro' thoſe extreme Points a, b, d, &c. it will be the *Ellipſis*'s *Periphery* requir'd.

2. *To*

2. *To describe an* Ellipsis *Geometrically by Points.*

Having the Transverse and Conjugate Diameters given, *viz.* TS and Nn, placed at *Right Angles* in their *Middles*, as before: Then from either End of the Conjugate, *viz.* N (or n) set off half the Transverse Diamater to x. That is, make $Nx = TC$ (*continuing the* Conjugate Nn *when it is shorter than* TC) Or, which is all one, make $Cx = TC - NC$. Then take any Point in the Line Cx at Pleasure; suppose it at G, and from that Point at G set off the Distance Cx to the Transverse (as at E) *viz.* make $GE = Cx$, and join the Points GE with a Right Line, produced so far beyond E as to make $EB = NC$. Consequently $GB = TC$.



Then, I say, where-ever the Point G was taken between C and x, the Point B will just touch (*or fall in*) the *Ellipsis's Periphery.*

Demonstration.

Draw the Right Line BA perpendicular to TS, *viz.* let BA be a *Semi-ordinate* rightly apply'd to the transverse Diameter TS; then $\triangle GCE$ and $\triangle BAE$ will be alike.

Consequently	1	$CE : AE :: EG : EB$, by Theorem 13.
1, and	2	$CE + AE : AE :: EG + EB : EB$. See *p.* 192.
But	3	$CE + AE = CA$. $EG \times EB = TC$. And $EB = NC$
Therefore	4	$CA : AE :: TC : NC$
6, in □'s	5	$\square CA : \square AE :: \square TC : \square NC$
5, ∴	6	$\frac{\square CA \times \square NC}{\square TC} = \square AE$
But	7	$\square NC - \square AB = \square AE$
That is,		$\square EB - \square AB = \square AE$
6, 7	8	$\frac{\square CA \times \square NC}{\square TC} = \square NC - \square AB$
8 × □ TS	9	$\square CA \times \square NC = \square NC \times \square TC - \square AB \times \square TC$
9 +	10	$\square NC \times \square TC - \square CA \times \square NC = \square AB \times \square TC$
10, *Analogy*	11	$\square TC : \square NC :: \square TC - \square CA : \square AB$
That is,	12	$TC \times CS : \square NC :: \overline{TC + CA} \times \overline{TC - CA} : \square AB$

which is according to the common *Properties* of the *Ellipsis:* Therefore the Point B is truly found. Q. E. D.

Hence

Hence it follows, that if a convenient Number of ſuch Lines as *G E B* be ſo drawn (as above directed) from the like Number of Points taken between *C* and *x*, &c. their extream Points (as at *B*) will be thoſe Points by which *(with an even Hand)* the *Ellipſis* may be truly deſcrib'd, as before.

But, if this be well underſtood, it will be very eaſy to conceive how to deſcribe an *Ellipſis* very readily, without drawing thoſe Lines, by having a thin, ſtreight, narrow *Ruler* juſt the Length of *T C*, made ſomewhat ſharp at both Ends, upon which, from one of its Ends, ſet off the Length of *N C*. Then, if the Point upon the Ruler which repreſents *E* be gradually or eaſily moved along the Tranſverſe *T S*, and at the ſame Time the Point or End repreſenting *G* be kept ſliding cloſe along the Conjugate *N n*, 'tis evident from the Work above, that the End of the Ruler repreſenting *B* will, by that Motion, aſſign the true *Periphery* of the *Ellipſis* required; for by that Motion the ſtreight Edge of the Ruler doth ſupply an infinite Number of the aforeſaid Lines; as will appear very plain and eaſy in Practice.

Scholium.

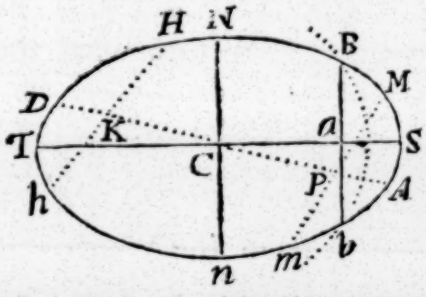
Now from hence was deduced the firſt Invention of that well-contrived *Inſtrument* for drawing an *Ellipſis* by one Motion, commonly called the *Elliptical Compaſſes*, being uſually made of Braſs, and compos'd of three Parts, two of which repreſent *(or rather ſupply)* the tranſverſe and conjugate Diameters ſet together at *Right Angles*; and the third Part is a moveable Ruler, which performs the Office of the laſt-mentioned thin Ruler. But becauſe the making of it is ſo well known to moſt Mathematical Inſtrument-makers, *eſpecially to that accurate and ingenious Artiſt Mr* JOHN ROWLEY, *Mathematical Inſtrument-maker, under St.* Dunſtan's *Church in* Fleet-ſtreet, London; *who, for his great* Skill *in contriving, framing, and graduating all kind of Mathematical Inſtruments, may, I believe, be juſtly called one of the beſt Workmen of his Trade in* Europe; I think it needleſs therefore to give a *particular* Deſcription of that *Inſtrument*.

Alſo from hence came that *ingenious Invention* of making *Engines* for turning all Sorts of elliptical or oval Work, as oval Boxes, Picture-Frames, &c.

Sect.

Sect. 5. *Any Ellipſis being given, to find its* Tranſverſe *and* Conjugate *Diameters.*

Suppoſe the given *Ellipſis* to be *T N S n* (in the annexed Scheme) in which let it be required to find the tranſverſe Diameter *T S* and its Conjugate *N n*. Draw within the *Ellipſis* any two *Right Lines* parallel to each other as *Hh* and *Mm*, and biſect thoſe Lines, *viz.* find the Middle Point of each, as at *K* and *P*; then thro' thoſe Points *K* and *P* draw a Right Line, as *D A*, and it will be a Diameter; for it will divide the Ellipſis into two equal Parts, [*See* Defin. 1, *Page* 363.] conſequently the Middle of *DA* will be the true Middle or common Center of the Ellipſis, as at *C*.



For 'tis the Nature and Property of all Diameters, *howſoever they are drawn in any* Ellipſis *(as 'tis in a* Circle) *to cut or croſs one another in the common Center or Middle of the Figure, as at C.*

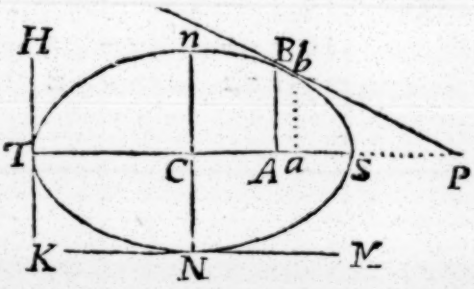
Upon the Point *C* deſcribe an *Arch* of any Circle that will cut the Ellipſis's Periphery in two Points, as at *B* and *b*: then join thoſe Points *B b* with a *Right Line*, and it will be an *Ordinate* thro' whoſe Middle (as at *a*) and the common Center *C*, the tranſverſe Diameter *T S* muſt paſs. For *B S* = *S b*, and *B a* is at Right Angles with *T S*; therefore the Line *B b* is an *Ordinate* rightly apply'd to *T S* the tranſverſe Diameter. And if thro' the Point *C* there be drawn the Right Line *N n* parallel to *B b*, it will become the Conjugate; as was requir'd.

Sect. 6. *To draw a* Tangent, *or Right Line that may touch the Ellipſis's Periphery in any aſſigned Point.*

The Drawing of *Tangents* to or from any *aſſigned Point* in the Ellipſis's Periphery, admits of three Caſes.

Caſe 1. If it be requir'd to draw a *Tangent* that may touch the Ellipſis in either of the extream Points of its tranſverſe Diameter, as at *T* or *S*, it is plain the *Tangent* muſt be drawn parallel to the conjugate Diameter *N n*; as *H K* in the following *Figure* is ſuppos'd to be.

Caſe 2. Or, if the *Tangent* muſt be drawn to touch the *Ellipſis* in either of the extream Points of its *Conjugate Diameter*, as at *N* or *n*, 'tis as evident that it muſt be drawn parallel to the *Tranſverſe Diameter TS*, as *KM*. Conſequently if that *Tangent* and the *Tranſverſe* were both infinitely continu'd, they would never meet.



Caſe 3. But if it be requir'd to draw a *Tangent* that may touch the *Ellipſis* in any other *Point*, as at *B*, &c. Then, if the *Tangent* and the *Tranſverſe Diameter* be both continu'd, they will meet in ſome *Point*, as at *P*; and thoſe two Points (viz. *B* and *P*) do ſo mutually depend upon each other, that one of them muſt be aſſigned in order to find the other, that ſo the *Tangent* may by them be truly drawn. Let $D = TS$, $y = AS$, and $z = AP$. Then, if y be given, z may be found by this *Theorem* $\left\{\frac{Dy - yy}{\frac{1}{2}D - y} = z\right.$. Or, if z be given, y may be found by this *Theorem* $\left\{\frac{D+z}{2} \pm \sqrt{\left\{\frac{DD+zz}{4}\right.} = y\right.$.

Demonſtration.

Draw the *Semi-ordinate ba*, as in the *Figure*; then will $\triangle BAP$ and $\triangle baP$ be alike. Put $x = Aa$ the *Diſtance* between the two *Semi-ordinates* (viz. *between BA* and *ba*) which we ſuppoſe infinitely ſmall.

Then	1	$z : z - x :: BA : ba$, by *Theorem* 13.
But	2	$\overline{D-y} \times y : \overline{D-y+x} \times \overline{y-x} :: \square BA : \square ba$
That is,	3	$Dy - yy : Dy - yy + 2bx - Dx - xx :: \square BA : \square ba$
1 in □'s	4	$zz : zz - 2zx + xx :: \square BA : \square ba$
Suppoſe	5	$x = 0$, that ſo x may be every-where rejected.
3, Then	6	$Dy - yy : Dy - yy + 2y - D :: \square BA : \square ba$
4, And	7	$zz : zz - 2z :: \square BA : \square ba$
6, 7	8	$Dy - yy : Dy - yy + 2y - D :: zz : zz - 2z$
8 ∴	9	$2yzz - Dzz = 2yyz - 2Dyz$
9 ÷ 2z	10	$yx - \frac{1}{2}Dz = yy - Dy$
10 ±	11	$\frac{1}{2}Dz - yz = Dy - yy$

11 ÷	12	$z = \frac{Dy - yy}{\frac{1}{2}D - y}$	which is the 1st *Theorem*, and gives the following *Analogy*.
Analogy	13	$\frac{1}{2}D - y : y :: D - y : z$.	Viz. $CA : SA : TA : AP$
10 — yz	14	$yy - Dy - yz = -\frac{1}{2}Dz$	
14 C □	15	$yy - Dy - yz + \frac{1}{4}DD - \frac{1}{2}Dz + \frac{1}{4}zz = \frac{1}{4}DD + \frac{1}{4}zz$	
15 ww	16	$y - \frac{1}{2}D - \frac{1}{2}z = \sqrt{\frac{1}{4}DD - \frac{1}{4}zz}$	
That is,	17	$y = \frac{1}{2}D + \frac{1}{2}z \pm \sqrt{\frac{1}{4}DD + \frac{1}{4}zz}$	which is the 2d *Theor.* Q. E. D.

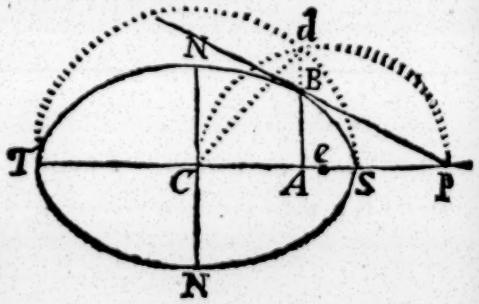
The *Geometrical* Performance of these two *Theorems* is very easy, as by the following *Figure*.

1. Suppose the *Point B* in the *Ellipsis Periphery* were given, and it were requir'd to find the *Point P*, &c.

Make TC *Radius*, and upon the common Center C describe the *Semicircle* TdS, and join the *Points C* and *d* with a *Right Line*; then bisect that *Line* (by *Prob.* 2, *Page* 287) and mark the *Point* where the bisecting *Line* would cross the *Transverse*, as at ‖ *e*. Upon that *Point* ‖ *e*, with the *Radius Ce (or Cd)* describe another *Semicircle*, producing the *Transverse Diameter* to its *Periphery*, and it will assign the *Point P*.

For if $D = TS$, $y = AS$, $z = AP$, as before.

Then	1	$\overline{D - y} \times y = \square\, dA$
And	2	$\overline{\frac{1}{2}D - y} \times z = \square\, dA$
For	3	$TA : dA :: dA : SA$
And	4	$CA : dA :: dA : AP$
But	5	$CA = \frac{1}{2}D - y$, &c.
1, 2	6	$\frac{1}{2}Dz - yz = Dy - yy$ as at the 11th *Step* before



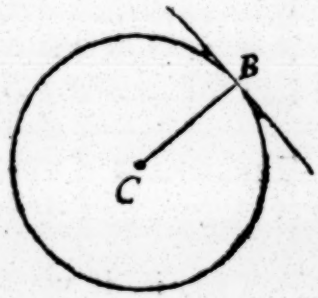
Therefore the *Point P* is truly found. Consequently, if a *Right Line* be drawn through those *Points B* and *P*, it will be the *Tangent* requir'd, according to the first *Theorem*.

2. The *Converse* of this is as easy, to wit, if the *Point P* be given, thence to find the *Point B* in the *Ellipsis Periphery*. Thus, circumscribe half the *Ellipsis* with the *Semicircle* TdS, as before; and bisect the Distance between the *Points C* and *P*, as at *e*, *viz.* Let $Ce = eP$. Then making Ce *Radius*, upon the Point *e*, describe the *Semicircle* CdP; and from the *Point* where the two *Semicircles* intersect or cross each other, as at *d*, draw the *Right Line* dA perpendicular to the *Transverse* TS, and it will assign the

the Point of Contact *B* in the *Ellipſis Periphery* through which the *Tangent* muſt paſs.

But the *Practical Method* of drawing *Tangents* to any aſſign'd Point in the *Ellipſis Periphery* may *(without finding the aforeſaid Point P)* be eaſily deduced from the following *Property* of *Tangents* drawn to a *Circle*, which is this:

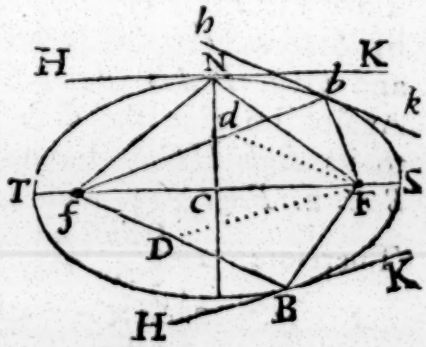
If to any *Radius* of a *Circle*, as *CB*, there be drawn a *Tangent Line (as HK)* to *touch* the *Radius* at the *Point B*; the two *Angles*, which the *Tangent* makes with the *Radius*, will always be two *Right Angles* (16, 17, 18, 19 *Euclid.* 3.) that is, $\prec HBC = \prec CBK = 90°$.



In like Manner the two *Angles*, made between the *Tangent* and the two Lines drawn from the *Foci* of any *Ellipſis* to the *Point* of *Contact*, will always be equal, but not *Right Angles*, ſave only at the two Ends of the *Tranſverſe Diameter*.

Theſe being well conſider'd, and compar'd with what hath been ſaid in *Page* 366, it muſt needs be eaſy to underſtand the following Way of drawing *Tangents* to any *aſſign'd Point* in the *Ellipſis Periphery*; which is thus:

Having by the *tranſverſe* and *conjugate Diameters* found the two *Foci f* and *F*, by *Sect.* 3. from them draw two *Right Lines* to meet each other in the *aſſign'd Point* of *Contact* as *fb* and *Fb* (or *fB* and *FB)* in the *annex'd Figure*. Next ſet off (viz. make) *bd* = *bF* (or *BD* = *BF)* and join the *Points Fd* (or *FD*) with a *Right Line*.



Then, I ſay, if a *Right Line* be drawn through the *Point* of *Contact b* (or *B)* parallel to *dF*, or *DF*, it will be the *Tangent* requir'd. For it is plain, that as the $\prec fNH = \prec FNK$ when the *Tangent* is parallel to the *Tranſverſe Diameter*, even ſo is the $\prec fbh = \prec FBk$, (and $\prec fBH = \prec FBK$) and will be every-where ſo, as the *Point* of *Contact b* (or *B)* and its *Tangent* is carried about the *Ellipſis Periphery* with the *Lines f b F* (or *f B F)*.

CHAP. III.

Concerning the Chief Properties of every Parabola.

NOTE, in every Parabola, the intercepted Diameter, or that Part of its Axis, which is between the Vertex and that Ordinate which limits its Length, as Sa or SA, &c. is call'd an Abſciſſa.

Sect. 1. The Plain or Figure of every Parabola is proportion'd by its Ordinates and Abſciſſæ, as in the following Theorem:

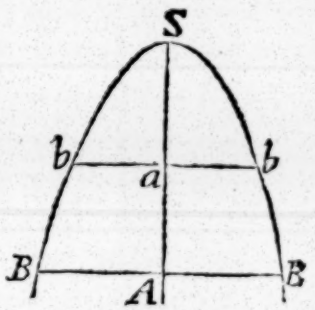
THEOREM { *As any one Abſciſſa : is to the Square of its Semi-ordinate : : ſo is any other Abſciſſa : to the Square of its Semi-ordinate.*

That is, if we ſuppoſe the *annex'd Figure* to be a *Parabola*, wherein Sa, and SA, are *Abſciſſæ*, and bab, BAB, *Ordinates* rightly apply'd, it will

be $Sa : \square ba :: SA : \square BA$ { whereſoever the *Points a*, *A*, are taken,

or $Sa : SA :: \square ba : \square BA$ {

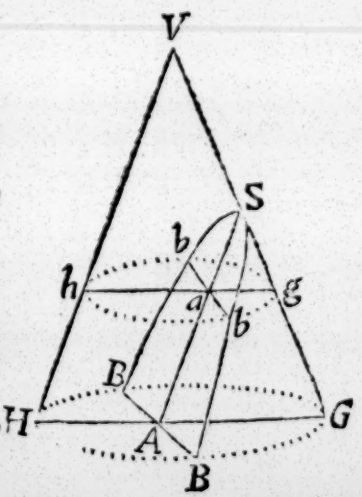
And ſo for any other *Abſciſſæ*, &c.



Demonſtration.

Let the following Figure HVG repreſent a *Right Cone* cut into two Parts by the *Right Line* SA, parallel to its Side VH. Then the *Plain* of that *Section*, viz. $BbSbB$, will be a *Parabola*, by *Sect* 4. *Page* 364. wherein let us ſuppoſe SA to be its *Axis*, and bab, BAB to be *Ordinates* righly apply'd to that *Axis*. Again, imagine the *Cone* to be cut by the *Right Line* hg parallel to its *Baſe* HG. Then will hg be the *Diameter* of a *Circle*, by *Sect.* 2. *Page* 363. and $\triangle Sag$ like to $\triangle SAG$.

Therefore	1	{ $Sa : ag :: SA : AG$ By *Theorem* 13.
1 ∴	2	$Sa \times AG = SA \times ag$
2 × ha	3	{ $Sa \times AG \times ha = SA \times ag \times ha$ By *Axiom* 3.
But	4	{ $HA = ha$, becauſe SA is parallel to VH
And {	5	$\square BA = AG \times HA$ } By Lem. P. 363 $\square ba = ag \times ha$ }
3, 4, 5,	6	{ $Sa \times \square BA = SA \times \square ba$ By *Axiom* 5.
6, *Analogy*	7	{ $Sa : \square ba :: Sa : \square BA$ Vide *Page* 194.



Q. E. D. Theſe

These *Proportions* being prov'd to be the common *Property* of every *Parabola*, all that is farther requir'd about that *Section*, or *Figure*, may from thence easily be deduced.

Sect. 2. *To find the* Latus Rectum *or Right* Parameter *of any Parabola.*

The *Latus Rectum* of a *Parabola* hath the same *Ratio* or *Proportion* to any *Abscissa*, and its *Semi-Ordinate*, as the *Latus Rectum* of any *Ellipsis* hath to its *Transverse* and *Conjugate Diameters*, and may be found by this *Theorem.*

THEOREM { *As any Abscissa : is in Proportion to its Semi-ordinate* : : *so is that Semi-ordinate : to the* Latus Rectum.

Let L = the *Latus Rectum.*

Then	1	$Sa : ba :: ba : L$	where ever the *Points a*, and
And	2	$SA : BA :: BA : L$	*A*, are taken in the *Axis.*
1 ∴	3	$\frac{\square ba}{Sa} = L$: Or $Sa \times L = \square ba$	
2 ∴	4	$\frac{\square BA}{SA} = L$: Or $SA \times L = \square BA$	
3 = 4	5	$\frac{\square BA}{SA} = \frac{\square Sa}{ba}$ Per *Axiom* 5.	
5 ×	6	$Sa \times \square BA = SA \times \square ba$, which gives this	
Analogy	7	$Sa : \square ba :: SA : \square BA$, the same as at the 7th	

Step of the last *Process*; therefore L (thus found) is the true *Latus Rectum*, by which all the *Ordinates* may be regulated and found, according to its *Definition* in *Section* 4, *Page* 364. For by the third Step $Sa \times L = \square ba$, and by the 4th Step $SA \times L = \square BA$. Consequently $\surd \overline{Sa \times L} = ba$ and $\surd \overline{SA \times L} = BA$; and so for any other *Ordinate.*

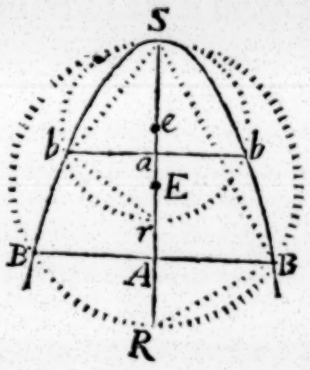
Or if the *Ordinates* are given, to find their *Abscissæ*; then it will be, $L : ba :: ba : Sa$, and $L : BA :: BA : SA$, &c.

Consequently $\frac{\square ba}{L} = Sa$, and $\frac{\square Ba}{L} = SA$, &c.

From the Consideration of these *Proportions*, it will be *easy* to conceive how to find the *Latus Rectum Geometrically*, thus:

Join

Join the *vertical Point S* of the *Axis*, and either *extream Point* of any *Ordinate* as *B* (or *b*) with a *Right Line*, viz. *SB* (or *Sb*) and biſect that Line (by *Problem* 2. *Page* 287.) marking the *Point* where the biſecting *Line* doth interſect or croſs the *Axis*, as at *E* (or *e*) and with the *Radius SE* (or *Se*) upon the *Point E* (or *e*) deſcribe a *Circle*; (as in the annex'd Figure) then will the Diſtance between the *Ordinate* and that *Point* where the *Circle's Periphery* cuts the *Axis*, viz. *AR* (or *ar*) be the true *Latus Rectum* required.



For $SA : BA :: BA : AR$, and $Sa : ba :: ba :$ or, by *Theor.* 13. therefore $AR = L$. And $ar = L$, by the 1ſt and 2d *Steps* above.

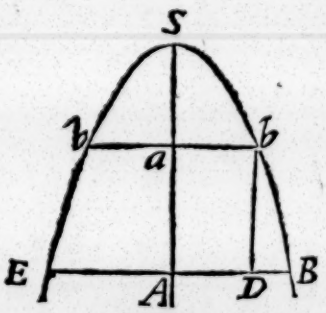
Conſectary.

From theſe *Proportions* of finding the *Latus Rectum*, it will be eaſy to deduce and demonſtrate the following *Theorem:*

THEOREM { *As the* Latus Rectum : *is to the Sum of any two Semi-ordinates* : : *ſo is the Difference of thoſe two Semi-ordinates* : *to the Difference of their Abſciſſæ.*

Suppoſe any *Right Line* drawn within the *Parabola*, as *b D*, parallel to its *Axis SA*; then will that *Line* (viz. *b D*) be a *Diameter* (by *Def.* 5, *Page* 365) which will make $ED = AB + ab$, $DB = AB - ab$, and $bD = SA - Sa$. Then it will be $L : ED :: DB : bD$, according to the *Theorem*.

Demonſtration.



Firſt	1	$SA = \frac{\square BA}{L}$, by *Step* 2. of the laſt *Proceſs*.
And	2	$Sa = \frac{\square ba}{L}$ by *Step* 1. of the laſt *Proceſs*.
1 — 2	3	$SA - Sa = \frac{\square BA - \square ba}{L}$
3 × *L*	4	$\overline{SA - Sa} \times L = \square BA - \square ba$
But	5	$\square BA - \square ba = \overline{BA + ba} \times \overline{BA - ba}$
4,=5,6	6	$\overline{SA - Sa} \times L = \overline{BA + ba} \times \overline{BA - ba}$
6, *Analogy*	7	$L : BA + ba :: BA - ba : SA - Sa$
Or	8	$L : ED :: DB : bD$

Which gives the following *Analogy*.

This

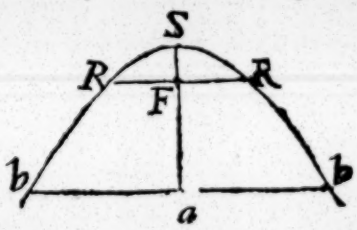
This peculiar Property of the Parabola was first publish'd, *Anno* 1684, by one Mr. *Thomas Baker*, Rector of *Bishop Nympton* in *Devonshire*, in a Treatise intitled, *The Geometrical Key: Or, the Gate of Æquations unlock'd*; wherein he hath shew'd the Geometrical Construction and Solution of all Cubick and Biquadratick Adfected Æquations by one general Method, which he calls a *Central Rule*, deduced from this peculiar Property of the Parabola.

Sect. 3. *To find the* Focus *of any Parabola.*

The *Focus* of every *Parabola* is that *Point* in its *Axis* through which the *Latus Rectum* doth pass. (See *Definition* 3. *Sect.* 4. *Page* 359.) Therefore its Distance from the *Vertex* of the *Parabola* may be easily found, either by the *Latus Rectum* itself, or by any other *Ordinate*, and its *Abscissæ*.

Thus, suppose the *Point* at *F* to be the *Focus*, *S* the *Vertex*, the *Ordinate* $RFR = L$ the *Latus Rectum*, and bab any other *Ordinate*. Then will $SF = \frac{1}{4}L$.

Or $SF = \frac{\square ba}{4Sa}$



Demonstration.

First	1	$SF \times L = \square FR$. by *Sect.* 2. *Page* 375.
And	2	$FR = \frac{1}{2}L$; for the Ordinate $RFR = L$ as above.
$2\ \odot^2$	3	$\square FR = \frac{1}{4}\square L = \frac{1}{2}L \times \frac{1}{2}L$
$1, = 3$	4	$SF \times L = \frac{1}{4}\square L$
$4 \div L$	5	$SF = \frac{1}{4}L$, as by *Definition* 4. *Sect.* 4. *Page* 359.
Again	6	$\frac{\square ba}{Sa} = L$, by the third *Step* in Page 375.
Conseq.	7	$\frac{\square ba}{4Sa} = \frac{1}{4}L$, &c. as above. Q. E. D.

Sect. 4. *To* describe, *or draw a Parabola several Ways.*

Note There are two or three Ways of drawing a Parabola instrumentally at one Motion; but because those Instruments or Machines are not only too perplex'd for a Learner to manage, but also a little subject to Error, I have therefore chosen to shew how that Figure may be (the best) drawn by a convenient Number of Points, *viz.* Ordinates found, either Numerically or Geometrically, according to the DATA; which, if the Work of the three last Sections be well consider'd, must needs be very easy.

1. If

1. If any *Ordinate* and its *Abſciſſa* are given, there may by them be found as many *Ordinates* as you pleaſe to aſſign or take *Points* in the *Parabola*'s *Axis*; (by *Sect.* 4. *Page* 380) and the *Curve* of the *Parabola* may be drawn by the extream Points of thoſe *Ordinates*, as the *Ellipſis* was *Page* 373.

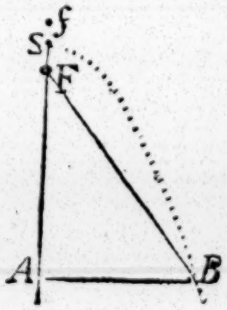
2. If the *Latus Rectum*, and either any *Ordinate*, or its *Abſciſſæ*, are given, then any aſſign'd Number of *Ordinates* may by them be found (by *Sect.* 2. *Page* 381) either *Numerically* or *Geometrically*, &c.

3. If only the Diſtance of the *Focus* from the *Vertex* of the *Parabola* be given, any aſſign'd Number of *Ordinates* may be found by it. For $SF = \frac{1}{4} L$ the *Latus Rectum*, and $\frac{1}{2} L = FR$ as in the laſt *Section*; and it will be, as SF: is to $\square FR$:: ſo is any other *Abſciſſa*, (*viz.* Sa or SA, &c.): to the *Square* of its *Semi-ordinate* (viz. $\square ba$, or $\square BA$) according to the common Property of the *Parabola*.

Altho' any of theſe Ways of finding the *Ordinates* are eaſy enough, yet that Way which may be deduced from the following *Propoſition* will be found much more eaſy and ready in *Practice*.

PROPOSITION. *The Sum of any Abſciſſa and focal Diſtance from the Vertex, will be equal to the Diſtance from the Focus to the extream Point of the Ordinate, which cuts off that Abſciſſa.*

For Inſtance, ſuppoſe S to be the *Vertex* of any *Parabola*, the Point F to be its *Focus*, and AB any *Semi-ordinate* rightly apply'd to its *Axis* SA: Then, I ſay, where-ever the *Point* A is taken in the *Axis*, it will be $SA + SF = FB$. Conſequently, if $Sf = SF$, it will be $fA = FB$.



Demonſtration.

Firſt	1	$SF = \frac{1}{4} L$ by the 7th *Step*, *Sect.* 3.
Ergo	2	$fA = FA + \frac{1}{2} L$ by *Conſtruction* above.
2 $\odot^2$	3	$\square fA = \square FA + FA \times L + \frac{1}{4} LL$
Again	4	$SA = FA + \frac{1}{4} L$ by the *Suppoſition* and *Figure*.
4 $\times L$	5	$SA \times L = FA \times L + \frac{1}{4} LL$, but $SA \times L = \square AB$
Ergo	6	$\square AB = FA \times L + \frac{1}{4} LL$
3 — 6	7	$\square fA - \square AB = \square FA$, conſe. $\square fA = \square FA + \square AB$
But	8	$\square FA + \square AB = \square FB$, by *Theorem* 11.
Ergo	9	$\square fa = \square FB$
9 $\sqrt{\ }^2$	10	$fA = FB$ Q. E. D.

This

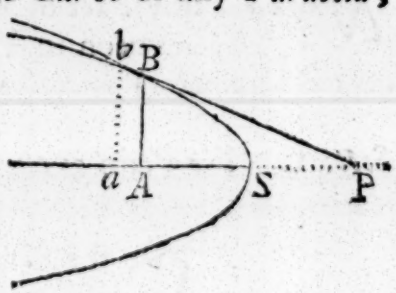
This *Propoſition* being well underſtood, 'twill be very eaſily apply'd to *Practice*, ſuppoſing the *Focal Diſtance* given, or any other *Data* by which it may be found. Thus draw any *Right Line* to repreſent the *Parabola's Axis*, and from its *vertical Point*, as at *S*, ſet off the *Focal Diſtance* both upwards and downwards, *viz.* make $Sf = SF$, the Diſtance of the given *Focus* from the *Vertex*; as in the *Scheme:* Then by the *Propoſition* 'tis evident, that, if never ſo many Lines be drawn *Ordinately* at *Right Angles* to the *Axis*, the true Diſtance between the Point *f* out of the *Parabola*, and any of thoſe Lines (or *Ordinates*) being meaſur'd or ſet off from the *Focus F* to the ſame Line or Ordinate, 'twill aſſign the true Point in that Line through which the *Curve* muſt paſs; that is, it will ſhew the true Limits or Length of that *Ordinate*; as at *B* in the laſt Scheme.

Proceeding on in the very ſame Manner from *Ordinate* to *Ordinate*, you may with great Expedition and Exactneſs find as many *Ordinates (or rather their Points only, like B)* as may be thought convenient, which, being all joined together with an even Hand, will form the *Parabola* requir'd.

N. B. *The more* Ordinates *(or their* Points*) there are found, and the nearer they are to one another, the eaſier and exacter may the* Curve *of the* Parabola *be drawn. The ſame is to be obſerv'd when any other* Curve *is required to be drawn by* Points.

Sect. 5. *To draw a* Tangent *to any given* Point *in the* Curve *of a* Parabola.

Tangents are very eaſily drawn to the *Curve* of any *Parabola*; For, ſuppoſing *S* to be its *Vertex*, *B* the Point of *Contact (viz.* the Point where the *Tangent* muſt touch the *Curve)* and *P* the Point where the *Tangent* will interſect *(or meet with)* the *Parabola's Axis* produced: Then if from the Point of *Contact B* there be drawn the *Semi-ordinate B A* at Right Angles to the Axis *S A*, whereſoever the Point *A* falls in the Axis, 'twill be $SP = SA$.



Demonſtration.

Draw the *Semi-ordinate b a (as in the Figure)* then will the *BAP* and $\triangle$ *b a P* be alike. Let $y = AS$ the *Abſciſſa*, and $z = SP$;

SP; put $x = Aa$ the Diſtance between the two *Semi-ordinates*, which we ſuppoſe to be infinitely near each other, as in the *Ellipſis*, Page 377.

	Then	1	$y+z : BA :: y+z+x : ba$, per *Theorem* 13.
1,	Or	2	$y+z : y+z+x :: BA : ba$. See *Page* 192.
	Again	3	$y : \square BA :: y+x : \square ba$, per *Theorem* Page 380.
3,	Or	4	$y : y+x :: \square BA : \square ba$
2 in □'s		5	$yy+2yz+zz : yy+2yz+2yz+2yx+zz+2zx+xx :: \square BA : \square ba$
4,	5	6	$y : y+x :: yy+2yz+zz : yy+2yz+2yx+zz+2zx+xx$
6	∴	7	$yy+2yz+yx+zz+2zx+\frac{zzx}{y} = yy+2yz+2yx+zz+2zx+xx.$
That is,		8	$\frac{zzx}{y} = yx+xx$, conſequently $\frac{zz}{y} = y+x$
Suppoſe		9	$x = 0$ and rejected, as in the *Ellipſis*, Page 377.
Then		10	$\frac{zz}{y} = y$, conſequently $zz = yy$
10	2	11	$z = y$, that is, $SP = SA$

Q. E. D:

CHAP. IV.

Concerning the chief Properties of the Hyperbola.

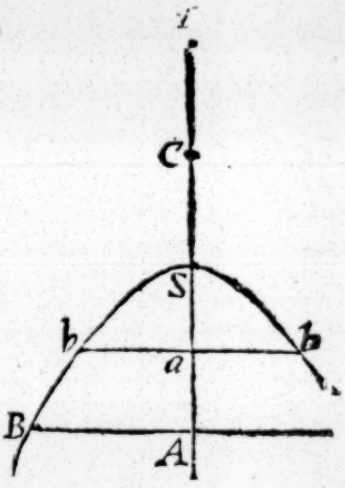
NOTE, any Part of the *Axis* of an *Hyperbola*, which is intercepted between its Vertex and any Ordinate (*viz.* any intercepted Diameter) is call'd an *Abſciſſa*; as in the *Parabola*.

Sect. 1. *The* Plain *of every* Hyperbola *is proportion'd by this general* Theorem.

THEOREM. As the Sum of the Tranſverſe and any *Abſciſſa* multiply'd into that *Abſciſſa*: is to the Square of its Semi-ordinate :: ſo is the Sum of the Tranſverſe and any other *Abſciſſa* multiply'd into that *Abſciſſa*: to the Square of its Semi-ordinate.

That

That is, if TS be the Tranſverſe Diameter,
And $\begin{cases} Sa, SA \text{ Abſciſſæ.} \\ ba, BA \text{ Semi-ordinates.} \end{cases}$
Then is $\begin{cases} Ta = TS + Sa \\ TA = TS + SA \end{cases}$

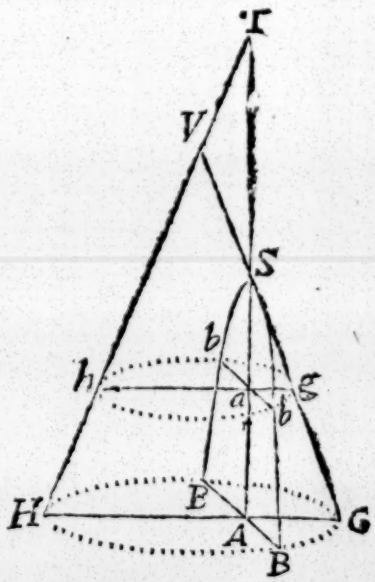


And it will be
$Ta \times Sa : \square ba :: TA \times SA : \square BA.$

That is,
$\overline{TS+Sa} \times Sa : \square ba :: \overline{TS+SA} \times SA : \square BA$
&c.

Demonſtration.

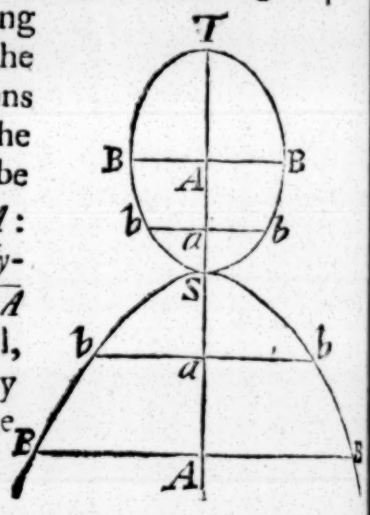
Let the following Figure HVG repreſent a *Right Cone* cut into two Parts by the Right Line SA; then will the Plain of that Section be an *Hyperbola* (by *Sect.* 5, *Chap.* 1.) in which let SA be its Axis, or intercepted Diameter, bab and BAB *Ordinates* rightly apply'd (as before in the *Parabola*) and TS its Tranſverſe Diameter. Again, if the Cone is ſuppos'd to be cut by hg, parallel to its Baſe HG, it will alſo be the Diameter of a Circle, *&c.* as in the *Ellipſis* and *Parabola*. Then will the $\triangle Sga$ and $\triangle SGA$ be alike; alſo the $\triangle Tab$ and $\triangle TAH$ will be alike; therefore it



will	be	1	$Sa : ag :: SA : AG$
	And	2	$Ta : ab :: TA : AH$
1	∴	3	$Sa \times AG = SA \times ag$
2	∴	4	$Ta \times AH = TA \times ab$
3	× 4	5	$\begin{cases} Sa \times Ta \times AG \times AH = \\ SA \times TA \times ag \times ab \end{cases}$
	But	6	$ag \times ab = \square ab$
	And	7	$\begin{cases} AG \times AH = \square AB \\ \text{per } \textit{Lemma} \text{ Page } 363. \end{cases}$
5,	6, 7	8	$\begin{cases} Sa \times Ta \times \square AB = \\ SA \times TA \times \square ab \\ \text{which give the following} \end{cases}$
8,	*Anal.*	9	$Sa \times Ta : \square ab :: SA \times TA : \square AB$, &c.

Q. E. D.

These *Proportions* are the common Property of every *Hyperbola*, and do only differ from those of the *Ellipsis* in the *Signs* + and —; as plainly appears in the following *Proportions*. That is, if we suppose TS the Transverse Diameter common to both Sections (*viz.* both the *Ellipsis* and *Hyperbola*) as in the annexed Scheme: then in the *Ellipsis* it will be $\overline{TS - Sa} \times Sa : \square ab :: \overline{TS - SA} \times SA : \square AB$ as by *Sect.* 1, *Chap.* 2. and in the *Hyperbola* it is $TS + Sa \times Sa : \square ab :: \overline{TS + SA} \times SA : \square AB$, as above. Therefore all, that is farther requir'd in the *Hyperbola*, may (in a manner) be found as in the *Ellipsis*, due Regard being had to changing of the *Sine*.



Sect. 2. *To find the* Latus Rectum, *or Right* Parameter, *of any Hyperbola.*

From the last Proportion take either of the Antecedents and its Consequent, *viz.* either $Ta \times Sa : \square ab$. Or $TA \times SA : \square AB$, to them bring in the Tranverse TS for a third Term, and by those three find a fourth Proportional (as in the Ellipsis) and that will be the *Latus Rectum*.

Thus	1	$Ta \times Sa : \square ab :: TS : \frac{\square ab \times TS}{Ta \times Sa}$ = the *Latus Rectum*, which call L (*as in the* Parabola.)
Then	2	$TS : L :: Ta \times Sa : \square ab$
But	3	$Ta \times Sa : \square ab :: TA \times SA : \square AB$, therefore
2, 3	4	$TS : L :: TA \times SA : \square AB$, &c.

Consequently L is the true *Latus Rectum*, or right Parameter, by which all the *Ordinates* may be found, according to its Definition in *Chap.* 1. And because $TS + Sa = Ta$, let it be $TS + Sa$ instead of Ta, then it will be $\frac{\square ab \times TS}{TS \times Sa + \square Sa} = L$ and in the *Ellipsis* it would be $\frac{\square ab \times TS}{TS \times Sa - \square Sa} = LR = L$

Sect.

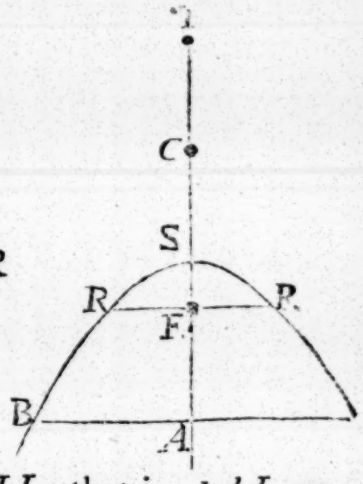
Sect. 3. *To find the* Focus *of any Hyperbola.*

The *Focus* being that Point in the *Hyperbola*'s Axis through which the *Latus Rectum* muſt paſs *(as in the* Ellipſis *and* Parabola) it may be found by this *Theorem.*

THEOREM. *To the Rectangle made of half the Tranſverſe into half the* Latus Rectum, *add the Square of half the Tranſverſe; the Square Root of that Sum will be the Diſtance of the* Focus *from the Center of the* Hyperbola.

Demonſtration.

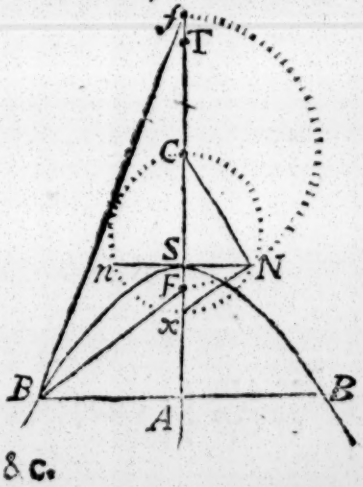
Suppoſe the Point at F, in the annex'd Scheme, to be the *Focus* ſought; then will $FR = \frac{1}{2}L$. Let $TC = CS$ be half the Tranſverſe; then is the Point C call'd the *Center* of the *Hyperbola* (for a Reaſon that ſhall be hereafter ſhew'd.) Again; let $CS = d$, and $SF = a$.



Then	1	$2d : L :: \overline{2d + a} \times a : \frac{1}{4}LL$
That is,	2	$TS : L :: \overline{TS + SF} \times FS : \square FR$
1	3	$\frac{1}{2}dL = 2da + aa$
3 + dd	4	$dd + \frac{1}{2}dL = dd + 2da + aa$
4 ω^2	5	$\surd\overline{dd + \frac{1}{2}dL} = d + a = FC$
Or 5, $-d$	6	$\surd\overline{dd + \frac{1}{2}dL} - d = a = SF$

In the *Ellipſis* 'tis, $2d : L :: \overline{2d - a} \times a : \frac{1}{4}LL$, that is, $\frac{1}{2}dL = 2da - aa$, &c.

The Geometrical Effection of the laſt Theorem is very eaſily perform'd, thus: make $Sx = \frac{1}{2}L$, *viz.* half the *Latus Rectum*; and let $CS = d$, as above. Upon Cx (as a *Diameter*) deſcribe a Circle, and at S the *Vertex* of the *Hyperbola* draw the Right Line nSN at Right Angles to Cx; then join the Points CN with a Right Line, and 'twill be $CN = d + a = FC$.



For	1	$CS : SN :: SN : Sx$, per *Fig.*
That is,	2	$d : SN :: SN : \frac{1}{2}L$.
2	3	$\frac{1}{2}dL = \square SN$
But	4	$dd + \square SN = \square CN$
3, 4	5	$dd + \frac{1}{2}dL = \square CN$
5 ω^2	6	$\surd\overline{dd + \frac{1}{2}dL} = CN = d + a$, &c.

Now

Now here is not only found the Diſtance of the *Hyperbola's Focus*, either from its Center *C*, or Vertex *S*, but here is alſo found that Right Line uſually call'd its Conjugate Diameter, *viz.* the Line nSN, which bears the ſame Proportion to the Tranſverſe and *Latus Rectum* of the *Hyperbola*, as the Conjugate Diameter of the *Ellipſis* doth to its Tranſverſe and *Latus Rectum*. For in the *Ellipſis* $TS : Nn :: Nn : LR$, per *Sect.* 2, *Page* 363. Conſequently $\frac{1}{2}TS : \frac{1}{2}Nn :: \frac{1}{2}Nn : \frac{1}{2}LR$. But $\frac{1}{2}TS = d$, $\frac{1}{2}Nn = SN$, and $\frac{1}{2}LR = {}^{1}L$. Therefore $d : SN :: SN : \frac{1}{2}L$. As at the 2d Step above.

What Uſe the aforeſaid Line nSN is of, in Relation to the *Hyperbola*, will appear farther on.

Sect. 4. *To deſcribe an* Hyperbola *in Plano.*

In order to the eaſy Deſcribing of an *Hyperbola* in *Plano*, it will be convenient to premiſe the following *Propoſition*, which differs from that of the *Ellipſis* in *Sect.* 3, *Chap.* 2, only in the *Signs*.

PROPOSITION. *If from the* Foci *of an* Hyperbola *there be drawn two Right Lines, ſo as to meet each other in any Point of the* Hyperbola's *Curve, the Difference of thoſe Lines* (in the *Ellipſis* 'tis their Sum) *will be equal to the Tranſverſe Diameter.*

That is, if *F* be the *Focus*, and it be made $Cf = CF$ (as in the laſt Scheme) then the Point *f* is ſaid to be a *Focus* out of the Section (or rather of the oppoſite Section) and it will be $fB - FB = TS$.

Demonſtration.

Suppoſe fC, or $Cf = z$, and $SA = x$, let CS, or $TC = d$, as before; then will $fA = d + x + z$, and $FA = d + x - z$. Again, let $FB = h$, and $fB = b$, then $2d = h - b$, by the *Propoſition*.

From theſe *ſubſtituted Letters* it follows,

That	1	$dd + 2dx + 2dz + xx + 2zx + zz = \square\, fA$
And	2	$dd + 2dx - 2dz + xx - 2zx + zz = \square\, FA$
But		$\square\, fA + \square\, AB = \square\, fB$, and $\square\, fA + \square\, AB = \square\, FB$
Per 4th of *laſt*	3	$dd \times \frac{1}{2}dL = da + 2da + aa = \square\, FC = zz$.

3 — *dd*

3 — dd	4	$zz - dd = \frac{1}{2} dL$	
4 ÷ $\frac{1}{2}d$	5	$\frac{zz - dd}{\frac{1}{2}d} = L$	
Again	6	$2d : L :: \overline{2d + x} \times x : \square AB$, by common Properties.	
5, 6	7	$2d : \frac{zz - dd}{\frac{1}{2}d} :: 2dx + xx : \square AB$	
7 ∴	8	$\frac{2dzzx + zzxx - 2dddx - ddxx}{dd} = \square AB$	
1 + 8	9	$\left\{ \frac{dd + 2dx + 2dz + xx + 2zx + zz + 2dzzx + zzxx - 2d^3x - ddxx}{dd} = \square fA + \square AB = hh \right.$	
2 + 8	10	$\left\{ \frac{dd + 2dx - 2dz + xx - 2zx + z + 2dzzx + zzxx - 2d^3x - ddxx}{dd} = \square FA + \square AB = bb \right.$	
9 + d	11	$d^4 + 2d^3z + 2ddzx + ddzz + 2dzzx + zzxx = ddhh$	
10 × dd	12	$d^4 - 2d^3z - 2ddzx + ddzz + 2dzzx + zzxx = ddbb$	
11 ww^2	13	$dd + dz + zx = db$	*Although the Æquation at the* 16*th* Step *be in itſelf impoſſible, becauſe* z *is greater than* d *(by the* 4*th* Step*) yet from thence it will be eaſy to conclude, that the Difference between* d *and* $z + \frac{zx}{d}$ *will give the true Value of* b, *as in the* 17*th* Step.
12 ww^2	14	$dd - dz - zx = db$	
13 ÷ d	15	$d + z + \frac{zx}{d} = b$	
14 ÷ d	16	$d - z - \frac{zx}{d} = b$	
16, or	17	$z + \frac{zx}{d} - d = b$	
15—17	18	$2d = b - b$	

But becauſe I would leave no Room for the Learner to doubt about changing the *Æquation*, $d - z - \frac{zx}{d} = b$ into that of $z + \frac{zx}{d} - d = b$, it may be convenient to illuſtrate the whole Proceſs in Numbers, whereby (I preſume) 'twill plainly appear that $b - b = TS$.

In order to that, let the Tranſverſe $TS = 2d = 12$, then $d = 6$ ſuppoſe the Abſciſſa $SA = x = 4$, and the Semi-ordinate $AB = 3$

Firſt	1	$\overline{TS + SA} \times SA : \square AB :: TS : L$, *per Sect.* 2.
1, *viz.*	2	$12 + 4 \times 4 = 64 : 9 :: 12 : 1,6875 = L$
Again	3	$\sqrt{dd + \frac{1}{2}dL} = d + a = CF$, *per* Sect. 3.
3, *viz.*	4	$\sqrt{36 + 5,0625} = 6,408 = CF = z$
Then	5	$d + x + z = 6 + 4 + 6,408 - 16,408 = fA$
And	6	$d + x - z = 6 + 4 - 6,408 = 3,592 = FA$

5 $\odot^2$	7	$269,2224 = \square fA$
6 $\odot^2$	8	$12,9024 = \square FA$
But	9	$9 = \square AB$, for $AB = 3$ by Suppoſition.
7 + 9	10	$278,2224 = \square fA + \square AB = \square fB$
8 + 9	11	$21,9024 = \square FA + \square AB = \square FB$
10 ww^2	12	$16,68 = fB$
11 ww^2	13	$4,68 = FB$
12—13	14	$12,00 = fB = FB - TS$. Which was to be prov'd.

If this *Propoſition* be truly underſtood, it muſt needs be eaſy to conceive how to deſcribe the Curve of any *Hyperbola* very readily by *Points* when the *Tranſverſe Diameter* and the *Focus* are given (or any other *Data* by which they may be found, as in the precedent Rules) thus:

Draw any ſtraight Line at Pleaſure, and on it ſet off the Length of the given *Tranſverſe TS*, and from its extreme Points or Limits, *viz. TS*, ſet off $Tf = SF$, the Diſtance of the given *Focus* (viz. *the Point f without, and F within the Section, as before)*: that done, upon the *Point f (as a Center)* with any aſſum'd *Radius* greater than *TS*, deſcribe an Arch of a Circle; then from that *Radius* take the Tranſverſe *TS*, making their Difference a ſecond *Radius*, with which, upon the Point *F*, within the Section, deſcribe another Arch to cut or croſs the firſt Arch, as at *B*; then will that Point *B* be in the Curve of the *Hyperbola*, by the laſt Propoſition. And therefore 'tis plain, that, proceeding on in this Manner, you may find as many Points (like *B)* as may be thought convenient (the more there are, and nearer they are together, the better) which being all join'd together with an even Hand (as in the Parabola) will form the *Hyperbola* requir'd.

There are ſeveral other Ways of delineating an *Hyperbola* in *Plano:* One Way is, by finding a competent Number of *Ordinates*, as by Section 1, *&c.* but I think none ſo eaſy and expeditious as this mechanical Way: I ſhall therefore, for Brevity's Sake, paſs over the reſt, and leave them to the Learner's Practice, as being eaſily deduced from what hath been already ſaid.

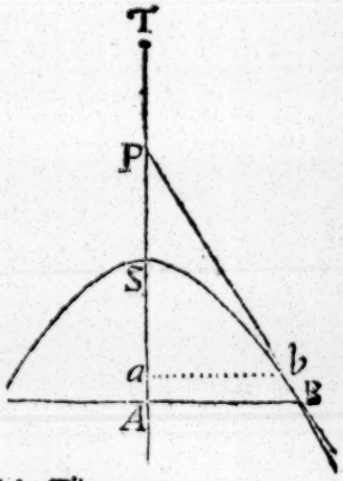
Sect.

Sect. 5. *To draw a* Tangent *to any given Point in the Curve of an* Hyperbola.

The drawing of a *Tangent*, that will touch any given Point in the Curve of an *Hyperbola*, may be eaſily perform'd by Help of a *Theorem*; as in the *Ellipſis*, Sect. 6, Chap. 2.

Let $\begin{cases} D = TS \text{ the } \textit{Tranſverſe Diameter} \\ L = \text{the } \textit{Latus Rectum} \\ y = SA \text{ the } \textit{Abſciſſa}. \end{cases}$

And $z = AP$ { the Diſtance between the *Ordinate* and that Point in the *Tranſverſe* cut by the *Tangent*.



Then, if y be given, z may be found by this *Theorem*. $\left\{ \frac{Dy+yy}{\frac{1}{2}D+y} = z \right.$ [which differs from that in the *Ellipſis* only in Signs. *Vide Page* 371.]

Or, if z be given, then y may be found by this *Theorem*.

THEOREM. $\sqrt{\frac{DD+zz}{4}} : +\frac{1}{2}z - \frac{1}{2}D = y.$

Demonſtration.

Draw the *Semi-ordinate* ba, as in the Figure, and put $x = Aa$ { an infinite ſmall Space between the two *Semi-ordinates*; as before in the *Ellipſis*, &c.

Then	1	$D : L :: Dy + yy : \square AB$
That is,	2	$TS : L :: \overline{TS + SA} \times SA : \square AB$
1 ∴	3	$\frac{DyL + yyL}{D} = \square AB$
Again	4	$D : L :: Dy + yy - 2yx - Dx + xx : \square ab$
That is,	5	$TS : L :: \overline{TS + Sa} \times sa : \square ab$
4 ∴	6	$\frac{DyL + yyL - 2yxL - DxL + xxL}{D} = \square ab$
Per Figure	7	$z : AB :: z - x : ab$, viz. $PA : AB :: Pa : ab$
7 in □'s	8	$zz : \square AB :: zz - 2zx + xx : \square ab$
Suppoſe	9	$x = 0$ and every-where rejected (as in the *Ellipſis*)
Then 3, 9	10	$zz : \frac{DyL + yyL}{D} :: zz - 2z : \square ab$
10 ∴	11	$\frac{DyLzz + yyLzz - 2DyLz - 2yyLz}{Dzz} = \square ab$

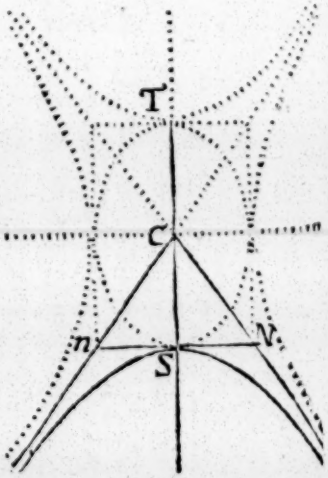
6, 11	12	$\left\{\begin{array}{l}\frac{DyL+yyL-2yL-DL}{D} = \\ \frac{DyLzz+yyLzz-2DyLz-2yyLz}{Dzz}\end{array}\right.$
12 reduced	13	$Dz+zy=Dy+yy$
13 *Analogy*	14	$D+y:y::D+y:z$, viz. $CA:SA::TA:AP$
$13\div\frac{1}{2}D+y$	15	$z=\frac{Dy+yy}{\frac{1}{2}D+y}$ which is the firſt *Theorem.*
13 — zy	16	$yy+Dy-zy=\frac{1}{2}Dz$
16 □ C	17	$yy+Dy-zy+\frac{DD-2Dz+zz}{4}=\frac{DD+zz}{4}$
17 ww	18	$y+\frac{1}{2}D-\frac{1}{2}z=\sqrt{\frac{DD+zz}{4}}$
18 $\pm$	19	$y=\sqrt{\frac{D+zz}{4}}:+\frac{1}{2}z-\frac{1}{2}D$ which is the ſecond *Theorem.*

Q. E. D.

The Geometrical Effection of the firſt of theſe *Theorems* is very eaſy; for, by the 14th Step, 'tis evident that there are three Lines given to find a fourth proportional Line. By *Problem* 3, *Page* 308.

Scholium.

From the Compariſons, which have been all-along made in this Chapter, between the *Hyperbola* and the *Ellipſis*, 'twill be eaſy (even for a Learner) to perceive the Coherence that is in (or between) thoſe two Figures; but, for the better underſtanding of what is meant by the *Center* and *Aſymptotes* of an *Hyperbola*, conſider the annex'd *Scheme*, wherein it is evident (even by Inſpection) that the oppoſite *Hyperbola*'s will always be alike, becauſe they will always have the ſame Tranſverſe Diameter common to both, *&c.* (ſee *Sect.* 1, of this *Chap.*) Alſo, that the middle Point, or common Center of the *Ellipſis*, is the common Center to all the four *conjugal Hyperbola*'s.



And the two Diagonals of the Right-angled *Parallelogram*, which circumſcribes the *Ellipſis* (or is inſcrib'd to the four *Hyperbola*'s) being continued, will be ſuch *Aſymptotes* to thoſe *Hyperbola*'s as are defined *Chap.* 1, *Sect.* 5, *Defin.* 4.

Sect.

Sect. 6. *To draw the* Asymptotes *of any* Hyperbola, *&c.*

Having found the *Latus Rectum* (by Sect. 2.) and the Conjugate Diameter in $n\,S\,N$ in its true *Position*, by Sect. 3. Then through the Center C of the *Hyperbola*, and the extream Points $n\,N$ of its Conjugate Diameter, draw two *Right Lines*, as $C\,N$ and $C\,n$, infinitely continued (as in the following Figure) and they will be the *Asymptotes* required. That is, they are two such *Right Lines* as, being infinitely extended, will continually incline to the Sides of the *Hyperbola*, but never touch them.

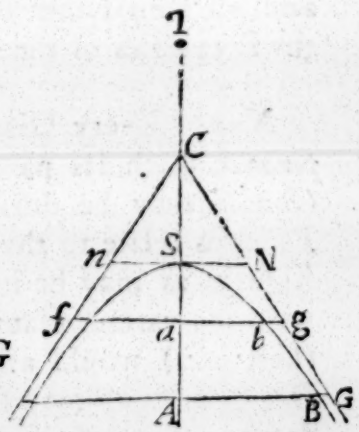
Demonstration.

Suppose the *Semi-ordinates* $a\,b$ and $A\,B$ to be rightly apply'd to the *Axis* $T\,A$; and produced both Ways to the *Asymptotes*, as at $f\,g$ and $F\,G$; then will the $\triangle C\,S\,N$, $\triangle C\,a\,g$, and $\triangle C\,A\,G$ be alike.

Let $d = CS = TC$. And $L =$ the *Latus Rectum*; as before,

Put $\left\{\begin{matrix} e = Sa \\ y = SA \end{matrix}\right\}$ the *Abscissæ*. Then $\left\{\begin{matrix} d+e = Ca. \\ d+y = CA. \end{matrix}\right.$

Then	1	$d : SN :: d+e : ag$. viz. $CS : SN :: Ca : ag$
1 in □'s	2	$dd : \square\,SN :: dd+2de+ee : \square\,ag$
But	3	$d = \square\,SN$. per Sect. 3.
2, 3 ∴	4	$\frac{ddL+2deL+eeL}{2d} = \square\,ag$
Again	5	$2d : L :: 2de+ee : \square\,ab$, per Sect. 2.
5 ∴	6	$\frac{2deL+eeL}{2d} = \square ab$
4 — 6	7	$\frac{dL}{2} = \square\,ag - \square\,ab$
But	8	$ag+ab = bf$ per *Fig.*
	9	$ag-ab = bg$ per *Fig.*
8 × 9	10	$\square ag - \square\,ab = bf \times bg$
7, 10	11	$bf \times bg = dL$
Again	12	$dd : \square SN :: dd+2dy+yy : \square AG$
That is,		$\square CS : \square SN :: \square CA : \square AG$
3, 12 ∴	13	$\frac{ddL+2dyL+yyL}{2d} = \square AG$
But	14	$2d : L :: 2dy+yy : \square AB$, per Sect 2.
14 ∴	15	$\frac{2dyL+yyL}{2d} = \square AB$



13 — 15	16	$\frac{dL}{2} = \square AG - \square AB$
Alſo	17	$AG + AB = BF$
	18	$AG - AB = BG$ } per *Fig.*
17 × 18	19	$\square AG - \square AB = BF \times BG$
16 19	20	$BF \times BG = \frac{1}{2} dL$
11, & 20 ∴	21	$bg = \frac{\frac{1}{2} dL}{bf}$. And $BG = \frac{\frac{1}{2} dL}{BF}$

From the laſt Step 'tis evident, that the *Aſymptotes* are nearer the *Hyperbola* at G than at g, and conſequently will continually approach to its Curve: For BF) $\frac{1}{2} dL$ ($= BG$ is leſs than bf); $\frac{1}{2} dL$ ($= bg$, becauſe the *Diviſor* BF is greater than the *Diviſor* bf; and it muſt needs be ſo where-ever the *Ordinates* are produc'd to the *Aſymptotes*, from the Nature of the *Triangles*.

Again; From the 7th and 16th Steps 'tis evident, that the *Aſymptotes* can never really meet and be co-incident with the Curve of the *Hyperbola*, although both were infinitely extended, becauſe $\frac{1}{2} dL$ will always be the Difference between the Square of any *Semi-ordinate* and the Square of that *Semi-ordinate*, when 'tis produc'd to the *Aſymptote*.

Conſectary.

From hence it follows, that every *Right Line* which paſſes thro' the *Center* and falls within the *Aſymptotes*, will cut the *Hyperbola*; and all ſuch Lines are call'd *Diameters* (as in the *Ellipſis*) becauſe the Properties of the *Hyperbola* and *Ellipſis* are the ſame.

Note. Every Diameter, both in the *Ellipſis*, *Parabola*, and *Hyperbola*, hath its particular *Latus Rectum* and *Ordinates*; which (ſhould they be diſtinctly handled, and the Effection of all ſuch Lines as relate to them, as alſo the Nature and Properties of ſuch Figures as may be inſcribed and circumſcribed to all the Sections, with the various Habitudes or Proportions of one *Hyperbola* to another, *&c.*) would afford Matter ſufficient to fill a large Volume. But thus much may ſuffice by way of *Introduction*; I ſhall therefore deſiſt purſuing them any farther, being fully ſatisfied, that, if what I have already done be well underſtood, the reſt muſt needs be very eaſy to any one that intends to proceed farther on that Subject.

A N

AN INTRODUCTION TO THE Mathematicks.

PART V.

THE Method of finding out any particular Quantity (*viz.* either any LINE, SUPERFICIES, or SOLID) by a regular Progreſſion, or Series of Quantities continually approaching to it, which, being infinitely continued, would then become perfectly equal to it; is what is commonly call'd *Arithmetick of Infinites*; which I ſhall briefly deliver in the following *Lemma's*, and apply them to Practice in finding the ſuperficial and ſolid Contents of Geometrical Figures farther on.

LEMMA I.

If any Series of equal Numbers (repreſenting Lines or other Quantities) as, 1. 1. 1. 1. &c. or 2. 2. 2. 2. &c. or 3. 3. 3. 3. &c. if one of the Terms be multiply'd into the Number of Terms, the Product will be the Sum of all the Terms in the Series.

This is ſo very plain, and eaſy to be underſtood, that it needs no *Example*.

LEMMA II.

If the Series of Numbers in Arithmetick Progreſſion begin with a Cypher, and the common Difference be 1; as, 0. 1. 2. 3. 4. &c. (repreſenting a Series of Lines or Roots beginning with a Point) if the laſt Term be multiply'd into the Number of Terms, the Product will be double the Sum of all the Series.

That is, putting $L =$ the laſt Term, $N =$ the Number of Terms, and $S =$ the Sum of all the Series:

Then

Then will $NL = 2S$. Conſequently, $\frac{1}{2} NL = S$.
viz. one Half of ſo many times the greateſt Term as there are Numbers of Terms in the Series.

Thus $\frac{0+1+2+3+4}{4+4+4+4+4} = \frac{10}{20} =$ the Sum of the Series $= \frac{1}{2} NL$.
$= NL$.

And this will always be ſo, how many Terms ſoever there are, by *Conſect.* 1, *Page* 185.

LEMMA III.

If a Series of Squares whoſe Sides or Roots are in Arithmetick Progreſſion, beginning with a Cypher, *&c.* (as in the laſt *Lemma)* be infinitely continued; the laſt Term, being multiply'd into the Numbers of Terms, will be Triple to the Sum of all the Series, *viz.* $NLL = 3S$, or $\frac{1}{3} NLL = S$.

That is, the Sum of ſuch a Series will be one Third of the laſt or greateſt Term, ſo many times repeated as is the Number of Terms in the Series.

Inſtances in the Square Numbers.

1. $\frac{0+1+4}{4+4+4} = \frac{5}{12} = \frac{1}{3} + \frac{1}{12}$
2. $\frac{0+1+4+9}{9+9+9+9} = \frac{14}{30} = \frac{7}{18} = \frac{1}{3} + \frac{1}{18}$
3. $\frac{0+1+4+9+16}{16+16+16+16} = \frac{30}{80} = \frac{3}{8} = \frac{9}{24} = \frac{1}{3} + \frac{1}{24}$ *&c.*

From theſe Inſtances 'tis evident, that as the Number of Terms in the Series does increaſe, the Fraction or Exceſs above does decreaſe, the ſaid Exceſs always being $\frac{1}{6N-6}$; which, if we ſuppoſe the Series to be infinitely continued, will then become infinitely ſmall, *viz.* in Effect nothing at all. Conſequently, NLL may be taken for the true or perfect Sum of ſuch an infinite Series of Squares.

LEMMA IV.

If a Series of Cubes, whoſe Roots are in Arithmetick Progreſſion, beginning with a Cypher, &c. (as above) be infinitely continu'd, the Sum of all the Series will be $\frac{1}{4} NLLL = S$.

That is, one Fourth of the laſt or greateſt Term ſo many times repeated as is the Number of Terms.

Inſtances

Inſtances in Cube Numbers.

If 0 . 1 . 2 . 3. *&c.* be the *Roots* of the *Cubes.*

Then 1. $\left\{\frac{0+1+8+27}{27+27+27+27}=\frac{36}{108}=\frac{4}{12}=\frac{1}{4}+\frac{1}{12}\right.$

2. $\left\{\frac{0+1+8+27+64}{64+64+64+64+64}=\frac{100}{320}=\frac{10}{32}=\frac{5}{16}=\frac{1}{4}+\frac{1}{16}\right.$

3. $\left\{\frac{0+1+8+27+64+125}{125+125+125+125+125+125}=\frac{225}{750}=\frac{45}{150}=\frac{3}{10}=\frac{6}{20}=\frac{1}{4}+\frac{1}{20}\right.$

From theſe *Examples* it plainly appears, that, as the Number of Terms in the Series increaſes, the Fraction or Exceſs above $\frac{1}{4}$ decreaſes, the Exceſs being always $\frac{1}{4N-4}$; which, if we ſuppoſe the Series to be infinitely continued, will become infinitely ſmall, or rather nothing; as in the laſt *Lemma*. Conſequently, $\frac{1}{4}NLL$ may be taken for the true and perfect Sum of all the Terms in ſuch an infinite Series of Cubes.

LEMMA V.

If a Series of Biquadrates, whoſe Roots are in Arithmetick Progreſſion, beginning with a Cypher, *&c.* (as before) be infinitely continued, the Sum of all the Terms in ſuch a Series will be $\frac{1}{5}NL^4$.

The Truth of this may be manifeſted by the like Proceſs, as in the foregoing *Lemma*'s, and ſo on for higher Powers. But if any one deſires a farther Demonſtration of theſe Series, he may (I preſume) meet with ample Satisfaction in Dr. *Wallis*'s Hiſtory of *Algebra*, Chap. 78 and 79, wherein the Doctor concludes with theſe Words:

" Thus having ſhew'd, that in the Progreſſion of Laterals (or " Arithmetical Proportionals) beginning at 0, the Sum of 2, 3, 4, " 5, 6 Terms, is always equal to half of ſo many times the great- " eſt; and there being no Pretence of Reaſon why we ſhould " then doubt it in a Progreſſion of 7, 8, 9, 10, *&c.* we conclude " it ſo to be, tho' ſuch Number of Terms be ſuppos'd infinite.

" Again; in a Progreſſion of their Squares having ſhew'd, that " in 2, 3, 4, 5, 6 Terms the Aggregate is always more than one " Third of ſo many times the greateſt, and the Exceſs always ſuch " aliquot

" aliquot Part of the greateſt, as is denominated by ſix times the " Number of Terms wanting 1. (As, if the Terms be 2, " it is $\frac{1}{3}+\frac{1}{6}$; if three, it is $\frac{1}{3}+\frac{1}{12}$; if 4, it is $\frac{1}{3}+\frac{1}{16}$; if 5, " it is $\frac{1}{3}+\frac{1}{24}$ of ſo many times the greateſt Term, and ſo onward) " we may well conclude (there being no Pretence of Reaſon " why to doubt it in the reſt) that it will be ſo, how many ſoever " be ſuch Number of Terms. And becauſe ſuch Exceſs, as the " Number of Terms do increaſe, will become infinitely ſmall (or " leſs than any aſſignable) we conclude (from the Method of Ex- " hauſtions) that, if the Number of Terms be ſuppos'd infinite, " ſuch Exceſs muſt be ſuppos'd to vaniſh, and the Aggregate of " ſuch infinite Progreſſion ſuppos'd equal to $\frac{1}{2}$ of ſo many times " the greateſt.

" In like manner having prov'd that ſuch Progreſſion of Cubes " doth (as the Number of Terms increaſe) approach infinitely near " to $\frac{1}{4}$ of ſo many times the greateſt, and of Biquadrates to $\frac{1}{5}$, and " ſo of Surſolids to $\frac{1}{6}$ of ſo many times the greateſt, and ſo on- " wards as we pleaſe to try; and there being no Pretence of Rea- " ſon why to doubt it as to the reſt, we may take it as a ſufficient " Diſcovery, that (univerſally) the Aggregate of ſuch infinite " Progreſſion is equal (or doth approach infinitely near) to ſuch " a Part of ſo many times the greateſt, as is denominated by the " Exponent (or Number of Dimenſions) of ſuch Power (as is " that according to which the Progreſſion is made) increas'd by " 1, (namely, of Laterals $\frac{1}{2}$; of Squares $\frac{1}{3}$; of Cubes $\frac{1}{4}$; of Bi- " quadrates $\frac{1}{5}$; of ſo many times the greateſt) and ſo onwards " infinitely."

This Diſcourſe of the Doctor's I thought convenient to inſert, ſuppoſing it may give ſome Satisfaction to the Learner, to hear ſo Great a Man as Dr. *Wallis*'s Argument about the Truth of theſe Series, which I have briefly deliver'd in the 'foregoing *Lemma*'s.

LEMMA VI.

If any two Series or Ranks of Proportionals have the ſame Number of Terms (whether Finite or Infinite) it will always be { As the firſt Term of one Series : is to the firſt Term of the other Series : : ſo is the Sum of all the Terms in the one Series : to the Sum of all the Terms in the other Series.

(12. *e.* 5)

As

As in these Numbers,			Or these Numbers,	
3	6		4	5
2	6		12	15
3	9		36	45
4	12		108	135
5	15		324	405
6	18		972	1215

That is, 1 : 3 :: 21 : 63. And 4 : 5 :: 1456 : 1820 *&c.*

The Application of these *Lemma*'s to Geometrical Quantities, *viz.* to Lines, Superficies, and Solids, wholly depends upon granting the following *Hypotheses.*

The Hypotheses.

1. That every Line is suppos'd to consist (or be compos'd) of an infinite Series of equidistant Points.

2. A Surface (*viz.* the Area of any Figure) to consist of an infinite Series of Lines, either streight or crooked, according as the Figure requires.

3. A Solid to consist of an infinite Series of Plains, or Superficies, according as its Figure requires.

Not that we suppose Lines, which have really no Breadth, can fill a Space or Superficies; or that Plains, which have not any Thickness, can constitute a Solid: But by what we here call Lines are to be understood small Parallelograms (or other Superficies) infinitely narrow, yet so as that their Breadths, being all taken and put together, must be equal to the Figure they are suppos'd to fill up. And those Plains or Superficies, which are here said to constitute a Solid, are to be understood infinitely thin; yet so as that their Depths or Thicknesses (which are hereafter also called Lines) being all taken together, must be equal to the Height of the propos'd Solid. Now, in order to render this Hypothesis as easy for a Learner to understand as I can, I shall here propose a very plain and familiar *Example*; *Viz.* Let us suppose any Book to be compos'd (or made up) of 100, 200, 309 (more or less) Leaves of fine Paper; such a Book, being close put together, will have Length, Breadth, and Depth or Thickness, and therefore may (not improperly) be called a Solid; and each of its Edges (being evenly cut) will be a Superficies compos'd of a Series of small Parallelograms, every one of their Breadths being only the Edge of a single Leaf of Paper; and if we conceive the Thickness of every one of those Leaves to be divided into 10, or

100, or 1000, *&c.* they will then become ſuch a Series of infinitely ſmall Lines as are (by the Hypotheſis) ſaid to compoſe or fill up a Superficies. And all the Superficies of thoſe infinitely thin or divided Leaves of Paper will become ſuch a Series of Plains, or Superficies, as are ſaid to conſtitute a Solid, *viz.* ſuch a Solid as the Bigneſs and Figure of that Book.

Now according to this Idea of Lines, Superficies, and Solids, one may, without the leaſt Prejudice to any *Demonſtration*, admit of the following *Definitions* and *Theorems*.

Definitions.

I. The Area's of Squares, and all other Parallelograms, are compos'd or fill'd up with an infinite Series of equal Right Lines.

II. The Area of every *plain Triangle* is compos'd of an infinite Series of Right Lines parallel to its Baſe, and equally decreaſing until they terminate in a Point at the vertical Angle.

III. The Area of a *Circle* may be compoſed either of an infinite Series of concentrick or parallel Circles, or of an infinite Series of Chord Lines parallel to its Diameter, or of an innumerable Multitude of Sectors.

IV. The Area of an *Ellipſis* may be compos'd either of an infinite Series of Ordinates rightly apply'd, or of an infinite Series of Right Lines parallel to its Tranſverſe Diameter.

V. The Area's of the *Parabola* and *Hyperbola* are compos'd of an infinite Series of Ordinates; or may alſo be compos'd of Right Lines parallel to its Axis, *&c.*

VI. A *Priſm* is a ſolid Body contain'd or included within ſeveral equal Parallelograms, having its Baſes or Ends equal and alike; and it is generally nam'd according to the Figure of its Baſe: That is,

VII. A *Cube* (or Solid like a Dye) is a Priſm bounded or included with ſix equal ſquare Plains.

VIII. A *Parallelopipedon* is a Priſm that hath its Sides bounded or included within four equal Parallelograms and two ſquare Baſes or Ends.

IX. A *Cylinder* (or Solid, like a Rolling-ſtone in a Garden) is only a round Priſm, having its Baſes or Ends a perfect Circle.

X. The

X. The Solidity of every Prism is compos'd of an infinite Series of equal Plains, parallel and alike to that of its Base.

XI. A *Pyramid* is a Solid bounded or included within several plain Triangles set upon any Polygonous Base, having their vertical Angles all meeting together in a Point, called the Vertex, and takes its Name from the Figure of its Base, *viz.* if it has a square Base, 'tis call'd a square Pyramid; if a triangular Base, 'tis call'd a triangular Pyramid, *&c.*

XII. A *Cone* is only a round Pyramid, which hath been already defined in *Page* 355, *&c.*

XIII. The Solidity of every Pyramid is compos'd or constituted of an infinite Series of Plains, parallel and alike to that of its Base, equally decreasing until they terminate in a Point at the Vertex.

XIV. A *Sphere* or *Globe*, (*viz.* a Ball) is a Solid bounded or included within one regular Superficies, being form'd or generated by the Rotation of a Semi-circle about its Diameter (call'd the Axis of a Sphere) and its Solidity is compos'd or constituted of an infinite Series of concentrick Circles, whose Diameters are the Chords of that Circle by which it was form'd.

XV. A *Spheroid* (or Egg-like Figure) is a Solid bounded with one regular Superficies, form'd by the Rotation of a Semi-ellipsis about its Transverse Diameter (call'd the Axis of the Spheroid) and its Solidity is constituted of an infinite Series of concentrick Circles, whose Diameters are the Ordinates of that Ellipsis by which it was form'd.

XVI. There is another Sort of Solid call'd an *Oblate Spheroid*, being form'd by the Rotation of an Ellipsis about its Conjugate Diameter, and it is like a flat Turnep.

XVII. If a Semi-parabola be turn'd about its Axis, 'twill form a Solid call'd a *Parabolick Conoid*, being compos'd or constituted of an infinite Series of Circles, whose Diameters are the Ordinates of a Parabola.

XVIII. If a Parabola be turn'd about its Base, or greatest Ordinate, 'twill form a Solid call'd a *Pyramidoid*, but most commonly a *Parabolick Spindle*, which will be constituted of an infinite Series of Circles, whose Diameters are Right Lines parallel to the Parabola's Axis.

XIX. If an Hyperbola be turn'd about its Axis, 'twill form a Solid call'd an *Hyperbolick Conoid*, being constituted of an infinite Series of Circles, whose Diameters are the Ordinates of the Hyperbola.

XX. The curve Superficies of all circular Solids (*viz.* Cylinders, Cones, Spheres, *&c.*) are compos'd of an infinite Series of the Peripheries of thoſe Circles which conſtitute their Solidities.

Upon theſe Definitions are grounded all the following *Theorems*; and therefore, if they were diligently compar'd with their reſpective Figures, it muſt needs be of great Help to the Learner, and would render all that follows very eaſy; wherein I ſhall begin with what hath been already demonſtrated, by way of introducing the reſt.

THEOREM I.

The Area *of every Right angled* Parallelogram *is obtain'd by multiplying the Length into its Breadth.*

That is, $BD \times FB =$ the Area of the Parallelogram $BDFG$, by *Lemma* 1, compar'd with Definition 1.

Example.

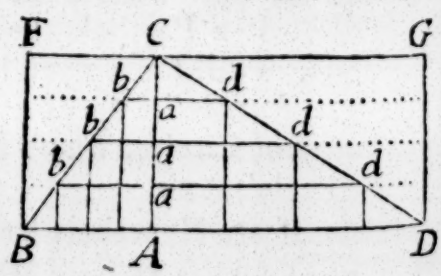
Suppoſe $BD = 26$, and $FB = 9$, then $26 \times 9 = 234$ the Area. See *Prob.* 1, *Page* 339.

THEOREM II.

The Area *of every plain Triangle is equal to half the* Area *of its circumſcribing* Parallelogram. *That is,* $\frac{BD \times CA}{2} =$ *the* Area *of* $\triangle BCD$, *in the following Figure.*

Demonſtration.

Suppoſe the Perpendicular CA to be divided into an infinite Number of equal Parts, as at the Points *a*, *a*, *a*, &c. and through thoſe Points there were drawn Right Lines parallel to the Baſe BD; (*viz.* *bad*, *bad*, *bad*, &c.) then will thoſe Lines be a Series of Terms in Arithmetick Progreſſion beginning at the Point C (*viz.* 0, *bd*, 2*bd*, 3*bd*, &c as is evident by the Figure, wherein BD the greateſt Term $= L$, and CA the Number of Terms $= N$.



But $\frac{1}{2} NL = S$, by Lemma 2. And S = the Triangle's Area by Definition 2. Q. E. D.

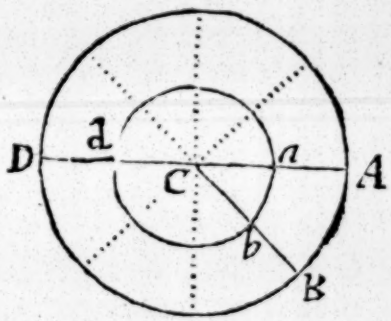
Example. Let $BD = 26$, and $CA = 9$, as above; then $\frac{26 \times 9}{2} = 117$, or $6\frac{1}{2} \times 9 = 117$. Or thus, $26 \times \frac{9}{2} = 117$, the Area requir'd. [See *Problem* 3, *Page* 330.]

THEOREM III.

The Peripheries of Circles are in Proportion one to another as their Diameters are.

Demonstration.

Let the Periphery of a Circle be divided into any Number of equal Arches by Right Lines drawn from the Center *(viz. Radii)* suppose 'em 8, as in the annexed Figure, wherein AB is one of them; then, if thro' any Point in the Radius there be drawn a concentrick or parallel Circle, its Periphery will also be divided into 8 equal Arches by those *Radii*, one whereof will be ab, and the $\triangle Cab$ will be like to $\triangle CAB$,



Therefore $Ca : ab :: CA : AB$, or $Ca : CA :: ab : AB$, consequently $2Ca : 2CA :: 8ab : 8AB$. But $2Ca = da$ the Diameter of the Circle, whose Periphery is $8ab$; and $2CA = DA$, the Diameter of the Circle, whose Periphery is $8AB$. Therefore, *&c.* as by the *Theorem*. Q. E. D.

Example.

In Chapter 6, Part III, it was found, that, if the Diameter of a Circle be 2, its Periphery will be 6,2831853, *&c.* Therefore, 2 : 6,2831853, *&c.* :: 1 : 3,14159265, *&c.* the Periphery of the Circle whose Diameter is 1.

Corollary.

Hence it follows, that because Unity, or 1, may be made the first Term in the Proportion, therefore 3,14159265, *&c.* may be made a constant or settled Factor; which, being multiply'd into any propos'd Diameter, will produce the Periphery of that Circle.

Note, Instead of 3,14159265, *&c.* it may be sufficient to take only 3,1416.

Or,

Or, in whole Numbers, the Proportion may be,

As 7 : 22 : : Diam. : Periphery
Or 113 : 355 :: Diam. : Periphery
} { theſe Numbers may ſerve, and are often uſed in common Practice.

THEOREM IV.

The Area *of any* Sector *of a* Circle *is equal to half the Rectangle of the* Radius *into its Arch.* That is, $\frac{CA \times AB}{2}$ = the Area of *ACP.*

Demonſtration.

Suppoſe the Radius *CA* to be divided into an infinite Series of equidiſtant Points, as *a*, *e*, *y*, &c. and thro' thoſe Points there were drawn concentrick or parallel Arches, as *ab*, *ed*, *yf*, &c. then they will be a Series of Arches in Arithmetick Progreſſion, beginning at the Point *C* (*viz.* 0, 1, 2, 3, *&c.*) as plainly appears by the Figure, wherein the greateſt Term is *AB* = *L*, and Number of Terms is *CA* = *N*. But $\frac{1}{2}NL = S$, the Sum of all the Series, by Lemma 2, and *S* = the Sector's Area, by Definition 3. Q. E. D.

Let the Radius *CA* = 12, and the Arch *AB* = 8, then $\frac{12 \times 8}{2}$ = 48. Or $12\frac{1}{2} \times 8 = 48$. Or $\frac{8}{2} \times 12 = 48$, the Area of the Sector *ACB.*

THEOREM V.

The Area *of every Circle is equal to half the Rectangle of the* Radius *into its* Periphery. *That is, according to* Archimedes, *a Circle is equal to a Right-angled Triangle, whoſe Sides containing the Right-angle are equal, one to the* Radius, *and the other to the* Perimeter *of that Circle.* Pro. 1. de Dimenſione Circuli.

The Truth of this *Theorem* may be eaſily deduced from the laſt, thus: If we ſuppoſe the laſt Sector to be one Eighth part of a Circle, then it follows, that $\frac{8\,4AB \times CA}{2} = 4AB \times CA$ will be the Area of the whole Circle. But $4AB$ = half the Circle's Periphery, and *CA* = half its Diameter; therefore, *&c.* as per Theorem. Q. E. D.

Example.

Example.

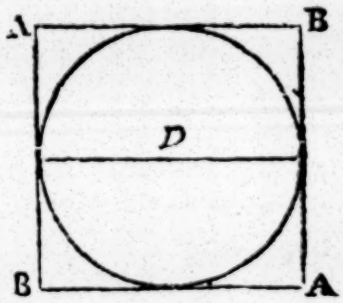
If the Diameter be Unity, or 1, the Periphery will be 3,14159265 *&c.* by *Theorem* 3. Then $\frac{3,14159265}{2} \times \frac{1}{2} = 0,78539816$, *&c.* (or 0,7854 for common Uſe) will be the Area of that Circle.

Scholium.

From hence naturally flows the following Proportion between the Square and its inſcrib'd Circle.

PROPORTION. { As the Perimeter (*viz.* the Sum of the four Sides) of any Square : is to its Area : : ſo is the Periphery of the inſcrib'd Circle : to its Area.

That is, ſuppoſing $AB = D =$ the Side of the Square, and the Diameter of its inſcrib'd Circle; then $4D =$ the Perimeter, $DD =$ the Area of the Square, and $3,1416\,D =$ the Periphery of the Circle, by *Theorem* 3. But $4D : DD :: 3,1416\,D : 0,7854DD =$ the Circle's Area. And if $D = 1$, then $4D = 4$, and $DD = 1 \times 1 = 1$, and the Periphery will be 3,1416. Then 4 : 1 : : 1 : 0,7854 &c. as in the *Example* above. And from hence may be eaſily deduced the following *Theorems*.



THEOREM VI.

The Area's *of all Circles are in Proportion one to another as the Squares of their Diameters.* (2. *e.* 12.)

For if $D =$ the Diameter of one Circle, and $d =$ the Diameter of another Circle, then will $0,7854\,DD$ be the Area of one Circle, and $0,7854\,dd$ will be the Area of the other Circle; as above. But $0,7854\,DD : 0,7854\,dd : : DD : dd$. Or thus, let $D =$ the Diameter, and $P =$ the Periphery of one Circle; $d =$ the Diameter, and $p =$ the Periphery of another Circle;

Then	1	$\frac{1}{2}D \times \frac{1}{2}P = \frac{1}{4}DP = A$,	the Area of one Circle.
And	2	$\frac{1}{2}d \times \frac{1}{2}p = \frac{1}{4}dp = a$,	the Area of the other Circle.
1 × 4	3	$DP = 4A$	(per laſt Theorem.
2 × 4	4	$dp = 4a$	
3 ÷ D	5	$P = \frac{4A}{D}$	

4 ÷ d

4 ÷ *d*	6	$p = \frac{4a}{d}$
But	7	$P : p :: D : d$, per *Theorem* 3.
5, 6, 7	8	$D : d :: \frac{4A}{D} : \frac{4a}{d}$
8 ∴	9	$4DDa = 4ddA$, that is, $DDa = ddA$
9, *Analogy*	10	$DD : A :: dd : a$, or $A : a :: DD : dd$

Q. E. D.

Corollary.

Hence it follows, that becauſe the Square of 1 is 1 (*viz.* 1×1 = 1) and 0,78539816, &c. or 0,7854 is the Area of the Circle whoſe Diameter is 1 (as before) therefore it will be 1 : 0,7854 : : ſo is the Square of any Circle's Diameter : to its Area. And becauſe 1 is the firſt Term in the Proportion, therefore 0,7854 may be made a conſtant Factor; which, being multiply'd into the Square of any propos'd Diameter, will produce the Area of that Circle.

Note, The four laſt Theorems do plainly ſhew the Reaſon of all the common or practical *Problems* about a Circle, which, for the Learner's farther Satisfaction, I have here inſerted together. Suppoſing as before,

That { D = the *Diameter*, P = the *Periphery*, A = the *Area* } of any propoſed Circle.

Probl. 1. D *being given, to find* P.

Then	1	1 : 3,1416 : : $D : P$. per Theorem 3.
1 ∴	2	$3{,}1416\,D = P$.
Examp.		Suppoſe $D = 32$. Then $3{,}1416 \times 32 = 100{,}5312$ the *Periphery*.

Probl. 2. D *being given, to find* A.

	3	1 : 0,7854 : : $DD : A$, per Theorem 6.
3 ∴	4	$0{,}7854\,DD = A$
		Suppoſe $D = 32$ (as before)
Examp.		$DD = 32 \times 32 = 1024$
Then		$0{,}7854 \times 1024 = 804{,}2496$, the Area requir'd.

Probl. 3. P *being given, to find* D.

And		
2 ÷	5	$D = \frac{P}{3{,}1416}$ } Or { becauſe $\frac{1}{3{,}1416} = 0{,}3183$ therefore $0{,}3183\,P = D$.

This, being only Converſe to the firſt, needs no Exam.

		Prob. 4. *P* being given to find *A*.
2 $\odot^2$	6	$9,8\ 965\ DD = PP$
6 $\div$	7	$DD = \frac{PP}{9,86965}$, or $0,10132\ PP = DD$
4 $\div$	8	$DD = \frac{A}{0,7854}$, or $1,2732\ A = DD$
For		$\frac{1}{0,7854} = 1,2732$
7, 8	9	$\frac{PP}{9,86965} = \frac{A}{0,7854}$, or $0,10132\ PP = 1,2732\ A$
9 $\times$ &c.	10	$\frac{PP}{12,5664} = A$, or $0,07957\ PP = A$
		Prob. 5. *A* being given to find *D*.
8 uu^2	11	$D = \surd \frac{A}{3,7854}$, or $D = \surd\ 1,2732\ A$
		Prob. 6. *A* being given, to find *P*.
10 $\times$ &c.	12	$PP = 12,5664\ A$, or $PP = \frac{A}{0,07957}$
12 uu^2	13	$P = \surd\ 12,5664\ A$, or $P = \surd \frac{A}{0,07957}$

Theſe ſix *Problems* contain all the Variety that can be propoſed about finding the Periphery, Diameter, and Area of any Circle.

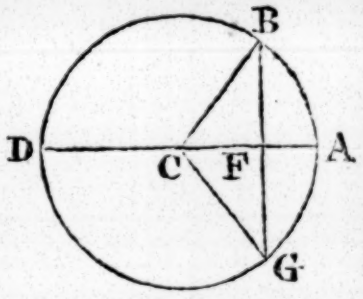
But if it be required to find the Area of any Segment, or Part of a Circle cut off by a Chord, that Work will require a farther Conſideration.

Firſt, As to the *Data* there muſt alwavs be given the *Diameter*; or, either the *Periphery* or *Area* of the *Circle*, in order to find the *Diameter*.

Secondly, There muſt alſo be given, either the *Chord* which is the *Baſe* of the *Segment*, or the *verſed Sine*, which is the Height of the *Segment*. That is, either *BG*, or *AF*, in the following *Scheme*, muſt be given, that ſo the *Area* of the $\triangle$ *BCG* may be found. Then it's evident (by the Figure) that, if the *Area* of the $\triangle$ *BCG* be taken from the *Area* of the *Sector CBAG*, the Remainder will be the *Area* of the *Segment BAG*. And if the *Area* of the *Segment BAG* be taken from the whole *Area* of the *Circle*, the Remainder will be the *Area* of the other *Segment DBG*.

Example in *Numbers.*

Let there be given $DA = 32$, as in *Prob.* 1. and the *verſed Sine* $AF = 6$; then $\frac{1}{2} DA = BC = CA = 16$, and $CA - AF = CF = 10$. But $\square BC - \square CF = \square BF$. Conſequently $\sqrt{\square BC - \square CF} = BF$, *viz.* $\sqrt{156} = 12{,}49 = BF$.



Then, by the Doctrine of plain *Triangles*, the *Arch* $BA = \prec BCA$ may be found in Degrees and Decimal Parts. Thus BC : *Radius* :: BF : *Sine* $\prec BCF = 51{,}31$ Degrees. And then it will always hold in this Proportion;

Viz. { *As the Circle's Periphery in Degrees: is to its Periphery in equal Parts* (according to the Dimenſions taken) : : *So is the Arch in Degrees* (viz. $\prec BCA$) : *to the ſame Arch in equal Parts.*

That is, $360^\circ : 100{,}5312 : : 51{,}31^\circ : 14{,}3284 = BA$. Then $14{,}3284 \times 16 = 229{,}2544$, the *Area* of the *Sector* $BCAG$; and $12{,}49 \times 10 = 124{,}9$, the *Area* of the $\triangle BCG$. Their Difference $104{,}3544$ = the *Area* of the *Segm.* BAG.

Or the *Area* of any *Segment* may be otherwiſe found (as moſt uſually it is) by a *Table* of the *Segments* of a *Circle*, whoſe *Area* is Unity, or 1. The *Conſtruction* or making of ſuch a *Table* is very well laid down in Mr. *Darie*'s Book of *Gauging*, *Chap.* 9. which he performs in this *Problem.*

PROBLEM.

In a Circle whoſe Area is Unity, and its Diameter cut by Chord Lines into 1000 *equal Parts, to find the Segment to any verſed Sine propos'd, not exceeding* 500 *of thoſe equal Parts.*

1. Multiply the *verſed Sine* propoſed by 0,002, and ſubtract the *Product* from an *Unit* or 1.

2. This Remainder you ſhall ſeek in the common *Table* of *Natural Sines*, (the Arch being divided into Degrees and Centeſimals) which being found, let its *Co-arch* be doubled, and called *A*.

3. You muſt find the *correſpondent Sine* to *A*; which *Sine* being found, you may call *S*, and then it holds $6{,}2831853) 0{,}017453^2 925 A - S$ (= the *Segment* required.

Now this *Segment* being thus found, if you ſubduct it from an *Unit*, you have the *Co-ſegment*, &c.

Note, Notwithſtanding what has been ſaid in the ſecond Precept of this *Problem*, it very often falls out that the Remainder there ſpoken of cannot be truly found in the Table of Natural Sines; therefore in this Caſe my Advice is, that you make two Operations, one with a Sine the next greater, and one with a Sine the next leſs; and in ſo doing you will be ſure to have the Segment requir'd bounded between the Reſults of thoſe two Operations.

Example, *Let it be propoſed to find the correſpondent Segment to the verſed Sine* 263.

Firſt, $263 \times 0{,}002 = 0{,}526$, and $1 - 0{,}526 = 0{,}474$, its *Arch* is 28,29° being leſs than juſt; its Complement is 61,71°, which, being doubled, is $123{,}42 = A$.

Then $,0174533\ A = 2{,}154086286$

$= 0{,}8346556 = S$ the *Sine* of A.

6,2831853) 1,319430686 (0,20993 the *Segment.*

Now I make a ſecond Work.

263 being *multiplied* with 0,002 is 526, and $1 - 526 = 0{,}474$ its *Arch* is 28,30° being greater than juſt; and its Complement is 61,70°, which being doubled is $123{,}4 = A$.

Then $0{,}0174533\ A = 2{,}1537372$

$- 0{,}8348478 = S$ the *Sine* of A

6,2831853) 1,3188894 (0,209907 the *Segment.*

So you ſee by theſe two Operations that the *Segment* is bounded, and 'tis very probable it may be 0,20995.

But to abbreviate this large *Factor*, and this large *Diviſor*, I ſhall here inſert two Tables of them, which will be ready for Uſe, and exact enough too.

Diviſor.		*Factor.*	
6,2832	1	,0174533	1
12,5664	2	,0349066	2
18,8495	3	,0523599	3
25,1327	4	,0698132	4
31,4159	5	,0872665	5
37,6991	6	,1047197	6
43,9823	7	,1221730	7
50,2655	8	,1396263	8
56,5487	9	,1570796	9

Thus far Mr. *Darie*, which I have here inſerted to ſhew the *Learner* how, by the Help of theſe two *Tables*, and a *Table* of *Natural Sines*, he may eaſily make a *Table* of *Segments*, whoſe Uſe ſhall be ſhewed farther on, *viz.* when I come to treat of practical Gauging. In the mean Time I ſhall here lay down another Method to find the *Area* of any *Segment* of a *Cir-*

cle (very near) by a new *Theorem*, without the Help either of a Table of *Sines* or *Segments*, having the ſame *Data* as before in Page 404.

Viz. Let $\begin{cases} R = \text{the } \textit{Radius}, \text{ or } \frac{1}{2} \textit{ Diameter} \text{ of the given } \textit{Circle}. \\ d = \text{the Difference between the } \textit{verſed Sine} \text{ and } \textit{Radius}. \\ C = \text{half the } \textit{Chord} \text{ of the } \textit{Segment}\text{'s } \textit{Baſe}. \end{cases}$

THEOREM. $\left\{ \frac{2\frac{1}{3}RR - 1\frac{1}{3}Rd - dd}{1\frac{1}{2}R + d} \times C = S, \text{ the } \textit{Area} \text{ of the } \textit{Segm.} \right.$

Example, Suppoſe $R = BC = 16$, $d = FC = 10$, and $C = BF = 12{,}49$; as before.

Then $2\frac{1}{3}RR = 597{,}3333$. $1\frac{1}{3}Rd = 213{,}3333$. $dd = 100$

$313{,}3333 = 1\frac{1}{3}Rd + dd$

$1\frac{1}{2}R + d = 34$ 284,0000 (8,3529. Laſtly, $8{,}3529 \times 1249 = 104{,}3276$ the *Area* of the *Segment* BAG, as before.

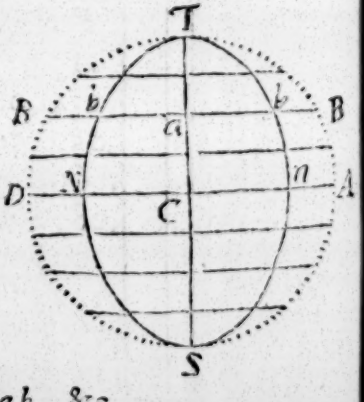
THEOREM VII.

As Squares are to the Area's *of their inſcribed Circles, ſo are Parallelograms to the* Area's *of their inſcribed Ellipſes.*

That is, $\begin{cases} \text{As the Square of the Diameter of any Circle : is to its } \textit{Area} \text{ : : ſo is the Rectangle of the Tranſverſe and Conjugate Diameters of any Ellipſis : to its } \textit{Area}. \end{cases}$

Demonſtration.

Circumſcribe any *Ellipſis* with a *Circle*; and ſuppoſe an infinite Number of *Chord Lines* drawn therein, all parallel to the *Conjugate Diameter*, as thoſe in the annexed Figure; then it will be $\begin{cases} \textit{As } (DA) \textit{ the Diameter of the Circle : is to } (Nn) \textit{ the Conjugate Diameter of the Ellipſis : : ſo is } (B\,a\,B) \textit{ any Chord in the Circle : to } (b\,a\,b) \textit{ its reſpective Ordinate in the Ellipſis}. \end{cases}$



For according to the Property of the *Circle*

it is	1	$\overline{TS - Ta} \times Ta = \square Ba$
And		by the Property of the *Ellipſis*.
it is	2	$\square TC : \square NC :: TS - Ta \times Ta : \square ba$
1, 2	3	$\square TC : \square NC :: \square BA : \square ba$
3, Hence	4	$TC : NC :: Ba : ba$
Conſeq.	5	$2TC : 2NC :: 2Ba : 2ba$
That is	6	$DA : Nn :: BaB : bab$
Put	7	$D = 2TC$, and $d = 2NC$
Then	8	$D : d ::$ *Chord* Bab : *Ordinate* bab, &c.

But

But the Sum of an *infinite Series* of ſuch *Chords*, as *B a B*, do conſtitute the *Area* of the *Circle*, by *Definition* 3 : and the Sum of the like *Series* of their reſpective *Ordinates*, as *b a b*, do conſtitute the *Ellipſis's Area* by *Definition* 4. Therefore *D* : *d* : : *Circle's Area* : *Ellipſis's Area*, by *Lemma* 6. But *D* : *d* : : *D D* : *D d* Whence it follows, that *D D* : *Circle's Area* : : *Dd* : *Ellipſis's Area* Q. E. D. Conſequently, as 1 : is to 0,7854 : : ſo is the *Rectangle* or *Product* of the *Tranſverſe* and *Conjugate Diameters* of any *Ellipſis* : to its *Area*.

Example, Suppoſe *TS* = 36, and *Nn* = 16 ; then 36 × 16 = 576, and 576 × 0,7854 = 452,3904 the *Area* of the *Ellipſis*.

Corollaries.

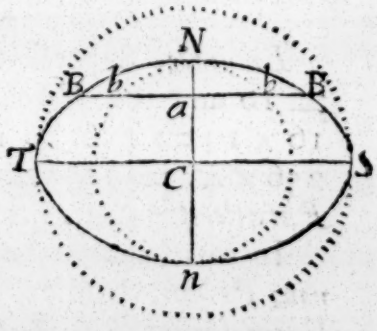
1. Hence it is eaſy to conceive, that the ſquare Root of the *Rectangle* or *Product* of the *Tranſverſe* and *Conjugate Diameters* will be the *Diameter* of a *Circle* whoſe *Area* will be equal to the *Ellipſis's Area*, viz. ✓ 576 = 24 the *Diameter* of a *Circle* = to the *Ellipſis*.

2. All *Segments* of an *Ellipſis* and its *circumſcribing Circle* (whoſe Baſes are parallel to the *Conjugate Diameter*, and of the ſame Height) are in Proportion one to another, as their *Baſes* are. That is, *B a b* : *b a b* : : *Area Segment B N B* : *Area Segment b N b* ; or *TS* : *Nn* : : *Area Segment B N B* : *Area Segment b N b*.

THEOREM VIII.

The Area *of every* Ellipſis *is a mean Proportional between the* Area's *of its* circumſcribing *and* inſcrib'd Circles.

The Truth of this *Theorem* may be eaſily deduced from the laſt; for ſuppoſing *D* = *TS*, and *d* = *Nn*, as before; then it is already proved, that *D D* : *D d* : : *circumſcribing Circle's Area* : *Ellipſis's Area*. But *D D* : *D d* : : *D d* : *d d*. Therefore *Ellipſis's Area* : *inſcrib'd Circle's Area* : : *D d* : *d d*. By *Theorem* 6.



Example, Let *TS* = *D* = 36, and *Nn* = *d* = 16, as before; then *D D* = 1296, and *d d* = 256.

Then

Then will $\begin{cases} 1296 \times 0{,}7854 = 1017{,}8784 \text{ the great } \textit{Circle}\text{'s Area} \\ 256 \times 0{,}7854 = 201{,}0624 \text{ the leſſer } \textit{Circle}\text{'s Area.} \end{cases}$

Suppoſe A = the *Ellipſis's Area*; then, according to the *Theorem*, it will be, $1017{,}8784 : A :: A : 201{,}0624$. *Ergo* $AA = 1017{,}8784 \times : 201{,}0624 = 204657{,}07401216$. Conſequently, $\surd\ 204657{,}07401216 = 452{,}3904 = A$, the *Area* of the *Ellipſis* as before in the laſt *Example*.

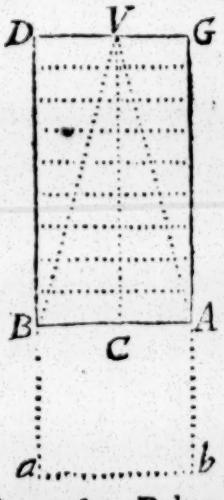
Corollary.

From hence it follows, that all *Segments* of an *Ellipſis* and its *inſcrib'd Circle*, whoſe *Baſes* are parallel to the *Tranſverſe Diameter*, and have the ſame Height, are in Proportion one to another as the *Area's* of the *Ellipſis* and *Circle* are. That is, *Area* of *Circle* : *Area* of *Ellipſis* : : *Segment bNb* : *Segment BNB*. Or, *Nn* : *TS* : : *Area Segment bNb* : *Area Segment BNB*.

THEOREM IX.

The Solid Content of any Priſm (what Figure ſoever its Baſe is of) *is obtained by multiplying the* Area *of its* Baſe *into its Height.*

For Inſtance, a *Parallelopipedon* (or ſquare *Priſm*) is conſtituted of an *infinite Series* of equal *Squares*; that of its *Baſe BA* being one of the *Terms*, and its Height *DB*, or *GA*, the *Number* of all the *Terms*. Conſequently, the *Area* of *BAba* × *DB* = the *Sum* of all the *Series* (by *Lemma* 1.) which is the *Solidity* of the *Parallelopipedon DBGA*, by *Definition* 10.



Example, Suppoſe the Side of the *Baſe BA* = 16 and the *Height DB* = 42; then will $16 \times 16 = 256$ be the *Area* of the Baſe, and $256 \times 42 = 10752$ the *Solid Content* of the *Parallelopipedon DBGA*.

In this Manner you may find the *Solidity* of all regular *Polygonous Priſms*, whoſe *Baſes* (or Ends) are parallel and alike, what Form ſoever they are of, that is, whether their *Baſes* are *Triangles*, *Pentagons*, *Hexagons*, or *Octagons*, &c.

THEO-

THEOREM X.

Every Pyramid is the third Part of the Prism, that hath the same Base and Height with it (7. *e.* 12.)

That is, the *Solid Content* of the *Pyramid BVA* (in the last Figure) is one *Third* of its circumscribing *Prism DBGA.*

Demonstration.

For every *Pyramid* that hath a square *Base* (as *BAba*, in the last Figure) is constituted of an *infinite Series* of *Squares*, whose *Sides* or *Roots* are continually increasing in *Arithmetick Progression*, beginning at the *Vertex* or *Point V* (See *Theor.* 2.) its *Base BABa*; being the greatest *Term* (= *LL*) and its *perpendicular Height VC*, or *DB*, is the Number of all the *Terms* = *N*; but $\frac{NLL}{3} = S$ the *Sum* of all the *Series*, by *Lemma* 3, and *S* = the *Solid Content* of the *Pyramid BVA*, by *Definition* 13.

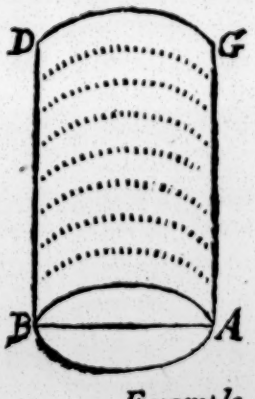
Example. Suppose the Side of a *Pyramid*'s *Base* be *BA* = 16, and its *Height* be *VC* = 42. Then $16 \times 16 = 256$ the *Area* of its *Base BABa* = *a*, and $\frac{256 \times 42}{3} = 3584$. Or $\frac{356}{3} \times 42 = 3584$ or thus, $256 \times \frac{42}{3} = 3584$, is the *Solidity* of that *Pyramid BVA.*

Corollary.

From hence it will be easy to conceive, that every *Pyramid* is $\frac{1}{3}$ of its *circumscribing Prism*, what Form soever its *Base* is of, *viz.* whether it be a *Square*, *Triangle*, *Pentagon*, &c.

THEOREM XI.

The Solid Content of every Cylinder is obtain'd by multiplying the Area *of its Base into its Height.*



For every *Right Cylinder* is only a round *Prism*, being constituted of an *infinite Series* of equal *Circles*; that of its *Base* or End being one of the *Terms*, and its *Height BD* is the Number of all the *Terms.* Therefore the *Area* of its *Base BA*, being multiply'd into *DB*, will be its *Solidity*, by *Lemma* 1. *viz.* Let *D* = *BA*, and *H* = *GA.* Then $0{,}7854\ DD \times H$ = its *Solidity.*

Example.

Example, Let the *Diameter* of its *Baſe* be $D = 16$, and its *Height* $H = 42$. Then 1 : 0,7854 : : $16 \times 16 = 256$: 201,0624 the *Area* of its *Baſe*. And $201,0624 \times 42 = 8444,6208$ the *Solid Content* of that *Cylinder D B G A*.

Corollary.

Hence it is evident that every ſquare *Parallelopipedon* is to its *inſcrib'd Cylinder*, as 1 . is to 0,7854. Or in whole *Numbers*, as 452 : to 355 very near. And that all *Priſms* are in Proportion to their *inſcrib'd Cylinders*, as the *Area's* of their *Baſes* are.

THEOREM XII.

The Curve Superficies of every Right Cylinder is equal to the Rectangle made of its Height into the Periphery of its Baſe.

That is, *D B* multiply'd into the *Periphery* of the *Diameter B A*, will produce the *Curve Superficies* of the laſt *Cylinder D B G A*. For the *Cylinder* is conſtituted of an *infinite Series* of equal *Circles* (according to the laſt *Theorem.*) Therefore its *Curve Superficies* is compos'd of the *Peripheries* of thoſe *Circles*, by *Definition* 20. But the *Periphery* of its *Baſe B A* is one of the *Terms*, and its Height *D B* is the Number of *Terms*. Therefore, *&c.* as by *Lemma* 1. To which, if there be added the *Area's* of both its Ends (or *Baſes*) the *Sum* will be the *Superficies* of the whole *Cylinder*.

Example. Suppoſe the *Diameter* of its *Baſe* to be $B A = 16$, and its Height $D B = 42$, as before; then 1 : 3,1416 : : 16 : 50,2656 the *Periphery* of its *Baſe*. Again, 1 : 0,7854 : : $16 \times 16 = 256$: 201,0624 the *Area* of each End or *Baſe*.

Then $50,2656 \times 42 = 2111,1552$ the *Curve Superficies*, to which add $201,0264 \times 2 = 402,1248$ both the End *Area's*.

The *Sum* = 2513,2800 is the *Superficies* of the whole *Cylinder*.

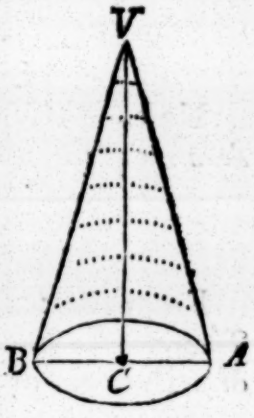
THEOREM XIII.

Every Cone is the third Part of a Cylinder, having the ſame Baſe with it, and their Altitudes equal. (10. *e.* 12.)

Demon-

Demonſtration.

The Truth of this *Theorem* may be eaſily conceiv'd by only conſidering, that a *Cone* is but a round *Pyramid*, and therefore it muſt needs have the ſame *Ratio* to its circumſcribing *Cylinder* as the ſquare *Pyramid* hath to its circumſcribing *Parallelopipedon*, *viz.* as 1 : to 3. However, to make it yet clearer, let it be farther conſidered, that every *Right Cone* is conſtituted of an *infinite Series* of *Circles*, whoſe Diameters do continually encreaſe in *Arithmetick Progreſſion* beginning at the *Vertex* or *Point V*, the *Area* of its *Baſe B A* being the greateſt *Term*, and its perpendicular Height *VC* the *Number* of all the *Terms*; therefore the *Area* of the Circle $BA \times \frac{1}{3} VC$ will be the Sum of all the *Series*, by *Lemma* 3, which is the *Cone*'s *Solidity*.



Example. Let the Diameter of its *Baſe* be $BA = 16$, and its Height $VC = 42$; Then $1 : 0{,}7854 :: 16 \times 16 = 256 : 201{,}0624$ the *Area* of the *Baſe*; and $\frac{201{,}0624 \times 42}{3} = 2814{,}8736$ the *Solidity* of the *Cone BVA*. Or thus, $201{,}0624 \times {}^{42}\!/_{3} = 2814{,}8736$, *&c.*

Corollary.

Hence it follows, that every ſquare Pyramid is to its inſcrib'd Cone, as 1 : 0,7854. (Or as 452 : 355) conſequently, that all Pyramids have the ſame *Ratio* to their inſcrib'd *Cones* as the *Area*'s of their Baſes have.

THEOREM XIV.

The Curve Superficies of every Right Cone is equal to half the Rectangle of the Periphery of its Baſe into the Length of its Side.

The Truth of this *Theorem* is ſelf-evident from the *Definition* of a Cone, *Chap.* 1, *Part* IV, where it appears, that the *Curve Superficies* of every Right Cone (as *BVA*) is equal to the *Area* of a *Sector* of that Circle whoſe *Radius* is the Side of the Cone (*VB*) and its *Arch* equal to the *Periphery* of the Cone's Baſe (*BA*) But the *Area* of any *Sector* is equal to half the *Rectangle* of the *Radius* into its *Arch*, by *Theorem* 4. Therefore, *&c.*

Example. Suppoſe the Length of the Cone's Side to be VB, or $VA = 42{,}7551$, and the Diameter of its Baſe, *viz.* $BA = 16$ (as before) then will 50,2656 be the *Periphery* of its Baſe, and $\frac{50{,}2656 \times 42{,}7551}{2} = 1074{,}5553$, *&c.* the *Curve* of the *Superficies*; to which if there be added the *Area* of its Baſe, the Sum will be the *Superficies* of the whole (viz. *all the*) Cone.

That is, 1074,5553
+ 201,0624 the *Area* of the *Baſe.*

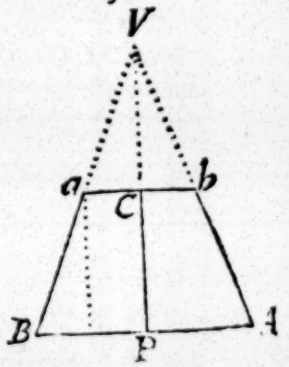
Sum 1275,6177 is the total *Superficies*, &c.

Note, *The Truth of this Theorem may be prov'd from the Conſideration of the laſt Theorem and Definition* 20.

Scholium.

From the 10th and 13th *Theorems* may be eaſily deduced ſeveral *Theorems* for finding the *ſolid Content* of any *Fruſtum* or *Part*, either of a *Pyramid* or *Cone*, cut by a plain Parallel to its *Baſe.*

Suppoſe a ſquare *Pyramid*, as BVA, to be cut by a *Plain* at ab, parallel to its *Baſe* BA, and it were requir'd to find the *Solidity* of the *Fruſtum* or *Part* $abAB$; let there be given $D = BA$ the Side of the greater Baſe. $d = ba$ the Side of the leſſer Baſe. $H = CP$ the perpendicular Height.



Firſt,	1	$D - d : H :: d : \frac{dH}{D-d} = VC$ by the Figure.
Then	2	$DD \times \frac{H + VC}{3} =$ the whole Pyramid BVA By *Theorem* 10.
And	3	$dd \times \frac{1}{3} VC =$ the Pyramid aVb cut off.
Viz. 1, 2	4	$\frac{DDDH}{3D-3d} =$ the whole Pyramid BVA.
And 1, 3	5	$\frac{dddH}{3D-3d} =$ the Pyramid aVb.
4 — 5	6	$\frac{DDDH - dddH}{3D-3d} =$ the Fruſtum $abAB$.
6. *Reduc.*	7	$\overline{DD + Dd + dd} \times \frac{1}{3} H =$ the Fruſtum $abAB$.

Which in Words gives this following *Theorem.*

THEO-

THEOREM XV.

To the Rectangle of the Sides of the two Bases, *add the Sum of their* Squares; *that Sum, being multiply'd into one Third of the* Frustum's *Height, will give its Solidity.*

Example. Suppose the Side of the greater Base $BA = 16$, and the Side of the lesser Base (or Top] $ab = 12$ the Height $CP = 9$. Then the $16 \times 12 = 192$. $16 \times 16 = 256$. and $12 \times 12 = 144$. Next $192 + 256 + 144 = 592$. and $\frac{592 \times 9}{3} = 1776$. Or $592 \times \frac{9}{3} = 1776$ the Content of the *Frustum* of a square *Pyramid*.

And if it were the like *Frustum* of a *Right Cone*, it may be found by the same *Theorem*. Supposing $D =$ the Diameter of the greater Base, $d =$ the Diameter of the lesser, and $H =$ the Height of the *Frustum*, then the Sum of all the *Squares* which constitute the *Frustum* of a square *Pyramid*, are to the Sum of all the Circles which constitute the like *Frustum* of a right Cone, in the *Ratio* of 1 : to 0,7854 (or of 452 : to 355) therefore it will be $1 : 0{,}7854 :: \overline{DD + Dd + dd} \times \frac{1}{3} H : 0{,}7854\, DD + 0{,}7854\, Dd + 0{,}7854\, dd \times \frac{1}{3} H =$ the Cone's *Frustum*, that is, in the last *Example*, 1 : 0,7854 : : 1776 : 1394,8704 the like *Frustum* of a right Cone. Or because $\frac{1}{0{,}7854} = 1{,}273236$, *&c.* Therefore it may be made $1{,}273236) \overline{DD + d + dd} \times \frac{1}{3} H$ (= the same *Frustum*; that is, 1,273236) 1776 (1394,87, *&c.* as before. And if you take the *Triple* of this Divisor, *viz.* $1{,}273236 \times 3$, it will be $3{,}8197) \overline{DD + Dd + dd} : \times H$ (= the *Frustum*, &c.

Again,

Suppose	1	$x = D - d$, and $F =$ the *Frustum*.
Then	2	$DD + Dd + dd = \frac{3F}{H}$, by the 7th Step of the last.
$1 \odot^2$	3	$xx = DD - 2Dd + dd$
$2 - 3$	4	$3Dd = \frac{3F}{H} - xx$
$4 \div 3$	5	$DD = \frac{F}{H} - \frac{1}{3} xx$, or $Dd + \frac{1}{3} xx = \frac{F}{H}$
$5 \times H$	6	$\overline{Dd + \frac{1}{3} xx} \times H = F$ the *Frustum ab AB*.

Hence we have another easy *Theorem* for finding the same *Frustum*.

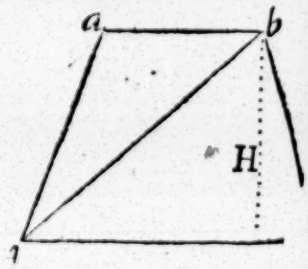
THEOREM XVI.

To the Rectangle of the Sides of the two Baſes, add one third Part of the Square of their Difference; that Sum, being multiply'd into the Height, will produce the Solidity.

Example. Let $D = 16$. $d = 12$. and $H = 9$, as before; then $Dd = 192$. $D - d = 4 = x$. $\frac{1}{3}xx = \frac{4\times4}{3} = 5,3333$, and $192 + 5,3333 = 197,3333$. Laſtly, $197,3333\times9 = 1775,9997$ the *Solidity* of the *Fruſtum* of the ſquare Pyramid, as before. And 3,81968) 1775,9997 (1394,87, *&c.* the like *Fruſtum* of a right Cone, as before.

Either of the two laſt *Theorems* (being rightly apply'd) will produce the true *ſolid Content* of all *Fruſtums* of any kind of Pyramids, that are intercepted between two parallel and alike *Plains* or *Baſes:* As above.

But if ſuch *Fruſtums* are cut through the Extremities of both Baſes by a Diagonal Plain (as *Ab* in the annexed Figure) into two Parts, *Aab* and *ABb*, call'd *Hoofs*; then the *Solidity* of thoſe *Hoofs* is uſually found by dividing the middle Term *D d* of the *Æquation* $Dd + Dd + dd$ into two Parts, and adding one of thoſe Parts to the *Square* of each Baſe. Thus, $\overline{DD + \frac{1}{2}Dd}\times\frac{1}{3}H =$ the great *Hoof ABb*, and $\overline{dd + \frac{1}{2}Dd}\times\frac{1}{3}H =$ the leſſer *Hoof Aab* of the *Fruſtum* of any ſquare Pyramid. Then 3,8197) $\overline{DD + \frac{1}{2}Dd}\times H$ (=) the greater *Hoof* of a *Cone*. And 3,8197) $\overline{dd + \frac{1}{2}Dd}\times H$ (=) the leſſer *Hoof*, &c.



Theſe are the *Theorems* made Uſe of by Mr. *Darie*, in his Book of *Gauging*, and are pretty near the Truth, but not exactly ſo; for they give the *Solidity* of the upper Hoof *Aab* a ſmall Matter too big, and the lower Hoof *ABb* as much too little.

Now, in order to rectify that ſmall Error, I ſhall here propoſe the two following *Theorems*, which come very near the Truth, and are more eaſily perform'd than thoſe propos'd in the *firſt Impreſſion of this Book.*

Firſt, $\overline{DD + \frac{1}{2}Dd + D - d}\times\frac{1}{3}H$ will be the *Solidity* of the greater *Hoof ABb.*

Secondly,

Secondly, $\overline{dd + \frac{1}{2}Dd + d - D} \times \frac{1}{3}H$ will give the *Solidity* of the leſſer *Hoof A a b*, of the *Fruſtum* of any ſquare *Pyramid*.

And for the like *Hoofs* of the *Fruſtum* of any *right Cone*, it will be

Thus, 3,8197) $\overline{DD + \frac{1}{2}Dd + D - d} \times$ * H (= the greater *Hoof*.

And 3,8197) $\overline{dd + \frac{1}{2}Dd + d - D} \times$ H (= the leſſer *Hoof*.

Note, *In order to avoid many Words in the following* Demonſtrations, *let* ⊙ *ſignify any* Circle *in general*; *and if any two Letters be join'd to it, thus,* ⊙ *B A*, &c. *it then denotes the* Area *of ſuch a* Circle *as thoſe two Letters repreſent the* Radius *of*.

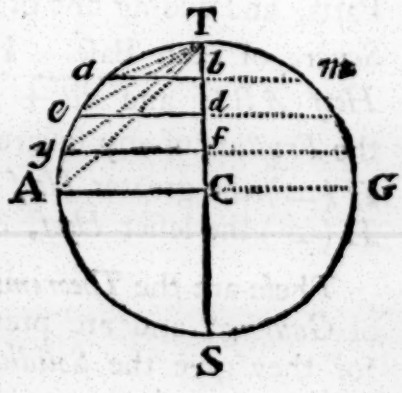
THEOREM XVII.

The Superficies of every Sphere *(or* Globe*) is equal to four Times the* Area *of its greateſt* Circle.

That is, of a *Circle* whoſe Diameter is the *Axis* of the *Sphere*.

Demonſtration.

If any *Semicircle* (as *A T G S*) be turn'd or moved about its Diameter (*T S*) it will deſcribe a ſolid Body call'd a *Sphere*, which will be conſtituted of an infinite Series of *concentrick* or *parallel Circles*, whoſe Diameters are Chords, *viz.* ⊙ *a b*, ⊙ *e d*, ⊙ *e f*, &c. by Definition 14. Conſequently, the *Superficies* of the *Sphere* will be compos'd of the *Peripheries* of thoſe Circles which conſtitute its *Solidity*, by Definition 20.



Let $D = TS$, the *Axis* of any *Sphere*. Then, according to the Property of a Circle, it

will be	1	$D - Tb \times Tb = \square ab$
That is,	2	$D \times Tb - \square Tb = \square ab$
Therefore	3	$D \times Tb = \square aT$, for $\square ab + \square Tb = \square aT$.
And	4	$D \times Td = \square eT$
	5	$D \times Tf = \square yT$, &c.

* The Error is here corrected, which Mr. *J. Robertſon* takes Notice of in his Book, entitled, *A Compleat Treatiſe of Menſuration*, Page 160.

Hence

Hence 'tis evident, that the *Series* □ *a T*, □ *e T*, □ *y T*, &c. are in the ſame *Ratio* with *T b*, *T d*, *T f*, &c. *viz.* in *Arithmetick Progreſſion*; whence it follows, that the ⊙ *a T* = the Sum of all the Circle's *Peripheries* between *T* and *b*, and ⊙ *e T* = the Sum of all the Circle's *Peripheries* between *T* and *d*, &c. Conſequently, that the ⊙ *AT* = the Sum of all the Circle's *Peripheries* included between *T* and *C*; that is, ⊙ *AT* = the *Superficies* of the *Hemiſphere*. And becauſe □ *AC* + □ *TC* = □ *AT*, and □ *AC* = □ *TC*. Therefore ⊙ *AT* = 2 ⊙ *AC* is the *Superficies* of the *Hemiſphere*. Conſequently, 4 ⊙ *AC* will be the *Superficies* of the whole *Sphere*. Q. E. D.

Example. Suppoſe the Axis *T S* = *D* = 16. Then *D D* = 256, and 1 : 0,7854 : : 256 : 201,0624 = ⊙ *AC*, for ½ *D* = *AC*. Then 201,0624 × 4 = 804,2496, the *Superficies* of the whole *Sphere*. Or, becauſe 3,1416 is four Times 0,7854, therefore it will always be 1 : 3,1416 : : *D D* : 3,1416 *D D*, the *Superficies* of the *Sphere* (as before); and it is equal to the *curve Superficies* of the *right Cylinder*, whoſe Diameter and Height are each = *D* the Axis of the *Sphere*. For 3,1416 *D* = the *Periphery* of the *Cylinder*'s Baſe, and that, multiply'd with *D* its Height, will be 3,1416 *D D* the *curve Superficies* of the *Cylinder*, by *Theorem* 12. And if to this there be added the *Area* of its two Baſes (or *Ends*) *viz.* 1,5708 *D D*, then 'tis evident, that the whole *Superficies* of the *Cylinder* will be to that of the *Sphere* in Proportion of 3 to 2.

Scholium.

From the Method here uſed in proving the laſt *Theorem* 'twill be eaſy to find the *curve Superficies* of any *Segment*, or *Part* of a *Sphere*, that is cut off by a Right Line or Plain, *viz.* ſuch as the Segment *a T m* in the laſt *Scheme*, whoſe *curve Superficies* is ⊙ *a T* (as above). Therefore (becauſe □ *a b* + □ *T b* = □ *a T*) it will be ⊙ *a b* + ⊙ *T b* = the *curve Superficies* of that *Segment*.

But if the Axis *T S*, and Height *T b*, of the *Segment* are given, then will it be *T S* × *T b* □ *a T*; as in the third Step above: Which gives this Proportion or Theorem;

Viz.

Viz. { *As the* Axis *of the* Sphere : *is to the whole* Superficies *of the* Sphere : : *ſo is the* Height *of any* Segment *to its curve* Superficies.

To which if there be added the *Area* of the Segment's Baſe, the Sum will be the Superficies of the whole Segment.

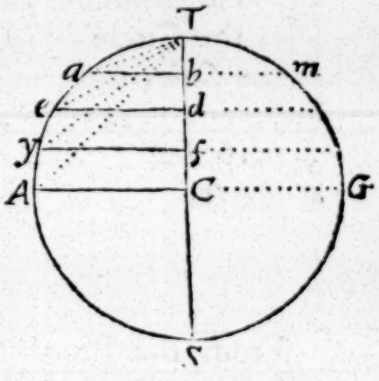
THEOREM XVIII.

Every Sphere is equal to two Thirds of its circumſcribing Cylinder.

That is, of a Cylinder whoſe Height and Diameter of its Baſe are each equal to the Axis of the Sphere.

Demonſtration.

According to the Work in the laſt *Theorem* it appears, that $\odot ab$, $\odot ed$, $\odot yf$, &c. do conſtitute the Solidity of the Sphere; and that $\square aT$, $\square eT$, $\square yT$, &c. are a Series of Terms in *Arithmetick* Progreſſion, $\square AT$ being the greateſt *Term*, and TC the *Number* of *Terms*; therefore $\odot AT \times \frac{1}{2} TC =$ the Sum of all the Series, per *Lemma* 2. And becauſe $\square aT - \square Tb = \square ab$, $\square eT - Td = \square ed$, $\square yT - \square Tf = \square yd$, $\square AT - \square TC = \square AC$, &c. wherein $\square Tb$, $\square Td$, $\square Tf$, &c. are a Series of Squares whoſe Roots Tb, Td, Tf, are in Arithmetick Progreſſion, $\square TC$ being the greateſt *Term*, and TC the *Number* of *Terms*; therefore $\odot TC \times \frac{1}{3} TC =$ the Sum of all that Series, *per* Lemma 3, conſequently, $\odot AT \times \frac{1}{3} TC - \odot TC \times \frac{1}{3} TC =$ the Sum of the *Series* $\odot ab$, $\odot ed$, $\odot yf$, &c. which conſtitute the Solidity of the half Sphere ATG. Put $D = 2\,TC$ the Axis of the Sphere; then $\frac{1}{4} D = \frac{1}{2} TC$, and $\frac{1}{6} D = \frac{1}{3} TC$. And becauſe $\square AT = 2 \square TC$; therefore $\odot AT = 2 \odot TC = 1{,}5708\ DD$. And $1{,}5708\ DD \times \frac{1}{4} D = 0\ 3927\ DDD$.



Again, $DTC \times \frac{1}{3} TC = 0{,}7854\ DD \times \frac{1}{6} D = 0{,}1309\ DDD$, then $0{,}3927\ DDD - 0{,}1309\ DDD = 0{,}2618\ DDD$ the Solidity of the *Semi-ſphere* ATG; conſequently, $0{,}2618\ DDD \times 2 = 0{,}5236$, DDD will be the ſolid Content of the whole *Sphere*, which is equal to two Thirds of the Cylinder whoſe Diameter of its *Baſe* and Height $= D$. For $0{,}7854\ DDD =$ the Solidity of the Cylinder, by *Theorem* 11. But $\frac{2}{3}$ of $0{,}7854\ DDD = 0{,}5236\ DDD$; as before. Therefore, *&c.* : as by *Theorem*.

Example. Suppoſe the *Axis* $D = 16$, then $DDD = 4096$, and $1 : 0{,}5236 :: 4096 : 2144{,}6656$ the *ſolid Content* of that *Sphere.*

Corollaries.

1. Hence it appears, that the ſolid Content of every Sphere is equal to its Superficies multiply'd into one ſixth Part of its *Axis.* For its Superficies is $3{,}1416\,DD$, by *Theorem* 17. But $3{,}1416 \times \frac{1}{6} D = 0{,}5236\,DDD$ the ſolid Content, as before.

2. And hence 'tis alſo evident, that there is the like *Ratio* or Habitude between the *Cube* and its *inſcrib'd Sphere*, as is betwixt the *Square* and its *inſcribed Circle*; and that is, as the Superficies of any *Cube* : is to the Superficies of its inſcrib'd Sphere : : ſo is the ſolid Content of that *Cube* : to the ſolid Content of the *Sphere.* [See the Circle's Proportion, *Page* 407.] For if $D =$ the Side of the *Cube*, then $6\,DD =$ its Superficies, and $DDD =$ its Solidity, and $3{,}1416\,DD =$ the *Sphere*'s Superficies. But $6\,DD : 3{,}1416\,DD :: DDD : 0{,}5236\,DDD$ the Solidity of the *Sphere*; as above.

Scholium.

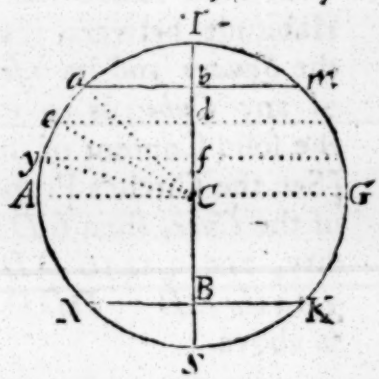
From the Proof of this *Theorem* 'twill be eaſy to deduce or raiſe *Theorems* for finding the ſolid Content of any *Fruſtum* or *Segment* of a *Sphere*, as $a\,T\,m$ in the laſt Figure. For we there ſuppoſe the Segment $a\,T\,m$ to be conſtituted of an infinite Series of Circles, which have the ſame *Ratio* with all thoſe Circles that conſtitute the *Semi-ſphere.* Therefore it follows, that $\odot\, a\,t \times \frac{1}{2} Tb - \odot\, b\,T \times \frac{1}{3} Tb$ will be the Sum of all the Circles intercepted between T and b. Conſequently 'twill be the Solidity of that Segment. And becauſe $\square\, ab + \square\, Tb = \square\, aT$: therefore $\odot\, ab + \odot\, Tb \times \frac{1}{2} Tb - \odot\, Tb \times \frac{1}{3} b =$ the ſame *Solidity.*

Let $c = ab$ half the *Segment*'s Baſe; $h = Tb$ its Height; and $S =$ the *Solidity* of the *Segment* or *Fruſtum:* Then $\odot\, ab = 3{,}1416cc$, and $\odot\, Tb = 3{,}1416hhh$. Conſequently, $\frac{3{,}1416cch + 3{,}1416hhh}{2} - \frac{3{,}1416hhh}{3} = S$, which being reduced will become $\frac{3cch + hhh}{2} \times 0{,}5236 = S$. Or $1{,}909855)\ 3cch + hhh\ (= S.$ for $0{,}5236)\ 1{,}0000\ (1{,}909855$. Which is one *Theorem* for finding the *Fruſtum*'s Solidity.

Note,

Note, Here we ſuppoſe the Height of the Segment, and the Diameter of its Baſe to be given; but if the *Axis* of the Sphere, and the Height of the Segment be given, then putting $D =$ the Sphere's Axis, $h =$ the Segment's Height, and c as before, 'twill be $\overline{D-h} \times h = cc$, viz. $Dh - hh = cc$. Therefore $3Dhh - 2hhh = 3cch + hhh$, conſequently $\overline{3Dhh - 2hhh} \times 0,5236 = S$, the *Fruſtum*'s Solidity. Or $1,90985) 3Dhh - 2hhh (= S$, as before. Which is a ſecond *Theorem* for finding the ſame *Fruſtum a T m*.

And if it be requir'd to find the middle Part *a m N K*, uſually call'd the middle *Zone* of a Sphere, then becauſe 'tis ſuppos'd that $a m = NK$, or which is all one, that $bC = CB$, therefore it is plain, that, if twice the Segment *a T m* be taken from the Solidity of the whole Sphere, there will remain the Middle *Zone a m N K*. But, becauſe that Work is a little troubleſome, I ſhall here ſhew how to raiſe a Theorem for the doing it.



Firſt, Becauſe $AC = yC = eC = aC = TC$. Therefore it will be $\square AC - \square Cf = \square yf$. $\square AC - \square Cd = \square ed$. $\square AC - \square Cb = \square ab$, &c. Here becauſe $\square AC$. $\square AC$. $\square AC$, &c. are a Series of *Equals*, and Cb the Number of all the Terms, therefore $\square AC \times Cb =$ the Sum of all that Series, by *Lemma* 1. And $\square Cf$. $\square Cd$. $\square Cb$, &c. being a Series of Squares whoſe *Roots* are in *Arithmetick Progreſſion*, beginning at the Center or Point C, viz. 0, Cf, Cd, Cb, &c. wherein the greateſt Term is $\square Cb$, and Number of Terms is Cb. *Ergo* $\square Cb \times \frac{1}{3} Cb =$ the Sum of all the Series by *Lemma* 3. Conſequently, the $\odot AC \times Cb - \odot Cb \times \frac{1}{3} Cb =$ the Sum of all the Series $\odot yf$. $\odot ed$. $\odot ab$, &c. which do conſtitute the Solidity of the *half Zone a m A G*. And becauſe $\square AC - \square Cb = \square ab$. *Ergo* $\odot AC - \odot ab = \odot Cb$. Conſequently $\odot AC \times Cb - \frac{\odot AC - \odot ab \times Cb}{3} = \frac{2 \odot AC + \odot ab \times \frac{1}{3} Cb}{}$ will be the Solidity of the *half Zone*.

Put $D = AG = 2AC$. $x = am$. and $H = bB = 2Cb$. Then $\odot AC = 0,7854 DD$. $\odot ab = 0,7854 xx$. And if we turn the common Factor 0,7854 into the Diviſor 1,27323,

and then take the Triple of that Diviſor, *viz.* 3,8197 (as before in the Fruſtum of Pyramids) the Reſult of the precedent Work will produce this following Theorem.

THEOR. XIX. $\left\{\frac{2DD+xx}{3,8197}:\times H=\right\}$ the *middle Zone a m NK.*

THEOREM XX.

Spheres *are in Proportion one to another as the* Cubes *of their Diameters.* (18. *e.* 12.)

Demonſtration.

Suppoſe $D =$ the *Diameter* or *Axis* of any *Sphere*, and $d =$ the Diameter of another *Sphere*, either greater or leſſer. Then is $0,5236\ DDD =$ the *Solidity* of one *Sphere*, and $0,5236ddd =$ the *Solidity* of the other *Sphere*, by *Theorem* 18. But $DDD : ddd ::$ $0,5236\ DDD : 0,5236ddd$. Q. E. D.

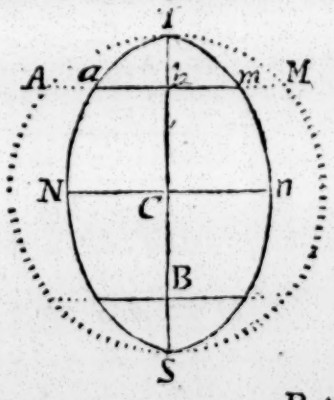
THEOREM XXI.

The ſolid Content of every Spheroid *is equal to two Thirds of its circumſcribing* Cylinder.

Demonſtration.

Suppoſe the Figure $NTnSN$, in the annex'd *Scheme*, to repreſent a *Spheroid*, form'd by the Rotation of the *Semi-Ellipſis* TNS, about its *Tranſverſe Axis* TS (as *by* Definition 15.)

Let $D = TS$, the Length of the *Spheroid*, and the *Axis* of its circumſcribing *Sphere*; and $d = Nn$, the Diameter of the greateſt Circle of the *Spheroid*. Then becauſe $\square TC : \square NC :: \square Ab : \square ab$, by Step 3 in *Theor.* 7, therefore it will be $DD : dd :: \square Ab : \square ab :: \odot Ab : \odot ab$, &c. But the Sum of an *infinite Series* of ſuch Circles as $\odot Ab$ (whoſe Diameters are Chords) do conſtitute the Solidity of the *Sphere*, (as before at *Theorem* 18) and the Sum of an *infinite Series* of ſuch Circles as $\odot ab$ (viz. whoſe *Diameters* are Ordinates of the Ellipſis) do conſtitute the Solidity of the *Spheroid*, by Definition 15. *Ergo* $DD : dd :: 0,5236\ DDD : 0,5236ddD =$ the Solidity of the *Spheroid*, by *Lemma* 6.



3 But

But 0,5236*dd D* = ⅔ of the *Cylinder* whose Diameter is = *d*, and Height = *D*, by *Theorem* 11. Q. E. D.

Now, from this *Proportion* between the *Sphere* and its inscrib'd *Spheroid*, 'twill be very easy to deduce *Theorems* for finding the *Solid Content* either of the *Segment* or middle *Zone* of any *Spheroid*, having the same Height with that of the *Sphere*.

For { *As the Solidity of the whole Sphere : is to the Solidity of the whole Spheroid : : so is any Part of the Sphere : to the like Part of the Spheroid*, by the *Converse* to *Lemma* 6.

As for Instance; suppose it were requir'd to find the middle *Zone* of any *Spheroid*: Let *D* = *TS*, and *d* = *Nn*, as above; and *H* = *b B*, *x* = *A M*, as in *Theorem* 19, and let *c* = *a m*. Then $\frac{2DD+xx}{3,8197} \times H$ = the middle *Zone* of the *Sphere*. And $0,5236DDD : 0,5236ddD :: \frac{2DD+xx}{3,8197} \times H : \frac{2dd \times H}{3,8197} + \frac{xx\,dd \times H}{3,8197DD}$ = the middle *Zone* of the *Spheroid*.

Again, *DD* : *dd* : : *xx* : *cc*, therefore $\frac{xx\,dd}{DD} = cc$. consequently, $\frac{xx\,dd}{DD} \times \frac{H}{3,8197} = \frac{cc}{3,8197} \times H$, which being taken instead of $\frac{xx\,dd \times H}{3,8197\,DD}$, there will arise this following

THEOREM XXII. $\left\{ \frac{2dd+cc}{3,8197} : \times H = \right.$ { the middle *Zone* of the *Spheroid* being the very same with *Theorem* 19.

Note, *In the same Manner you may raise* Theorems *for finding the* Segment *of a* Spheroid, *cut off at either of its Ends*, &c.

THEOREM XXIII.

The Area *of every* Parabola *is equal to two Thirds of its circumscribing* Parallelogram.

Demonstration.

Let the Figure *S A B* represent half a *Parabola*. Make *D B* parallel to the *Axis S A*, and *S d* parallel to the *Semi Ordinate A B*, and suppose *S d* to be divided into an *infinite Series* of equidistant

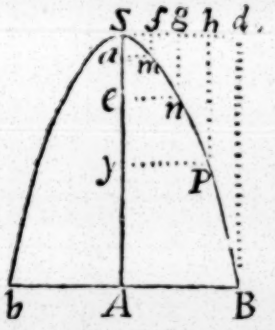
Points, as *f*, *g*, *h*, &c. and from thoſe *Points* imagine a Series of *parallel Lines*, viz. *f m*, *g n*, *h p*, &c. to touch the *Curve* of the *Parabola*, and meet the *Semiordinates m a*, *n e*, *y p*, &c. Then, according to the Property of the *Parabola*, it will

be
1 $SA : \square AB :: Sa : \square am$
2 $SA : \square AB :: Se : \square en$
3 $SA : \square AB :: Sy : \square yp$, &c.

But $Sa = fm$. $Se = gn$. $Sy = hp$. $SA = dB$

Therefore alternately it wil be

3, 4 $\square AB : dB :: \square yp : hp$
2, 5 $\square AB : dB :: \square en : gn$
1, 6 $\square AB : dB :: \square am : fm$, &c.



In theſe *Proportions* $\square a m$, $\square e n$, $\square y p$, &c. are a *Series* of *Squares* whoſe *Roots* $S f$, $S g$, $S h$, &c. are in *Arithmetick Progreſſion*, beginning at the Point *S*. And becauſe the Lines $h p$, $g n$, $f m$, &c. have the ſame *Ratio*, therefore they are as ſuch a *Series* of *Squares*, wherein $d B$ is the greateſt *Term*, and $S d$ the *Number* of *Terms*. Conſequently $\frac{dB \times Sd}{3}$ = the *Sum* of all thoſe Lines, by *Lemma* 3. But $SA \times AB = dB \times Sd$. Therefore $\frac{SA \times AB}{3}$ = the *Sum* of all that *Series* of *Lines*; but all thoſe *Lines* do conſtitute the *Area* of the *Semi Parabola's Complement*, viz. the *Area* of what half the *Parabola* wants of compleating or filling up the *Parallelogram S d A B*. Wherefore $SA \times AB - \frac{1}{3} SA \times AB = \frac{2SA \times AB}{3}$ will be the *Area* of half the *Parabola S A B*. Conſequently, $\frac{2}{3} SA \times bB$ will be the *Area* of the whole *Parabola b S B*. Q. E. D.

Example. Suppoſe the *Baſe*, or greateſt *Ordinate*, of a *Parabola* to be $b B = 24$, and its intercepted Diameter (or Axis) be $S A = 33$; then $2 S A \times b B = 66 \times 24 = 1584$. and 3) 1584 (528 the *Area* of that *Parabola*.

THEOREM XXIV.

Every Parabolick Conoid is equal to one Half of its circumſcribing Cylinder.

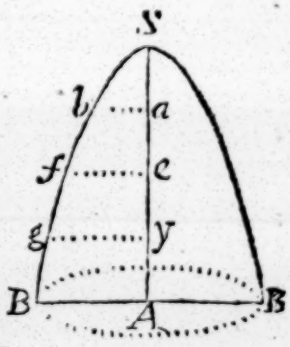
Demon-

Demonstration.

If any *Semi-Parabola* (as *B S A*) be turn'd or mov'd about its *Axis* (*S A*) 'twill form a *ſolid Parabolical Conoid*, conſtituted of an *infinite Series* of *Circles*, viz. $\odot\, b\,a$, $\odot\, f\,e$, $\odot\, g\,y$, &c. by *Definition* 17.

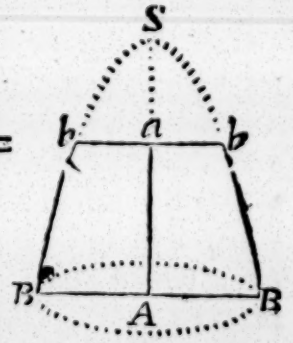
Now, according to the *Property* of every *Parabola*, it will be, $SA : AB :: AB : \frac{\square\, AB}{SA} = L$, the *Latus Rectum*.

$$\text{Then} \begin{cases} Sa \times L = \square\, ba \\ Se \times L = \square\, fe \\ Sy \times L = \square\, gy, \text{ \&c.} \end{cases}$$



Here $Sa \times L$, $Se \times L$, $Sy \times L$, &c. are a *Series* of *Terms* in Arithmetick *Progreſſion:* therefore $\square\, ba$, $\square\, fe$, $\square\, gy$, &c. are alſo a *Series* of *Terms* in the ſame *Progreſſion*, beginning at the Point *S*; wherein $\square\, AB$ is the greateſt *Term*, and *SA* the *Number* of all the *Terms*. Therefore $\square\, AB \times \frac{1}{2}\, SA =$ the *Sum* of all the *Series* by *Lemma* 2. Conſequently, $\odot\, AB \times \frac{1}{2}\, SA =$ the *Sum* of all the *Series* $\odot\, ba$, $\odot\, fe$, $\odot\, gy$, &c. which do conſtitute the *Solidity* of the *Conoid*. And putting $D = 2\,AB$, and $H = SA$. Then $0{,}7854\, DD \times \frac{1}{2} H = 0{,}3927\, DDH$ will be the ſolid *Content* of the *Conoid*, which is juſt half the *Cylinder* whoſe *Baſe* $= D$ and *Height* $= H$. [See *Theorem* 11.] Q. E. D.

This being underſtood, 'twill be eaſy to raiſe a *Theorem* for finding the lower *Fruſtum* of any *Parabolick Conoid*. For ſuppoſing $h = a\,A$ the *Height* of the *Fruſtum*, and $p = S\,a$ the *Height* of the Part $b\,S\,b$ cut off; then $h + p = S\,A$, the *Height* of the whole *Conoid*. Conſequently, $\frac{\odot\, AB \times h + .\, AB \times p}{2} =$ *Solidity* of the whole *Conoid*. And $\frac{ba + p}{2} =$ the *Solidity* of the Part cut off.



Ergo	1	$\left\{ \frac{\odot\, AB \times h + \odot\, AB \times p - \odot\, ba \times p}{2} = \right.$ the *Solidity* of the *Fruſtum*.
But	2	$h + p : \square\, AB :: p : \square\, ba$
Conſeq.	3	$h + p : \odot\, AB :: p : \odot\, ba$
3	∴ 4	$\odot\, AB \times p = \odot\, ba \times h + \odot\, ba \times p$

4 — ⊙ $ba \times p$	5	⊙ $AB \times p$ — ⊙ $ba \times p$ = ⊙ $ba \times h$
1 × 2	6	⊙ $AB \times h$ + ⊙ $AB \times p$ — ⊙ $ba \times p$ = $2F$
6 — 5	7	⊙ $AB \times h$ = $2F$ — ⊙ $ba \times h$
7 + ⊙ $ba \times h$	8	⊙ $AB \times h$ + ⊙ $ba \times h$ = $2F$
8 ÷ 2	9	$\frac{\odot AB + \odot ba}{2} \times h = F$ the *Fruſtum*'s Solidity.

Let $D = AB$, as before, and $d = 2ba$ the Diameter of the *Part* cut off; then we ſhall have this following

THEOREM XXV. { $\overline{0{,}3927\ DD + 0{,}3927\ dd} \times h$ = the *Solidity* of the *Fruſtum* requir'd.

Or { $\frac{DD + dd}{2{,}5464} \times h$ = the *Fruſtum*; for ,3927) 1,0000 (= 2,5464

and becauſe $2{,}5464 + \frac{2{,}5464}{2} = 3{,}8196$; therefore it may be made 3,8196) $\overline{DD + dd} \times \frac{1}{2} h$ (= the ſame *Fruſtum*, &c.

Note, *The Reaſon why I have reduced this* Theorem *to have the ſame* Diviſor *with thoſe at the* Fruſtums *of Pyramids*, &c. *will beſt appear farther on*, viz. *when they all come to be apply'd to Practice in* Gauging.

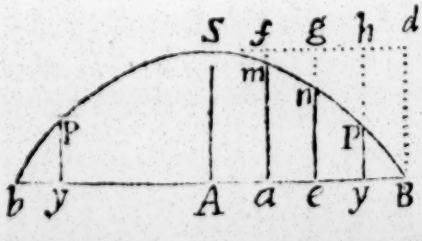
THEOREM XXVI.

Every Parabolick Spindle (or Pyramidoid) *is equal to eight Fifteenths of its circumſcribing Cylinder.*

Demonſtration.

If any *acute Parabola*, as $b\,S\,B$, be turn'd or mov'd about its greateſt Ordinate $b\,AB$, it will form a Solid call'd a *Parabolick Spindle*, conſtituted of an *infinite Series* of ⊙ ma, ⊙ ne, ⊙ py, &c. by Definition 18.

Let us ſuppoſe the Line $S\,d$, parallel to AB, &c. (as at *Theorem* 23) then it hath already been prov'd, that the Lines $f\,m$, $g\,n$, $h\,p$, &c. are a *Series* of *Squares* whoſe *Roots* are in *Arithmetick Progreſſion*: conſequently their *Squares*, viz. □ $f\,m$, □ $g\,n$, □ $h\,p$. &c. will be a *Series* of *Biquadrates*, whoſe *Roots* will be in *Arithmetick Progreſſion*: which being premiſ'd, we may proceed thus:



First, {	1	$SA - fm = ma$
	2	$SA - gn = ne$
	3	$SA - hp = py$ &c.

1 ⊙²	4	$\square SA - 2SA \times fm + \square fm = \square ma$
2 ⊙²	5	$\square SA - 2SA \times gn + \square gn = \square ne$
3 ⊙²	6	$\square SA - 2SA \times hp + \square hp = \square py$, &c.

In these *Æquations* the $\square SA$, $\square SA$, $\square SA$ being a *Series* of *Equals*, and AB the *Number* of all the *Terms*; therefore it will be $\square SA \times AB =$ the *Sum* of the *Series*, by *Lemma* 1.

2. Because fm, gn, hp, &c. are as a *Series* of *Squares* wherein SA is the *greatest Term*, and AB the *Number* of all the *Terms*; therefore $\frac{2SA \times SA \times AB}{3} = \frac{2\square SA \times AB}{3}$ will be the *Sum* of all that *Series*, by *Lemma* 3.

3. And the $\square fm$, $\square gn$, $\square hp$, &c. will be a *Series* of *Terms* in the *Ratio* of *Biquadrates*, as above; $\square dB = \square SA$ being the *greatest Term*, and AB the *Number* of all the *Terms*; therefore it will be $\frac{\square SA \times AB}{5} =$ the *Sum* of all that *Series*, by *Lemma* 5.

Whence it follows, that $\square SA \times AB - \frac{2\square SA \times AB}{3} + \frac{\square SA \times AB}{5} =$ the *Sum* of all the *Series* of $\square ma$, $\square ne$, $\square py$ &c. That is, $\frac{8\square SA \times AB}{15} =$ the *Sum* of all the *Series* of $\square ma$, $\square ne$, $\square hp$, $\square dB$. &c. consequently, $\frac{8 \odot SA \times AB}{15} =$ the *Sum* of all the *Series* of $\odot ma$, $\odot ne$, $\odot py$, &c which do constitute the *Solidity* of half the *Spindle*, viz. of SAB. Therefore putting $D = 2SA$, and $H = 2AB$, (viz. bAB) it will be $0{,}41888DDH$ = the *Solidity* of the whole *Parabolick Spindle* bSB, being $\frac{8}{5}$ of $0{,}7854\ DDH$ the Solidity of its *circumscribing Cylinder*. Q. E. D.

From hence we may also raise a *Theorem* for finding the *Frustum* $SApy$ of the last Figure. For $\odot SA$ being the greatest Term, $\odot py$ the least Term, and Ay the *Number* of all the Terms or Circles included between A and y,

Therefore	1	$\left\{\begin{array}{l}\overline{\square SA - \frac{2SA \times hp}{3} + \frac{\square hp}{5}} \times Ay = z \text{ the } Sum \\ \text{of all the } Series\ \square SA,\ \square ma,\ \square gn,\ \square py\end{array}\right.$
1×3	2	$\overline{3\square SA - 2SA \times hp + \frac{3\square hp}{5}} \times Ay = 3z$

$2 \div Ay$

$2 \div Ay$	3	$3 \square SA - 2SA \times hp + \frac{3 \square hp}{5} = \frac{3z}{Ay}$
But	4	$\square SA - 2SA \times hp = \square py - \square hp$, by 6th *Step.*
3 — 4	5	$2 \square SA + \frac{3 \square hp}{5} = \frac{3z}{Ay} - \square py + \square hp$
5 + &c.	6	$2 \square SA + \square py - \frac{2}{3} \square hp = \frac{3z}{Ay}$
Conſeq.	7	$2 \odot SA + \odot py - \frac{2}{5} \odot hp \times \frac{1}{2} Ay = z$, the *Sum*

of all the *Series* of $\odot SA$, $\odot ma$, $\odot ne$, $\odot py$, which do conſtitute the *Solidity* of the *Fruſtum* $SApy$. Therefore putting $D = 2SA$, as before. $C = 2py$. $x = 2hp$. and $H = Ay$, it will be $1,5708\,DD + 0,7854\,CC - 0,31416\,xx \times \frac{1}{3} H =$ the *Fruſtum* $SApy$. And if we make $L = 2H$. Then $1,5708\,DD + 0,7854\,CC - 0,31416\,xx \times \frac{1}{3} L =$ Double of that *Fruſtum*, being the *middle Zone*. And by turning theſe *Factors* into one common Diviſor, as in the *Fruſtum* of the *Conoid* at *Theorem* 25, *Page* 430, there will ariſe this following *Theorem.*

THEOREM XXVII.

$\left\{ \begin{array}{l} 3,8196) \, 2\,DD + CC - 0,4\,xx \times L \, (= \\ \text{the } \textit{middle Zone} \text{ of a } \textit{Parabolick Spindle.} \end{array} \right.$

It may be here expected that I ſhould now proceed to ſhew how the *Area* of any *Hyperbola*, and the *Contents* of ſuch *Solids* as may be form'd by the *Rotation* of that Figure about its *Axis*, &c. may be found; but becauſe thoſe Things cannot be exactly perform'd by any certain or ſettled *Theorem*, as theſe of the *Circle*, *Ellipſis*, and *Parabola* have been, I have therefore omitted them, and refer the Reader to Dr. *Wallis*'s *Algebra*, Chap. 90, *&c.* or to the *Philoſoph. Tranſact.* Numb. 34, wherein he may find the Method of forming *infinite Series* relating to the ſquaring of an *Hyperbola*, &c. which are too tedious to be fully explain'd and demonſtrated in this ſmall Tract, it being only intended as an *Introduction*, the which I ſhall here conclude.

F I N I S.

AN APPENDIX OF *Practical Gauging.*

THE Art of *Gauging* is that Branch of the Mathematicks called *Stereometry*, or the Meaſuring of Solids, becauſe the Capacities or Contents of all Sorts of Veſſels uſed for Liquors, *&c.* are computed as tho' they were really ſolid Bodies; which any one that hath made himſelf Maſter of the 'foregoing Parts of this Treatiſe may eaſily underſtand, without any farther Directions.

However, becauſe 'tis not to be ſuppos'd that every one, who deſigns to undertake the Office or Employment of a Gauger, hath made ſo great a Progreſs in Mathematical Learning, I have therefore preſented the young Gauger with this Appendix, wherein I have only inſerted ſuch Rules as are uſeful in Gauging, and have been already demonſtrated in this Treatiſe. But herein, I preſuppoſe that he hath acquir'd (or if not, 'tis very neceſſary he ſhould acquire) a competent Knowledge both in Arithmetick and Geometry: That is,

I. In Arithmetick he ſhould underſtand the principal Rules very well, eſpecially Multiplication and Diviſion, both in whole Numbers and Decimal Parts, (which may be eaſily learnt out of the 2d, 3d, and 5th Chapters of Part 1.) that ſo he may be ready at computing the Contents of any Veſſel, and caſting up his Gauges by the Pen only, *viz.* without the Help of thoſe Lines of Numbers upon Sliding-Rules, ſo much applauded, and but too much practis'd, which at beſt do but help to gueſs at the Truth; I mean ſuch Pocket-Rules as are but nine Inches (or a Foot) long, whoſe Radius of the double Line of Numbers is not ſix Inches; and therefore the Graduations or Diviſions of thoſe Lines are ſo very cloſe, that they cannot be well diſtinguiſh'd. 'Tis true, when the Rules are made two or three Feet long (I had one of ſix Feet) there they may be of ſome Uſe, eſpecially in ſmall Numbers; altho' even then the Operations may be much better (and almoſt as ſoon) done by the Pen: For, indeed, the chief Uſe of Sliding-Rules is only in taking of Dimenſions, and for that Purpoſe they are very convenient.

II. In Geometry the Gauger ſhould underſtand not only how to take Dimenſions (which is beſt learnt by Practice) but alſo how to divide any irregular Figure or Superficies, as Brewers Backs or Coolers, *&c.* into the eaſieſt and feweſt regular Figures they will admit of, that ſo their Area's may be truly computed with the leaſt Trouble. And this may be learn'd (with a little Care and Diligence) out of the 1ſt, 2d, and 5th Chapters of Part III, which the Gauger ſhould be well acquainted with. Alſo he ought to have ſo much Skill in Solids, as to be able, even at ſight (but this muſt be acquired by Experience) to determine what ſort of Figure any Veſſel is of (*viz.* any Tun or cloſe Caſk) or what Figures it may be beſt reduced to, ſo that its Dimenſions may be truly taken, and the Content thereof computed with the leaſt Error. I ſay, with the leaſt Error, becauſe 'tis very difficult, if not impoſſible, to do it exactly; for there is not any Tun, or cloſe Caſk, *&c.* ſo regularly made, as by the Rules of Art 'tis requir'd to be.

III. Beſides the aforemention'd, the young Gauger muſt know, that all Dimenſions uſeful in Gauging are to be taken in Inches, and Decimal Parts of an Inch; and if they are taken in any other Meaſures, as Feet, Yards, *&c.* thoſe Meaſures muſt be reduced to Inches, (ſee Sect. 4. Page 42.) becauſe the Contents of all Sorts of Veſſels (taken Notice of in Gauging) are computed by the Standard Gallon of its Kind, whoſe Content is known to be a certain Number of Cubick Inches: That is, the Beer or Ale Gallon contains 282, the Wine 231, and the Corn Gallon 268, 8 Cubick Inches. [See the five Tables, *&c.* in Pages 34, 35, 36, which I here ſuppoſe the Gauger to have learnt perfectly, by heart.] Conſequently, if either the ſuperficial or ſolid Content of any Veſſel, as Back, Tun, Caſk, *&c.* be once computed in Cubick Inches, 'twill be eaſy to know how many Gallons, either of Ale, Wine, or Corn, that Veſſel will hold.

Note, I have here ſaid, the Superficial Content in Cubick Inches, which may ſeem to be very improper, according to the Definition given of a Superficies in Page 279; but you muſt know, that, in the Buſineſs of Gauging, all Superficies or Area's are always underſtood to be one Inch deep, otherwiſe it could not be ſaid (as in the Gauger's Language it is) that the Area of ſuch a Back, or of ſuch a Circle, *&c.* is ſo many Gallons.

Theſe Things being very well underſtood, the young Gauger will be fitly prepar'd to underſtand the following *Problems*, which are ſuch as have (moſt of them) been already propos'd in the 'foregoing Parts of this Treatiſe, and only are here apply'd to Practice; and therefore I ſhall, for Brevity's Sake, often refer to thoſe Theorems and Problems.

Sect.

Sect. 1. To find the *Area* of any right-lined Superficies in Gallons.

PROBLEM I.

To find the Area *of any* ſquare Tun, Back, *or* Cooler, *&c. either in* Ale, Wine, *or* Corn Gallons.

RULE. { Multiply the given Length or Breadth (being here equal) into itſelf, and the Product will be the Area in Inches; then divide that Area by 282, or 231, or 268,8 and the Quotient will be the Area requir'd.

Example. Suppoſe the Side of a Square Tun, Back, or Cooler be 124,5 Inches, what will its Area be in Gallons?

Firſt $124,5 \times 124,5 = 15500,25$ the Area in Inches.

Then 282, And 231, Or 268,8 } 15500,25 { 54,96 *&c.*, 76,10 *&c.*, 57,66 *&c.* } the *Area* in { *Ale Gallons.*, *Wine Gallons*, *Corn Gallons.*

But if any one would rather work by Multiplication than by Diviſion, he may turn or change any Diviſor into a Multiplicator, if he divide Unity, or 1, by that Diviſor. (Vide Probl. 3. Pag. 402.)

Thus 282, And 231, Or 268,8 } 1,000000 { 0,003546, 0,004329, 0,003722 } the Multipli. for { *Ale Gallons.*, *W. Gallons.*, *C. Gallons.*

Conſequently $15500,25 \times 0,003546 = 54,96$ *&c.* the Area in Ale Gallons; as before and ſo on for the reſt.

PROBLEM II.

To find the Area *of any* Tun, Back, *or* Cooler *in the Form of a Right-angled Parallelogram in* Ale Gallons, *&c.*

See the Rule for finding its Area in Inches, at Probl. 1. P. 339. then either divide (or multiply) that Area as above, and you will have the Area in Gallons.

Example. Suppoſe the Length of a Brewer's Tun, Back, or Cooler be 217,5 Inches, and its Breadth 85,6 Inches, what will its Area be in Ale or Beer Gallons, *&c.*?

Firſt $217,5 \times 85,6 = 18648$. Then 282) 18648 (66,12, *&c.*

Or $18648 \times 0,003546 = 66,12$, *&c.* the *Area* requir'd, *&c.*

PROBLEM III.

To find the Area *of any* Triangular Tun, Back, *or* Cooler, *in* Ale Gallons, *&c.*

See the Rule for finding its Area in Inches at Prob. 3, p. 340; then divide (or multiply) that Area as before, and you will have the Area requir'd.

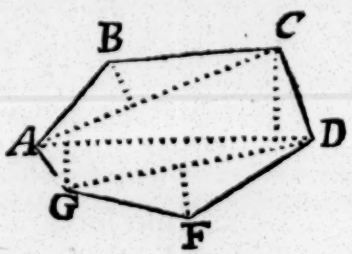
Example. If the Length of the Baſe of a Triangular Cooler be 86,4 Inches, and its Perpendicular Breadth be 57 Inches, what will its Area be in Ale Gallons?

Firſt, 86, $4 \times \frac{67}{2} = 2462{,}4$. Then 282) 2462,4 (8,73 *&c.* Or $2462{,}4 \times 0{,}003546 = 8{,}73$ *&c.* the Area in Ale Gallons.

Proceeding thus, you may eaſily find the Area of any Tun, Back, or Cooler, whether it be in the Form of a Rhombus, Rhomboides, Trapezium, or any other Polygon, either regular or irregular, in Ale or Beer Gallons, *&c.* if you firſt divide it into Triangles, and then find the Area's of thoſe Triangles; (as in the 2d, 4th, 5th, and 6th Problems in Chap. 5, Part III.) the Sum of thoſe Area's being divided (or multiply'd) by its proper Diviſor (or Multiplicator) as above, will give the Area requir'd.

Now, the Practical Way of dividing any Polygonous Tun, Back, *&c.* into Triangles, is by help of a chalk'd Line, ſuch as the Carpenters uſe, and may be thus perform'd.

Suppoſe any Brewer's Tun, Back, or Cooler in the Form of the annex'd Figure *ABCDFG*. Let one End of the chalk'd Line be faſten'd with a Nail (or otherwiſe) in any Corner or Angle of the Back, as at *A*; then ſtraining it to the Angle at *C*, ſtrike the Diagonal Line *AC*, upon the Bottom of the Back; and ſtraining it again to the Angle *D*, ſtrike another Diagonal Line, as *AD*, and ſo on for the Diagonal Line *GD*, &c. Then having mark'd out all the Diagonals, the Perpendiculars may be thus found: Faſten (as before) one End of the chalk'd Line in the Angle *B*, and then, by moving it *to* and *fro* upon the Stretch, find out the neareſt Diſtance between the Angle at *B* and the Diagonal Line *AC*; and there ſtrike a Line, and it will mark out the Perpendicular from *B* to the Line *AC*, and ſo on for the other Perpendiculars: Which being all mark'd out upon the Bottom of the Back, meaſure them and each Diagonal by a Line of Inches,



Inches, &c. and then the Area of that Back may be computed, as directed above.

And here, by the Way, it may be obſerved, that the Number of Triangles will always be leſs by two, and the Number of the Diagonals leſs by three, than the Number of the Sides of any Right-lin'd Figure that is ſo divided.

Having found (as above) the true Area of any Brewer's Back or Cooler (which, according to the Laws of Exciſe, ought always to be fix'd or immoveable) the next Thing will be to find out the true dipping or gauging Place in that Back, that ſo the true Quantity of Worts may be computed or (caſt up) at any Depth; which may be thus done.

1. When the Bottom of the Back is covered all over (of any Depth) either with Worts or Liquor (viz. *Water)* then dip it in eight or ten ſeveral Places (more or leſs according to the Largeneſs of the Back) as remote and equally diſtant one from another as you well can, noting down the wet Inches and decimal Parts of every Dip.

2. Divide the Sum of all thoſe Dips or wet Inches by the Number of Places you dipp'd in, and the Quotient will be the mean Wet of all thoſe Dips.

3. Laſtly, find out ſuch a Place by the Side of the Back (if you can) that juſt wets the ſame with that mean Dip, and make a Notch or Mark there, for the true and conſtant Dipping-place of that Back. Then if any Quantity of Worts (which do cover the whole Back) be dipp'd or gaug'd at that Place, and the wet Inches ſo taken be multiply'd into the Area of the Back in Gallons, the Product will ſhew that Quantity *(viz.* how many Gallons) of Worts are in that Back at that Time, provided the Sides of the Back do ſtand at Right Angles with its Bottom.

Sect. 2. *To find the* Area *of any* Circular *and* Elliptical Superficies *in Gallons.*

1. I have demonſtrated in Cap. 6, Part III, and Theorem 3, 5, 6. Part V. that the Periphery of the Circle whoſe Diameter is Unity, or 1, is 3,14159265 *&c.* (or for common Uſe 3,1416) and that its Area is 0,78539816 *&c.* (or 0,7854 *fere.*)

2. Alſo, that the Peripheries of all Circles are in Proportion one to another as their Diameters are; and their Area's are in Proportion to the Squares of the Diameters. That is, as 1 : 3,1416 : : the Diameter of any Circle : to its Periphery. And 1 : 0,7854 : : the Square of the Diameter : to the Area.

Upon

Upon theſe two Proportions depend the Solutions of all the common or practical Queſtions about a Circle. [*See* Page 408, 409.]

PROBLEM IV.

The Diameter *of any* Circle *being given in Inches, to find the* Periphery.

RULE. { Multiply the given Diameter with 3,1416, and the Product will be the Periphery requir'd. [*See* Prob. 1. p. 408.]

Example. Suppoſe the Diameter of a Circle be 54,5 Inches, and it were requir'd to find its Periphery. Then 54,5×3,1416 = 171,21, *&c.* Inches is the Periphery requir'd. The Converſe of this is eaſy, *viz.* by having the Periphery given, to find the Diameter. [*See* Prob. 3. Page 408.]

PROBLEM V.

The Diameter *of any* Circle *being given* (in Inches) *to find its* Area *in Gallons.*

RULE. { Multiply the Square of the propos'd Diameter into 0,7854, and the Product will be the Area in Inches; [*See* Probl. 2, P. 408.] that Area being divided by 282, or 231, *&c.* the Quotient will be the Area required.

Example. Suppoſe the given Diameter be 54,5 Inches as above. Firſt, 54,5 × 54,5 = 2970,25. And 2970,25 × 0,7854 = 2332,83 the Area in Inches.

Then 282, And 231, Or 268,8 } 2332,83 { 8,2724, 10,0988, 8,6788 } the Area in { *Ale* or *Beer* Gallons. *Wine* Gallons. *Corn* Gallons.

But theſe Area's in Gallons may be much eaſier found without knowing the Circle's Area in Inches, as above, by having the Square of the Diameter of that Circle whoſe Area is one Gallon; which may be thus found, by Theorem 6, Page 407.

0,785398 : 1 : : 282 : 359,05 the Square of the Diameter of the Circle whoſe Area is 282 cubick Inches, *viz.* one *Ale* Gallon.

And from this Proportion will ariſe the following Diviſors;

viz. 0,785398 { 282,000000 (359,05, 231,000000 (294,12, 268,800000 (342,24 } will be a *Diviſor* for { *A. G.* *W. G.* *C. G.*

If

If the Square of the Diameter of any Circle be divided by any one of theſe conſtant or fixed Diviſors, the Quotient will ſhew that Circle's Area in their reſpective Gallons. As for Inſtance, in the laſt Circle, whoſe Square of its Diameter is 2970,25.

Then 359,05
And 294,12
Or 342,24
} 2970,25 {
8,2725
10,0988
8,6788
} the *Area* in {
A. G.
W. G.
C. G.
} as before.

Now theſe Diviſors may be turn'd into Multiplicators by dividing Unity or 1, as in Page 435: Or rather by dividing the Area in Inches of that Circle whoſe Diameter is 1.

That is, 0,785398 by 282. Or by 231, *&c.*

Thus 282
And 231
Or 268,8
} 0,785398 {
0,002785
0,003399
0,002922
} the *Multiplicator* for {
Ale Gal.
Wine Gal.
Corn Gal.

Theſe Multiplicators are the reſpective Area's of a Circle whoſe Diameter is 1; and therefore, if the Square of the Diameter of any Circle be multiplied with any of theſe Numbers, the Product will be that Circle's Area in Gallons of the ſame Name:

Viz. 2970,25×0,002785 = 8,2725 the Area in *A. G.* as above.
And 2970,25×0,003399 = 10,0988 the Area in *W. G. &c.*

Thus you ſee, that if the Diameter of any Circle be given in Inches, there are three ſeveral Ways of finding its Area in Gallons, and all equally true; but that which is perform'd by the conſtant Diviſors is moſt generally practis'd.

PROBLEM VI.

The Tranſverſe (or longeſt Diameter) *and the Conjugate* (or ſhorteſt Diameter) *of any* Elliptical Superficies *being given, to find its* Area *in Gallons.*

RULE. Multiply the two Diameters (*viz.* the Length and Breadth) together, and divide their Product by 359,05 for Ale Gallons, or 294, 12 for Wine Gallons, *&c.* the Quotient will be the Area requir'd. [*See* Theorem 7, Page 412.]

Example. Suppoſe the longeſt Diameter to be 73,5 Inches and the ſhorteſt Diameter to be 51,6 Inches; what will the *Area* be in Ale Gallons?

Firſt 73,5×51,6 = 3792,6. Then 359,05) 3792,6 (10,56 the Area in Ale Gallons. Or 294,12) 3792,6 (12,89 the Area in Wine Gallons, *&c.*

Note, The two laſt Problems are of a great Uſe in Gauging of Worts amongſt Country Victuallers, who generally brew but ſhort Lengths of Ale (perhaps between 20 and 60 Gallons at a Brewing) and cool their Worts in ſeveral ſmall open Veſſels or Tubs, whoſe Baſes or Bottoms are either a Circle, or an Ellipſis, having their Sides but low, and are moſt commonly wider at the Top than at the Bottom.

Now a practical Way of computing the Quantity of Worts, that are at any Time in one of thoſe open Tubs, is briefly thus: When the Tub is dry, find the true Area of its Bottom according to its Figure (as above) and either mark that Area on the Outſide of the Tub (which was the Way I generally us'd to order, becauſe the Victuallers did often lend their cooling Tubs one to another) or elſe number the Tub, and enter its Area (and its Number) into the Stock-book; then, when any of thoſe Tubs hath Worts in it, take the Diameter of the Surface or Top of the Worts, and find that Area, adding it and the bottom Area together. If either the half Sum of thoſe two Area's be multiply'd with the Depth of the Worts (taken as near the Middle of the Tub as you well can) or, if the Sum of thoſe two Area's be multiply'd with half the Depth (ſo taken) the Product will ſhew the Quantity of thoſe Worts very near the Truth.

PROBLEM VII.

The Diameter *of any* Circle, *and the verſed Sine*, (viz. the Height *of any Segment*) *being given, to find the* Area *of that Segment in* Gallons.

In the 410th and 412th Pages you have two Ways (and their Examples) of finding the Area of any Segment of a Circle in Inches; then if that Area in Inches be divided by 282, or 231 *&c.* the Quotient will be its Area in Gallons. But becauſe the Area of any ſuch Segment may be readily found in Gallons (without finding its Area in Inches) by help of a Table of Segments, whoſe Conſtruction is laid down in the Problem, Page 411, *&c.* I have here inſerted a Compendium of ſuch a Table, which will ſerve very well for common Practice, not only to find the Area of any Segment of a Circle in Gallons, but alſo to find the Number of Gallons that are either drawn out, or remaining in any Cylindrick Veſſel lying along; or of any cloſe Caſk (being firſt reduced to a Cylinder) its Axis lying parallel to the Horizon, uſually call'd the *Ullage* of a Caſk; as ſhall be ſhew'd farther on.

A Table

A Table of the Segments of a Circle whoſe Area is Unity or 1, the Diameter being divided by parallel Chord-Lines into 100 equal Parts.

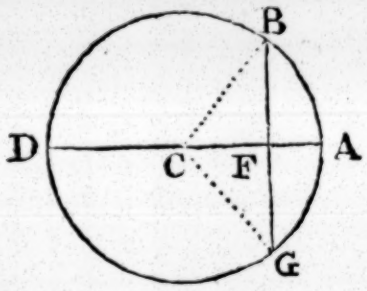
V.S.	*Segment.*	*V.S.*	*Segment.*	*V.S.*	*Segment.*	*V.S.*	*Segment.*
1	0,0017	26	0,2066	51	0,5127	76	0,8155
2	0,0048	27	0,2178	52	0,5255	77	0,8262
3	0,0087	28	0,2292	53	0,5382	78	0,8369
4	0,0134	29	0,2407	54	0,5509	79	0,8474
5	0,0187	30	0,2523	55	0,5635	80	0,8576
6	0,0245	31	0,2640	56	0,5762	81	0,8677
7	0,0308	32	0,2759	57	0,5888	82	0,8776
8	0,0375	33	0,2878	58	0,6014	83	0,8873
9	0,0446	34	0,2998	59	0,6140	84	0,8968
10	0,0520	35	0,3119	60	0,6265	85	0,9059
11	0,0598	36	0,3241	61	0,6389	86	0,9149
12	0,0680	37	0,3364	62	0,6514	87	0,9236
13	0,0764	38	0,3486	63	0,6636	88	0,9320
14	0,0851	39	0,3611	64	0,6759	89	0,9402
15	0,0941	40	0,3735	65	0,6881	90	0,9480
16	0,1032	41	0,3860	66	0,7002	91	0,9554
17	0,1127	42	0,3986	67	0,7122	92	0,9625
18	0,1224	43	0,4112	68	0,7241	93	0,9692
19	0,1323	44	0,4238	69	0,7360	94	0,9755
20	0,1424	45	0,4365	70	0,7477	95	0,9813
21	0,1526	46	0,4491	71	0,7593	96	0,9866
22	0,1631	47	0,4618	72	0,7708	97	0,9913
23	0,1738	48	0,4745	73	0,7822	98	0,9952
24	0,1845	49	0,4873	74	0,7934	99	0,9983
25	0,1955	50	0,5000	75	0,8045	100	1,0000

The Uſe of this Table of Segments depends upon the following Proportion,

viz. As the Diameter of any propos'd Circle : is to 100 (the Diameter of the tabular Circle) : : ſo is the Height of any Segment of the propos'd Circle : to a verſed Sine in the Table.

Then, if the tabular Segment, which ſtands againſt that verſed Sine, be multiply'd into the Circle's Area (either in Inches or Gallons) the Product will be the Area of the Segment requir'd [of the ſame Name] *viz.* If the Circle's Area be Inches, the Segment will be Inches; if Gallons, the Segment will be Gallons.

Example. Let the Diameter of the given Circle be $DA = 62,5$ Inches, and the Height of the Segment ſought be $FA = 20$ Inches; What will its Area be in Ale Gallons?



Firſt, the Area of the whole Circle will be 10,8793 Ale Gallons (by Problem 5.) and the Proportion will ſtand thus, 62,5 : 100 :: 20 : 32 the verſed Sine of the Table whoſe Segment is 0,2759. Then, $10,8720 \times 0,2759 = 3,0016$ Ale Gallons, being the Area of the Segment *BAGF*, as was requir'd. The like may be done for Wine Gallons, Corn Gallons, or Inches.

And, upon Occaſion, the like Segments of any Ellipſis may be eaſily found. See the *Proportions* in the Corollaries to the 7th and 8th Theorems, Page 412, *&c.* to which I here, for Brevity's Sake, refer the Reader.

Sect. 3. To compute the Contents of ſuch Veſſels *(viz.* Tuns, *&c.)* as are in the Form of the following Solids.

Note, Before the young Gauger proceeds to theſe Computations, he ſhould be well acquainted with ſuch Solids as are defin'd in P. 402 and 403, and then he may eaſily underſtand what Sort of Figures are meant in the following Problems, without the Repetition of many Words.

PROBLEM VIII.

To find the Content of any Priſm *whoſe Sides are* Parallelograms, *what Form ſoever its Baſe is of.*

That is, to compute the Content (in Gallons) of any Tun, &c. whoſe Sides are Parallelograms which ſtand upright, or at Right Angles with its Bottom.

Firſt, find its ſolid Content in Inches, by Theorem 9, Page 414; then divide that Content by 282, or 231, or by 268,8; the Quotient will ſhew the Content in their reſpective Gallons, *viz.* in Ale, Wine, or Corn Gallons.

Or elſe multiply the Content in Inches with 0,003546, or 0 004329, *&c.* [See the Multiplicators, Page 435] thoſe Products will be the Content in their reſpective Gallons.

Or otherwiſe thus:

Find the true Area of the Tun's Baſe or Bottom, as directed in Sect. 1, P. 435; that Area being multiply'd with the Tun's Height (*viz.* Depth within) will produce the Content in Gallons, as before. I take

I take the Work of this Problem to be ſo very eaſy, that it needs no Example.

PROBLEM IX.

To find the Content of any Pyramid (in Gallons) *whoſe Baſe is bounded with Right Lines.*

Every Pyramid is one Third-part of its circumſcribing Priſm, by Theorem 10, Page 415. Therefore, if the Area of the Baſe of any Pyramid, in Gallons, be multiply'd in one Third of its perpendicular Height; or if one Third of that Area be multiply'd with the whole Height, either of thoſe Products will be the Content of the Pyramid in Gallons, *&c.* But the Content of any ſquare Pyramid may be eaſily found in Gallons by this Rule:

RULE. Square the Side of its Baſe, and multiply that Square with the perpendicular Height; then divide that Product by $846 = 282 \times 3$ for Ale Gallons, or by $693 = 231 \times 3$ for Wine Gallons, or by $806{,}4 = 268{,}8 \times 3$ for Corn Gallons, the Quotient will be the Content requir'd.

Or, if you multiply the ſaid Product with 0,001182 for *A G.* or with 0,001443 for *W G.* or, laſtly, with 0,001241 for *C. G.* the Reſult will be the Content requir'd, as before.

PROBLEM X.

To find the Content (in Gallons) *of the* Fruſtum *of any ſquare Pyramid, cut off by a Plain parallel to its Baſe.*

Firſt, either by Theorem 15, Page 419, or Theorem 16, P. 420, find the propos'd Fruſtum's Solidity in Cubick Inches; then divide that Content in Cubick Inches by 282 or 231, *&c.* and the Quotient will be the Content of the Fruſtum in their reſpective Gallons.

But, from the foreſaid Theorem 15, there may be eaſily deduced the following general Rule for finding the Content of the like Fruſtum of any Pyramid, what Form ſoever its Baſes are of (ſuppoſing them to be parallel) whether they are alike or unlike.

RULE. Firſt, find the Area of each Baſe, (*viz.* the top and bottom Area's of the propos'd Fruſtum;) then find a Geometrical Mean between thoſe two Area's (by Lemma 1, Page 83;) the Sum of thoſe two Area's and their Mean, being multiply'd into one Third of the Fruſtum's Height, will produce the Content required.

Example. Suppoſe a Tun in the Form of the lower Fruſtum of a Pyramid, whoſe Baſes are Equilateral Triangles: Let the Side of the Top be 42 Inches, the Side of the Bottom be 63,4 Inches, and its Height [*viz* Depth] be 33 Inches; What will the Content of that Tun be in Ale Gallons?

Firſt, find the Area of that Baſe in Inches, by Probl. 7, P. 343; then find what thoſe Area's are in Ale Gallons, by Probl. 3, P. 436. Multiply thoſe two Area's together and the ſquare Root of their Product will be the mean Area, *&c.* as in this Example:

Example. The { Top / Bottom / Mean } Area is { 2,71 / 6,12 / 4,07 } Ale Gallons.

Their Sum 12,90

Then $12,9 \times \frac{23}{3} = 141,9$. Or $\frac{12,9}{3} \times 33 = 141,9$ the Content required.

PROBLEM XI.

To find the Content of any right Cylinder in Gallons.

That is, to compute the Content of any round Tun, *&c.* whoſe Diameters at Top and Bottom are equal, and at Right Angles with its Sides.

The Content of ſuch a Tun may be found by Theorem 11, Page 415; or otherwiſe by the following Rule.

RULE. Multiply the Square of the Diameter into the Height, and divide the Product by 359,05 (or multiply with 0,002785) *&c.* as in Page 439, that Quotient (or Product) will be the Content required.

Exam. Suppoſe the Diameter be 42,5, and the Height 31,5 Inches. Firſt $42,5 \times 42,5 = 1806,25$. And $1806,25 \times 31,5 = 56896,875$. Then 359,05) 56896,875 (158,46 the Content in Ale Gal. *&c.*

PROBLEM XII.

To find the Content of any Cone *or* round Pyramid *in Gallons.*

Becauſe every Cone is one Third of its circumſcribing Cylinder, [See Theorem 13, Page 416] therefore its Content may be truly found by the following Rule.

RULE. Multiply the Square of the Diameter of its Baſe into the perpendicular Height, then divide their Product by $1077,15 = 359,05 \times 3$ for Ale Gallons, or by $882,36 = 294,12 \times 3$ for Wine Gallons, *&c.* and the Quotient will be the Content required.

Or if the ſaid Product be multiply'd with 0,000928 = $\frac{0,002785}{3}$ or with 0,001133 = $\frac{0,0034}{3}$ thoſe Products will be the Content in their reſpective Gallons.

Example. Suppoſe the Diameter of the Baſe be 42,5, and the perpendicular Height be 31,5 Inches, what will the Content be in Ale Gallons? (as before.

Firſt 42,5×42,5 = 1806,25. And 1806,25×31,5 = 56896,875
Then 1077,15) 56896,875 (52,82. Or 56896,25×0,000928 = 52,82 the Content in Ale Gallons. And ſo on for Wine or Corn Gallons.

PROBLEM XIII.

To find the Content of the lower Fruſtum *of any Cone in Gallons.*

That is, to compute the Content of any round Tun, *&c.* whoſe Diameters at Top and Bottom are parallel, but unequal.

The Content of ſuch a Tun may be found by the Rule at Problem 10; but from Theorem 16, Page 420, 'twill be eaſy to deduce this following Rule.

RULE. To the triple Product of the Top and bottom Diameters add the Square of their Difference; multiply that Sum into the Height (or Depth): then divide the laſt Product by 1077,15 for Ale Gallons, or by 882,36 for Wine Gallons; the Quotient will be the Content requir'd.

Example. Suppoſe the Diameter at the Top to be 52,4 Inches, the Diameter at the Bottom 45,6, and the Height 30 Inches.

Firſt, 52,4×44,6 = 2337,04; and 2337,04×3 = 7011,12
Alſo, 52,4−44,6 = 7,8; and 7,8 × 7,8 = 60,84 } Add
The Height 30 × 7071,96 = 212158,8.

Then 1077,15) 212158,8 (196,96
Or 212158,8×0,000928 = 196,96 } the *Content* in *Ale* Gallons.

And ſo on for either Wine or Corn Gallons, as Occaſion requires. But if the Tun (or Veſſel) be not truly circular, that is, if either its Top or Bottom (or both of'em) be Elliptical, whether they are alike or unlike, it matters not, the Content of ſuch a Tun may be truly found by the general Rule at Problem 10.

PROBLEM XIV.

The Axis *or* Diameter *of any Sphere or Globe being given in* Inches, *to find its Content in* Gallons.

Every Sphere is two Thirds of its circumſcribing Cylinder, by Theor. 18, Page 423; from whence and Theor. 20, Page 426, 'tis proved,

proved, that if the Cube of the Axis of any Sphere (taken in Inches) be multiply'd into 0,5236, the Product will be the Content of that Sphere in Inches. Conſequently, if that Content be divided by 282, or by 231, *&c.* the Quotient will be the Content in Gallons.

But thoſe two Works of multiplying with 0,5236, and then dividing by 282, or by 231, *&c.* may be contracted into one.

Thus 282 / And 231 } 0,5236 { 0,001856 / 0,002266 } will be a *Multiplicator* for { *A. G.* / *W. G.*

Or 0,5236 { 282 / 231 { 538,57 / 441,17 } will be a *Diviſor* for { *Ale* Gallons. / *Wine* Gallons.

From hence ariſes this following Rule.

RULE. If the Cube of the Axis of any Sphere be divided by 538,57; or multiply'd with 0,001856: or divided by 441,17; or elſe multiply'd with 0,002266; the Quotient, or Product, will be the Sphere's Content in their reſpective Gallons.

Example. Suppoſe the Axis or Diameter of a Sphere or Globe be 22 Inches, how many Ale Gallons may it hold?

Here 22×22×22 = 10648; and 538,57) 10648 (19,76 *A.G.*
Or 10648 × 0,001856 = 19,76 Ale Gal. the Content required.
And ſo for either Wine or Corn Gallons, as Occaſion requires.

PROBLEM XV.

To find the Content of a Segment *of a* Sphere *in Gallons.*

In the Scholium, P. 424, there are two Theorems for reſolving this Problem according to the Data.

1. If the Diameter of the Segment's Baſe and its Height are given, the Content may be found by the firſt of thoſe Theorems, which gives this Rule:

RULE 1. To the Triple Square of half the Diameter add the Square of the Height; then multiply that Sum into the Height, and divide the Product by 538,57 for *AG*, or by 441,17 for *WG*, &c. as above.

2. But if the Axis of the Sphere and the Height of the Segment are given, the Content may be found by the Second of thoſe Theorems.

RULE 2. From the triple Product of the Axis into the Height ſubtract twice the Square of the Height; then multiply the Remainder into the Height, and divide that Product by 538,57, *&c.* as in the laſt Problem.

Either

Either of theſe Rules will produce the Content of the Segment in Gallons.

Example. Suppoſe the Diameter of the Segment's Baſe be 28 Inches, and its Height be 8 Inches, what may it contain in Ale Gallons?

Firſt 2) 28 (14. Then (by Rule 1.) $14\times14\times3 = 588$. And $6\times6 = 36$. Next $588 + 36 = 624$. Again $624\times6 = 3744$. Laſtly, 538,57) 3744 (6,95 the Content required.

Note, This Problem may be of Uſe in Gauging the Crowns of Brewers Coppers, *&c.*

Sect. 4. *The Practical Method of Gauging any fix'd Tun or Copper, and making a Table to ſhew what it will hold at every Inch deep, uſually called Inching of a Tun*, &c.

Firſt, you muſt know, that moſt (if not all) Brewers Tuns are ſo fix'd as to lean a little for Conveniency of cleanſing their Drink, which is uſually call'd the Drip or Fall of the Tun. Now this Drip or Fall of any Tun is the Hoof of ſuch a Solid as that Tun is ſuppos'd to repreſent, and under that Conſideration it may be found, as in Theor. 16, P. 420: But the practical (and indeed the beſt) Way is, to meaſure into the Tun (when 'tis dry) ſo much Liquor as will juſt cover its Bottom; for by that means you do not only find the true Fall, but alſo a true horizontal or level Plain over the Bottom of the Tun; from which if the Depth of the Tun (*viz.* the neareſt Diſtance from the Top of the Tun to the Surface of the Liquor) be ſet off upon every one of its Sides, you will then have a true parallel Plain at the Top of the Tun to that of the Liquor. Then, if the Sides of the Tun are ſtreight from the Top to the Bottom, take as many Dimenſions in the aforeſaid two Plains as are needful to find the true Area of each; and by thoſe two Area's and the foreſaid Depth find ſo much of the Tun's Content (by the general Rule at Problem X.) as is betwixt thoſe two Plains.

Next, to inch that Tun, divide the Difference between the Top and Bottom Area's by the aforeſaid Depth, and the Quotient will be an Addend or fixed Number; which being added to the leſſer Area, the Sum will be the Area of the next Inch; and, being added to that Area, their Sum will be the Area of the third Inch; and ſo on from Inch to Inch, until the Area of every ſingle Inch be found; the Sum of thoſe Area's (if the Work be true) will amount (or be equal) to the Content found, as above. And if the

the Tun's Drip or Fall be added to the Sum of all thoſe Area's, that Sum will be the whole or full Content of that Tun.

Now, from hence it muſt needs be eaſy to conceive, that if 1, 2, 3, or any Number of thoſe Area's accounted from the Bottom, be added to the Fall, that Sum will ſhew the Quantity of Liquor or Drink that is in the Tun, to ſuch a Number of wet Inches from the Bottom as there were Area's added together. Or, if the Sum of any Number of thoſe Area's (being accounted from the Top) be ſubtracted from the Tun's whole Content, the Remainder will ſhew what Quantity of Liquor or Drink is in the Tun, when there is ſuch a Number of dry Inches from the Top as there were Area's ſubtracted.

This being well conſider'd, it will be eaſy to make a Table either to every wet or dry Inch of any regular Tun (*viz.* whoſe Sides are ſtreight from Top to Bottom) what Form ſoever its Baſes are of, and whether it ſtand upon the greater or leſſer Baſe.

But if the Sides of the Tun are irregular (*viz.* not ſtreight from its Top to the Bottom) then the beſt and eaſieſt Way will be to divide or part the Tun into ſeveral Fruſtums, each of ten Inches deep; and finding the Content of every ſingle Fruſtum, by taking the Diameters in the Middle of every one of thoſe ten Inches (that is, the firſt Diameter at 5 Inches from the Top; the ſecond Diameter at 15 Inches from the Top, *&c.*) and multiplying their reſpective Area's with 10, (which is done by only removing the ſeparating Comma's one Place forward to the right Hand) if the Sum of all thoſe Fruſtums to be added to the Fall, (as before); that Sum will be the whole Content of the Tun.

Note, If you take the Height of the 'foreſaid ten Inch Fruſtums in the Side of the Tun, you muſt allow for the Difference between the ſlant Height and the Perpendicular Height in every Fruſtum.

Laſtly, If from the whole Content of the Tun you ſubtract the mean Area of the firſt Fruſtum ten Times, and from the Remainder ſubtract the mean Area of the ſecond Fruſtum ten Times, and from the laſt Remainder ſubtract the mean Area of the third Fruſtum, *&c.* until there remain nothing but the Fall or Hoof of the Tun, you will then by that Means have a Table that will ſhew what Quantity of Drink is in the Tun to any Number of dry Inches.

And this is alſo the Method of Gauging and Inching Brewers Coppers, *viz.* by firſt meaſuring into the Copper ſo much Liquor as will juſt cover its Crown, and then dividing its perpendicular Height into Fruſtums, and its Sides into four equal Parts; that ſo croſs-Diameters may be taken in the Middle of each Fruſtum:

But

but if the Copper be much wider at the Top than at the Bottom, and its Sides ſpheroidal or arching, as generally all large Coppers are; then, inſtead of taking thoſe mean Diameters in the Middle of every ten Inches, as above, you muſt take them in the Middle of every ſix Inches, and proceed on as before.

Now the Quantity of Liquor, that would cover the Crown of the Copper, may be found without meaſuring it, as above. In order to that, I do ſuppoſe the Crown to be the Segment of a Sphere, and the lower Part of the Copper wherein the Crown ariſeth, to be the Fruſtum of a parabolick Conoid; then, if the Diameter at the Top of the Crown and its perpendicular Height are given, the Quantity of Liquor may be found by this following Rule:

RULE. From the Area of the Plain at the Top of the Crown ſubtract $\frac{1}{3}$ of the Area of the Crown's Height; the Remainder, being multiply'd into half the Height of the Crown, will produce the Quantity or Number of Gallons that will cover the Crown.

This *Rule* is deduced from *Scholium*, *Page* 424, and *Theorem* 15. *Page* 430.

Sect. 5. To compute the Content of any cloſe Caſk in Gallons, *viz.* of any Butt, Pipe, Hogſhead, Barrel, *&c.*

In order to perform this difficult Part of Gauging, the three following Dimenſions of the propoſed Caſk muſt be truly taken in Inches, and Decimal Parts of an Inch.

Viz. The Bulge or Bung Diameter within the Caſk.
Either of the Head Diameters ſuppoſing them both equal.
And the Length of the Caſk within.

Note, In taking of theſe Dimenſions, it muſt be carefully obſerved,

1. That the Bung-hole be in the Middle of the Caſk; alſo that the Bung-ſtaff and the Staff over-againſt the Bung-hole are both regular or even within.

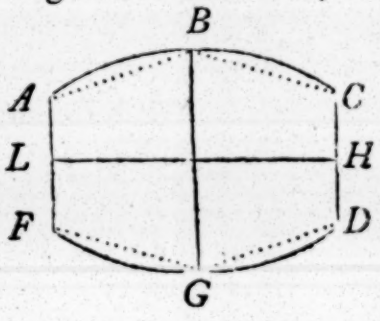
2. That the Heads of the Caſk are equal and truly circular; if ſo, the Diſtance between the Inſide of the Chine to the Outſide of its oppoſite Staff will be the Head Diameter within the Caſk, very near.

3. With a ſliding pair of Calipers (made on purpoſe for that Uſe) take the ſhorteſt Diſtance at Length between the Outſides of the two Heads; (ſuppoſing them even) from that Length ſubtract 1 $\frac{1}{2}$ Inch (more, or leſs, according to the Largeneſs of the

Caſk) for the Thickneſs of the two Heads, the Remainder will be the Length of the Caſk within.

Now by theſe Dimenſions, one would ſuppoſe the Content of the Caſk were perfectly limited; but it will be eaſy to perceive, by the following Figure, that the Diameters (aboveſaid) and the Length of one Caſk may be equal to thoſe of another, and yet one of thoſe Caſks may contain or hold ſeveral Gallons more than the other.

As for Inſtance, ſuppoſe the annex'd Figure *ABCDGF*, to repreſent a Caſk; then it is plain, that, if the outward and curved Lines *ABC* and *FGD* are the Bounds or Staves of the Caſk, it muſt needs hold more than if the inner ſtreight or prick'd Lines were its Bounds or Staves; and yet the Bung Diameter *BG*, Head Diameter *CD* and *AF*, and the Length *LH*, are the ſame in both thoſe Caſks.



Whence it plainly appears, that no one certain or general Rule can be preſcrib'd to find the true Content of all Sorts of Caſks, and therefore Gaugers do uſually ſuppoſe every Caſk to be in Form of ſome one of theſe following Solids.

Viz.
I. The middle Zone or Fruſtum of a Spheroid.
II. The middle Zone or Fruſtum of a Parabolick Spindle.
III. The lower Fruſtums of two equal Parabolick Conoids.
IV. The lower Fruſtums of two equal Cones.

Now the Way of Gueſſing at the Caſk's Form, and computing its Content, according to its ſuppos'd Form, I ſhall here ſhew in their Order.

I. If the Staves of the Caſk are very much curved or arching (as the outward Lines of the laſt Figure) then the Caſk is ſuppos'd to be in the Form of the middle Zone or Fruſtum of a Spheroid, whoſe Content may be computed, by *Theorem* 22. *Page* 427, which gives theſe two *Rules*.

RULE I. To twice the Square of the Bung Diameter add the Square of the Head Diameter; multiply that Sum into the Length, and divide the Product by 1077,15. *Viz.* 3, 8197×282 for Ale Gallons; and by 882,36 *Viz.* 3, 8197×231 for Wine Gallons. Or thus,

RULE 2. To twice the Area of the Bung Circle, add the Area of the Head Circle; multiply their Sum into one Third of the Length, and the Product will be the Content in their reſpective Gallons.

Example 1. Suppoſe a Caſk in the Form of the middle Zone of a Spheroid, whoſe Bung Diameter is 31,5, Head Diameter 24,5, and its Length 42 Inches.

Firſt 31,5×31,5×2=1984,5. And 24,5×24,5=600,25
Again 1984,5+600,25=2584,75. And 2584,75×42=108559,5
Then 1077,15)108559,5(100,78 the Content in Ale Gallons.
And 882,35) 108559,5 (123,03 the Content in Wine Gallons.

Or thus, by the Second Rule.

Bung Diameter 31,5 twice its Circle's Area is 5,5270
Head Diameter 24,5 its Circle's Area is 1,6718
The Length 42 divided by 3 is 14. 7,1988=their Sum.
Then 7,1988×14=100,78, the Content in A. Gallons as before.
And ſo the Content in Wine Gallons may be found.

II. If the Staves of the Caſk are not quite ſo much curved or arching, as was ſuppos'd before, the Caſk is then taken for the middle Fruſtum of a parabolick Spindle, and its Content is computed, as by *Theorem* 27. *Page* 432. Which gives this *Rule*.

RULE. To twice the Square of the Bung Diameter add the Square of the Head Diameter; from their Difference ſubtract four Tenths of the Square of the Difference of the Diameters; multiply the Remainder into the Length, and divide the Product by 1077,15, *&c.* as above.

Example 2. Suppoſe the Dimenſions the ſame as before. Then 31,5 × 31,5 × 2 : + 24,5 × 24,5 = 2584,75. And 31,5 — 24,5=7. Again 7×7×0,4=19,6. And 2584,75 — 19,6×42 = 107736,3. Then 1077,15) 107736,3 (100,01 the Cont. in *A. G. &c.* for *W. G.*

III. When the Staves of the Caſk are but very little curved or arching, then it's ſuppos'd to be in the Form of the Fruſtums of two equal parabolick Conoids, abutting or joining together upon one common Baſe at the Bulge, and the Content may be found by *Theorem* 25. *Page* 430. which gives theſe *Rules*.

RULE 1. To the Square of the Bung Diameter add the Square of the Head Diameter; multiply their Sum into the Length, and divide the Product by 718,08 (*viz.* 2,5464×282) for Ale Gallons: or by 588,22 (*viz.* 2,5464×231) for Wine Gallons. Or thus,

RULE 2. To the Area of the Bung Circle add the Area of the Head Circle; multiply the Sum into half the Length, and the Product will be the Content required.

Example 3. With the ſame Dimenſions as before. Then

315×31,5 + 24,5×24,5 = 1592,5. And 1592,5×42 = 66885
And 718,08) 66885 (93,01 the Content in Ale Gallons.
Or, 588,22) 66885 (113,7 the Content in Wine Gallons.

IV. If the Staves of the Caſk are ſtreight from the Bulge to the Head, as the inner prick'd Lines in the laſt Figure (if ſuch a Caſk can be made) it is then taken for the lower Fruſtums of two equal Cones, abutting or joining together upon one common Baſe at the Bulge. And its Content may be computed as at Problem 13. *Page* 445. or by *Theorem.* 15. *Page* 419. Thus,

RULE. To the Sum of the Squares of the Head and Bung Diameters add their Product; then multiply that Sum into the Length, and divide the laſt Product by 1077,15. Or by 882,36. The Quotient will be the Content, *&c.*

Example 4. With the ſame Dimenſions as before.

Firſt 31,5×31,5 + 24,5×24,5 + 31,5×24,5 = 2364,25
And 2364,25×42 = 99298,5. Then 1077,15) 99298,5 (92,18 the Content in Ale Gallons, and ſo on for Wine Gallons.

Thus you have the Methods of computing the true Contents of the four Solids, in whoſe Form all Caſks are ſuppos'd to be. And by the Examples it appears, that four ſuch Caſks as have their Dimenſions all equal, and the ſame with thoſe above-mention'd, their Contents will be as in the Margin.

	Ale Gallons.	
I.	100,78	Differ.
II.	100,01	0,77
III.	93,01	7,00
IV.	92,18	0,83

From the Diſproportion or Inequality of theſe Differences it will be eaſy to conceive, that there may be ſeveral Caſks whoſe Contents cannot be truly found, according to the aforeſaid ſuppos'd Forms; and therefore, in order to rectify the ſaid Inequalities, ſome Authors (that have written upon this Subject) have laid down *Theorems* of their own Invention; and yet call'd them by

by theſe Names) others have propos'd Tables for the ſame Purpoſe. But ſince it is ſo, that we can only gueſs at the Truth, the plaineſt and eaſieſt Way is to be preferr'd in Practice; and that is, by finding ſuch a mean Diameter as will reduce the propos'd Caſk to a Cylinder.

Thus, { Multiply the Difference between the Head and Bung Diameters, with 0,7. or with 0,65. or with 0,6. or with 0,55. according as the Staves of the Caſk are more or leſs arching; add the Product to the Head Diameter, and the Sum will be the mean Diameter required. Then find the Content, as at *Prob.* 11. *Page* 444.

Example. With the ſame Dimenſions as before. Then the Bung Diameter leſs the Head Diam. is 31,5—24,5 = 7. And

	MD.		*AG.*	*Cont.*	*Dif.*
24,5 + 7×0,7	= 29,40	its *Area*	2,4073×42	= 101,10	
7×0,65	= 29,05	——	2,3504×42	= 98,71	2,39
7×0,6	= 28,70	——	2,2941×42	= 96,35	2,36
7×0,55	= 28,35	——	2,2385×42	= 94,02	2,32

From theſe it may be obſerv'd, that the Difference between each Caſk's Content is regular, and very near equal; which plainly ſhews, that there is not ſo much Room left for Error this Way of computing their Contents, as was by the aforeſaid Forms.

Now the firſt of theſe four (*viz.* with 0,7) is very commonly uſed among *Gaugers* for all Sorts of Caſks; but I did never gauge any Caſk that would contain quite ſo much as that Rule did make it; and the Reaſon doth appear very plain from *Theorem* 22. *Page* 427. being compar'd with *Theorem* 19. *Page* 426. and the laſt Figure; *viz.* that no (Caſk being regularly made) can hold more than the middle Fruſtum of a Spheroid. But I always found by Experience, that if the ſecond and third of theſe Rules (*viz.* with 0,65 and 0,6) were duly apply'd, they would anſwer very near the Truth amongſt the common Sort of Caſks; and the fourth Rule (*viz.* with 0,55) will come pretty near the Truth in computing the Contents of Caſks, whoſe Staves are almoſt ſtreight betwixt the Head and Bung, *viz.* ſuch as Wine Pipes, *&c.*

Sect. 6. To find what Quantity of Liquor is either drawn forth, or remaining in any ſpheroidical Caſk, uſually call'd the Ullage of a Caſk; hath two Caſes.

Caſe 1. To find what Quantity of Liquor is in the Caſk, when its Axis is perpendicular to the Horizon, *viz.* when it ſtands upright upon one of its Heads.

In

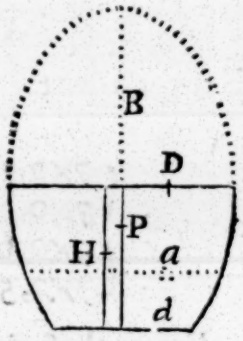
In order to perform this the eaſieſt Way, it will be convenient to know how to calculate the Area of any Circle betwixt the Bung and Head, whoſe Diſtance from the Bung or middle of the Caſk is given. Now that may be done by this Proportion.

Viz. As the Square of half the Length of the Caſk: is to the Difference between the Bung and Head Area's :: ſo is the Square of any Circle's Diſtance from the Bung: to the Difference between the Bung Area, and the Area of the Circle, *viz.* the Area of the Liquor's Surface.

Demonſtration.

Let H = Half the Length of the Caſk
D = Half the Bung Diameter.
d = Half the Head Diameter.

And P = the Diſtance of any Circle from the Bung
a = Half the Diameter of that Circle.



Then according to the common Property of the Ellipſis, *Page* 368, it will be,

$BB : DD :: BB - HH : dd$. And $BB : DD :: BB - PP : aa$.

Ergo $\frac{DDHH}{DD-dd} = BB$. And $\frac{DDPP}{DD-aa} = BB$.

Conſequently, $\frac{DDHH}{DD-dd} = \frac{DD-aa}{DDPP}$.

This Æquation, being brought out of the Fractions, will become $DDHH - aaHH = DDPP\ ddPP$, which gives this *Analogy* $HH : DD - dd :: PP : DD - aa$. Then $DD - aa$, being ſubtracted from DD, will leave aa. But Circles Area's are in Proportion to the Squares of their Diameters, by *Theorem* 6. *Page* 407. Therefore, *&c.* Q. E. D. Then, from the Bung Area ſubtract one third Part of the aforeſaid Difference, *viz.* between the Bung Area and the Area of the Liquor's Surface; multiply the Remainder with the Liquor's Diſtance from the Bung, and the Product will ſhew what Quantity of Liquor is either above or under half the Content of the Caſk.

Example. Let us ſuppoſe a Caſk of the ſame Dimenſions with that in the firſt *Example*, *Page* 451. and let it be required to find what Quantity of Liquor is in it (of Ale Meaſure) when there is but 9 Inches wet. Here half the Length of the Caſk is 21 Inches,

Inches, whoſe Square is 441, and the Liquor's Diſtance from the Bung is 21—9 = 12. Its Square is 144. The Difference between the Bung and Head Area's is 1,0917 (= 2,7635 — 1,6718.) Then 441 : 1,0917 :: 144 : 0,3564.
And 2,7635 — 0,3564 = 2,4071 the Area of the Liquor's Surface.

Again 3) 0,3564 (0,1188. And 2.7635 — 0,1188 = 2,6447 Then 2,6447×12 = 31,7364, what the Caſk wants of being half full. Conſequently 50,39 — 31,73 = 18,66 will be the Quantity of Liquor in the Caſk at 9 Inches wet in Ale Gallons.

And if the Caſk had wanted but 9 Inches of being full; then 50,39 + 31,73 = 82,12 would have been the Quantity of Liquor in the Caſk.

Note, Becauſe the two firſt Terms (*viz.* 441 and 1,0917) in the Proportion are fix'd, *viz.* continue the ſame for any Diſtance, 'twill be very eaſy to calculate the Area's of all the Circles betwixt the Bung and Head to every Inch, and by that Means to make a Table that will ſhew what Quantity of Liquor is either drawn out or remaining in the Caſk at any Depth.

Caſe 2. To find what Quantity of Liquor is in any Caſk, when its Axis is parallel to the Horizon, *viz.* when it lies along.

There are Variety of Tables to be found in Books of *Gauging* for this Purpoſe; but I always obſerved, that the following Method of computing the Ullage, by a Table of the Segments of a Circle, came very near the Truth in all Sorts of Caſks, which is thus perform'd:

1. By the Bung and Head Diameters, find ſuch a mean Diameter as you judge will reduce the propos'd Caſk to a Cylinder, by the Method laid down in *Page* 453. And then find its full Content, as in thoſe *Examples*.

2° From the Bung Diameter ſubtract the mean Diameter and half the Difference, (*viz.* divide it by 2.)

3. From the wet Inches of the propoſed Ullage, ſubtract the ſaid half Difference, and call it *x*; then obſerve this Proportion.

Viz. { As the mean Diameter: is to 100 (the Diameter of the tabular Circle) :: ſo is the laſt Difference (*viz. x*) : to a verſed Sine in the Table. (*Page* 441.)

Then if the tabular Segment, which ſtands againſt that verſed Sine, be multiply'd into the Content of the Caſk, the Product will ſhew the Ullage, *viz.* what Quantity of Liquor is either in the Caſk, or drawn forth.

Example

Example 1. Let the Caſk be that of the ſecond Sort, in *Page* 453. *viz.* whoſe Bung Diameter is 31,5 Inches, mean Diameter 29,05, and the Content 98,71 Ale Gallons; and ſuppoſe there were 10,5 Inches wet in it, it is required to find the wet and dry Gallons?

Here 31,5—29,05=2,45; its half is 1,12. And 10,5—1,22=9,28
Then 29,05 : 100 : : 9,28 : 0,319 = V. Sine; its Segm. is 0,2748
And 98,71×0,2748 = 27,12 the Number of wet Gallons.

Again 31,5—10,5=21 the dry Inches; and 21—1,22=19,78
Then 29,05 : 100 : : 19,78 : 0,68; its Segment is 0,7241
And 98,71×0,7241=71,48 the Number of dry Gallons.
Proof 71,48+27,12=98,6 the Contents of the Caſk very near; which plainly ſhews the Truth of this Method.

Thus far may ſuffice concerning Gauging of Backs or Coolers, Tuns, Coppers, and Caſks, *&c.* To which I ſhall only add, that as the Contents of all Brewers Utenſils are to be computed by the Ale Gallons, ſo the Contents of all Diſtillers Utenſils (*viz.* all their Waſh-Backs, Stills, and Caſks, *&c.*) muſt be computed by the Wine Gallon.

And in gauging of Malt (upon which there is now a Duty of four Shillings per Buſhel) you muſt obſerve, That a Corn or Malt-Buſhel doth contain 2150,42 cubick Inches; (See *Page* 42.) and therefore in gauging of Malt Ciſterns, or other Veſſels, 2150,42 will be a conſtant or fix'd Diviſor for finding the Area's of right-lin'd Figures in Buſhels at one Inch deep, and 2738 will be a conſtant or fix'd Diviſor for finding the Area's of circular Figures.

I have omitted the Buſineſs of gauging Maſh-Tuns, and taking an Account of the Goods or Grains, in order to eſtimate what Quantity of Worts were produced from them, *&c.* becauſe I could never find (by all my Obſervations) any Certainty therein; nor is it poſſible there ſhould be any, by Reaſon of the great Difference that is in Malt (and its Grinding too) for the beſt Malt (well ground) will yield or produce the moſt Worts, and leaſt Grains; on the contrary, bad Malt (being ill ground) yields the leaſt Worts and moſt Grains.

A

SUPPLEMENT

Not in any of the former Editions of this

BOOK.

Containing the

HISTORY

OF

LOGARITHMS,

WITH

Several eaſy Methods of Conſtructing the *Tables* of the LOGARITHMS and SINES, *&c.* Alſo the Demonſtration of the AXIOMS and Doctrine of *Plane*

TRIGONOMETRY.

Extracted from the

Philoſophical TRANSACTIONS, and the WORKS of Dr. *KEIL*, *RONAYNE*, *WARD*, &c.

Cuncta Trigonus *habet*, *ſatagit quæ docta* Matheſis,
Ille *aperit clauſum quicquid* Olympus *habet*.

THE

PREFACE.

THE *Mathematicks formerly received conſiderable Advantages; firſt, by the Introduction of the* Indian *Characters, and afterwards by the Invention of Decimal Fractions; yet has it ſince repeated at leaſt as much from the Invention of Logarithms, as from both the other two. The Uſe of theſe, every one knows, is of the greateſt Extent, and runs through all Parts of Mathematicks. By their Means it is that Numbers almoſt infinite, and ſuch as are otherwiſe impracticable, are managed with Eaſe and Expedition. By their Aſſiſtance the Mariner ſteers his Veſſel, the Geometrician inveſtigates the Nature of the higher Curves, the Aſtronomer determines the Places of the Stars, the Philoſopher accounts for other Phænomena of Nature; and laſtly, the Uſurer computes the Intereſt of his Money.*

The Subject of the following Treatiſe has been cultivated by Mathematicians of the firſt Rank; ſome of whom, taking in the whole Doctrine, have indeed wrote learnedly, but ſcarcely intelligible to any but Maſters. Others, again, accommodating themſelves to the Apprehenſion of Novices, have ſelected out ſome of the moſt eaſy and obvious Properties of Logarithms, but have left their Nature and more intimate Properties untouched. My Deſign therefore, in the following Tract, is to ſupply what ſeemed ſtill wanting, viz. *to diſcover and explain the Doctrine of Logarithms, to thoſe who are not yet got beyond the Elements of Algebra and Geometry.*

The wonderful Invention of Logarithms we owe to the Lord Neper, *who was the first that constructed and published a Canon thereof, at* Edinburgh, *in the Year* 1614. *This was very graciously received by all Mathematicians, who were immediately sensible of the extreme Usefulness thereof. And tho' it is usual to have various Nations contending for the Glory of any notable Invention, yet* Neper *is universally allowed the Inventor of Logarithms, and enjoys the whole Honour thereof without any Rival.*

The same Lord Neper *afterwards invented another and more commodious Form of Logarithms, which he afterwards communicated to Mr.* Henry Briggs, *Professor of Geometry at* Oxford, *who was hereby introduced as a Sharer in the completing thereof: But, the Lord* Neper *dying, the whole Business remaining was devolved upon Mr.* Briggs, *who, with prodigious Application, and an uncommon Dexterity, compassed a Logarithmic Canon, agreeable to that new Form for the first twenty Chiliads of Numbers (or from* 1 *to* 20000) *and for eleven other Chiliads, viz. from* 90000 *to* 10 1000. *For all which Numbers he calculated the Logarithms to fourteen Places of Figures. This Canon was published at* London *in the Year* 1624.

Adrian Vlacq *published again this Canon at* Goudæ *in* Holland *in the Year* 1628, *with the intermediate Chiliads before omitted, filled up according to* Briggs's *Prescriptions; but these Tables are not so useful as* Briggs's, *because the Logarithms are continued but to* 10 *Places of Figures.*

Mr. Briggs *also has calculated the Logarithms of the Sines and Tangents of every Degree, and the hundredth Parts of Degrees to* 15 *Places of Figures, and has subjoined to them the natural Sines, Tangents, and Secants, to* 15 *Places of Figures. The Logarithms of the Sines*

and Tangents are called Artificial Sines and Tangents, but the Sines and Tangents themſelves are called natural. Theſe Tables, together with their Conſtruction and Uſe, were publiſhed after Briggs's *Death, at* London, *in the Year* 1633, *by* Henry Gellibrand, *and by him called* Trigonometria Britannica.

Since then, there have been publiſhed, in ſeveral Places, compendious Tables, wherein the Sines and Tangents, and their Logarithms, conſiſt of but ſeven Places of Figures, and wherein are only the Logarithms of the Numbers from 1 *to* 100000, *which may be ſufficient for moſt Uſes.*

The beſt Diſpoſition of theſe Tables, in my Opinion, is that, firſt thought of by Nathaniel Roe, *of* Suffolk; *and, with ſome Alterations for the better, followed by* Sherwin *in his Mathematical Tables publiſhed at* London *in* 1705; *wherein are the Logarithms from* 1 *to* 10 1000 *conſiſting of* 7 *Places of Figures. To which are ſubjoined the Differences and proportional Parts, by Means of which may be found eaſily the Logarithms of Numbers to* 10000000, *obſerving at the ſame Time that theſe Logarithms conſiſt only of* 7 *Places of Figures. Here are alſo the Sines, Tangents, and Secants, with the Logarithms and Differences for every Degree and Minute of the Quadrant, with ſome other Tables of Uſe in practical Mathematicks.*

THE

THE CONSTRUCTION OF LOGARITHMS.

THESE moſt excellent and uſeful Numbers were firſt invented by the famous and never to be forgotten Lord *Neper*, Baron of *Merchiſton* in *Scotland*, aforeſaid (*Ann.* 1614.) who ingeniouſly contriv'd to perform Multiplication and Diviſion of Natural Numbers, by only adding or ſubtracting certain Artificial Numbers, which he called *Logarithms*, and the Extraction of Roots by dividing the Log. by 2 for the Square: by 3 for the Cube: by 4 for the Biquadrate, *&c.*

This Invention of his (no doubt) proceeded from a mature Conſideration of the Coherence that is betwixt Numbers in Geometrical Proportion and thoſe in Arithmetical Progreſſion.

As in theſe following:

Viz. { 1 . 2 . 4 . 8 . 16 . 32 . 64 . 128, *&c.* Geometrical.
0 . 1 . 2 . 3 . 4 . 5 . 6 . 7, *&c.* Arithmetical.

It is very perceptible, that, as the Numbers in the Geometrical Proportionals are produced by *Multiplication* or *Diviſion*, thoſe in the Arithmetical Progreſſion are produced by *Addition* or *Subtraction*: As doth appear in this Example:

Viz. { $4 \times 32 = 128$ / $2 + 5 = 7$ } or { $128 \div 32 = 4$ Geometr. / $7 - 5 = 2$ Arithmet.

Again, { 1 . 10 . 100 . 1000 . 10000 . 100000, *&c.* Geometr.
0 . 1 . 2 . 3 . 4 . 5, *&c.* Arithmet.

The ſame Coherence is betwixt theſe latter, as was between the two firſt Ranks.

Viz. { $1000 \times 10 = 10000$ / $3 + 1 = 4$ } or { $100000 \div 1000 = 100$ Geometr. / $5 - 3 = 2$ Arithmet.

Either of theſe Examples do ſufficiently ſhew the Reaſon and very Ground of Logarithms.

And from the latter of theſe it was, that the prime Logarithms or Characteriſtics were firſt aſſigned.

As

As in this Table:

Natural Num.	*Logarithms.*
1	0,0000000
10	1,0000000
100	2,0000000
1000	3,0000000
10000	4,0000000
100000	5,0000000

Having laid this Foundation, the next Work was to find out the Logarithms of the intermediate Numbers ſituated betwixt 1 and 10, *viz.* of 2, 3, 4, 5, 6, 7, *&c.* and of thoſe betwixt 10 and 100, *viz.* of 11, 12, 13, 14, 15, *&c.* and ſo on for the reſt. This was a Work of ſome Difficulty, and very laborious.

The firſt Step in order thereunto (as I conceive) was to find out a Rank of continual Means betwixt 10 and 1, ſo as that the laſt (and leaſt thereof) might be a mixed Number leſs than 2, and ſo near 1, as to have ſuch a Number of Cyphers before the ſignificant Figures thereof, as was intended the Places of Logarithms in the Table ſhould conſiſt of. Which Means are to be found, by extracting the ſquare Root of 10 (having firſt annexed a competent Number of Cyphers thereunto;) then extracting the Root of that Root, and ſo by a continued Extraction of Root out of Root, until there be a Root ſo qualify'd as before-mention'd: Which, to make a Table to ſeven Places in the Logarithm, will require twenty-five ſeveral Extractions, the laſt of which will produce this Number, 1,00000006862238.

The next Step was to find out a Number betwixt (1) and (0) in Arithmetical Progreſſion, that might truly correſpond with the Mean before found (betwixt 10 and 1) ſuch a Number muſt conſequently be its Logarithm. And this may be found by a continual biſecting (or halving) of 1, ſo often as was the Number of the foregoing Extractions (to wit, twenty-five) the laſt of which Biſections will produce 0,00000002980232, *&c.* the true Logarithm of 1,00000006862238.

For as 1,00000006862238 by twenty-five continued Involutions (*viz.* firſt into itſelf, then that Product into itſelf, and ſo on ſucceſſively) will produce 10; ſo will 0,00000002980232, by the like Number of Doublings and Redoublings, produce 1.

This Mean (or Number) and its Logarithm being thus found, it will follow by Proportion, *As the ſignificant Figures of this Mean : are to the ſignificant Figures of its Logarithm : : ſo are the ſignificant*

ficant Figures of any Mean, betwixt any given Number and 1 : (having ſeven Cyphers before ſuch Figures, as this hath) *to the ſignificant Figures of its Logarithm.* To which muſt be prefixed ſeven Cyphers to complete it. After which, being doubled, and redoubled according to the Number of Extractions required to produce its correſponding Mean, will at laſt diſcover the true Logarithm of the given Number. For the clearing of this, take an Example.

Suppoſe it were required to find the Logarithm of the Number 2, to ſeven Places. Firſt, by a continued Extraction of Root out of Root, beginning at 2, find ſuch a Mean, or Root as before, betwixt 2 and 1, as will have ſeven Cyphers before its ſignificant Figures; which, after twenty-three ſeveral Extractions, will be this Number 1,00000008262958. Then, according to the foregoing Proportions, it will be 6862238 : 2980232 : : 8262958 : 3588557. To which prefix ſeven Cyphers, as before directed, then will 1,00000008269958 have for its Logarithm, ,00000003588557; which being doubled and redoubled, as aboveſaid, will produce 0,30102997958658, the true Logarithm of 2; which being contracted to ſeven Places, according to the firſt Deſign, and agreeable to the ſeven Places of Cyphers, then it will become 0,3010299: But, in all the Tables that I have ſeen, the Logarithm of 2 is 0,3010300: I conceive the Reaſon is, becauſe the remaining Figures 7958658 come ſo near Unity of the laſt Place in the retained Figures.

And, by the ſame Method that this Logarithm of 2 is made, may the Logarithm of any other Number be found. But when once the Logarithms of a few of the prime Numbers, *viz.* of 3. 7. 11. 13. *&c.* (that is, of ſuch Numbers as cannot be produced by the multiplying of two Integer Factors) are obtained, the reſt may be eaſily compoſed by *Addition* and *Subtraction* only. For as $3 \times 2 = 6$ ſo Log. of $3 +$ Log. of $2 =$ Log. of 6. And as $10 \div 2 = 5$ ſo Log. of $10 -$ Log. of $2 =$ Log of 5. The like of all Numbers that have aliquot Parts (that is, ſuch Integer Numbers as may be divided by Integers.) And indeed the Logarithms of ſeveral of the prime Numbers may alſo be obtained by *Addition* or *Subtraction*, as might eaſily be ſhewed, and is not difficult to conceive by any one, who but duly conſiders the Nature and Deſign of Logarithms, *&c.* of which I ſhall forbear ſaying any thing in this Place, and keep to my firſt Deſign herein, which was to give a brief Account of the ingenious Author's Method, as I conceive it, of making the ſame; who undoubtedly found it a very difficult Work, by Reaſon there are required ſo many ſeveral Extractions of Roots out of Roots, which muſt needs render it both troubleſome and laborious. Then to propoſe a different Method of raiſing the Logarithms

rithms of ſuch prime Numbers before-mentioned, which require the Extraction of Roots to obtain their reſpective Means, with one tenth Part of the Trouble and Time required by the foregoing Method. And not only ſo, but more exact; for, by our preſent Method of converging Series, the Root of any Power, how high ſoever it be, is eaſily found at one ſingle Extraction: and thereby the Errors which would ariſe by extracting a *Surd Root* out of a *Surd Root*, eſpecially when often repeated, are avoided; and conſequently ſuch a Mean, as may be required betwixt any Number and Unity, is thereby more exactly found.

Now, how this may be performed, I here intend to ſhew, as briefly as I can. In order thereunto, take this as a Model.

Let a = the Root, or Mean required betwixt any Number and Unity:

Then $\begin{cases} a^2 = \square\, a \;.\; a^4 = \square\, a^2 \;.\; a^8 = \square\, a^4 \\ a^{16} = \square\, a^8 \;.\; a^{32} = \square\, a^{16} \;.\; a^{64} = \square\, a^{32} \\ a^{128} = \square\, a^{64} \;.\; a^{256} = \square\, a^{128} \;.\; a^{512} = \square\, a^{256} \end{cases}$

And ſo on ſucceſſively with the Indices in Geometrical Progreſſion, until the Power of a be made equal to ſuch a Term in that Progreſſion, as that the Root, or Value of a may have, betwixt Unity and its ſignificant Figures, ſo many Cyphers, as are the intended Number of Places in the Logarithms.

For Inſtance, let it be required to find the Mean between 10 and 1, then the Power of a muſt be $a^{33554432} = 10$, this Index 33554432 being the 25th Term in Geometrical Pregreſſion, which may be thus determined.

Let 1, the Characteriſtic or Logarithm of 10, be divided by ſuch a Term in Geometrical Progreſſion, as will cauſe ſuch a Number of Cyphers to be before the ſignificant Figures in the Quotient, as are required to be before the Figures of the Root a; ſuppoſe 7, as before. Then $1 \div 33554432 = ,00000002980232$, *&c.* which is the true Arithmetical Mean (as before found, by a continual biſecting of 1) correſpondent to that ſignify'd by a; and therefore the Value of a found by extracting the reſpective Root of $10 = a^{33554432}$ will be the Mean required; *viz.* 1,00000006862238 whoſe Log. is ,00000002980232. Theſe, being found, are the Foundation of the reſt, as before.

Then ſuppoſe it be required to find the Logarithm of any of the prime Numbers; if you pleaſe, that of two. In order thereunto, let a = the Root or Mean ſought betwixt 2 and 1, as before; then muſt a be continually involved, as by the above Model, until its Index be equal to the greateſt Term in Geometrical Progreſſion, whoſe Number of Places of Figures are to be equal to the Number of required Cyphers before a, to wit 7. According to which, the

Power

Power of a will be $a^{8388608} = 2$ (this 8388608 being the 23d Term in Geometrical Progreſſion) conſequently the reſpective Root of $2 = a^{8388608}$ will be the Mean requir'd.

Example.

Let $r + e = a$

Then will $r^{8388608} + 8388608\, r^{8388607}\, e$
$+35184367894528\, r^{8388606}\, ee = a^{8388606} = 2$

Suppoſe $r = 1$

Then $1 + 8388608e + 35184367894528ee = 2$

That is $8388608e + 35184367894528\, ee = 1$

Each Part being divided by the Co-efficient found prefixed to ee, viz. 351843, &c. then it will become

$$,00000023e + ee = ,0000000000000284 = D$$

$$\textit{Conſequently} \left\{ \frac{D}{,00000023 + e} = e \right.$$

. . . . . .

,0000000000000284 = D

,00000023 248 (,00000008 = e

$+ e =$,00000008 — 36

Diviſor ,00000031

Firſt $r = 1$,

$+ e =$,00000008

New $r = 1$,00000008

which being duly involved, in the ſame Order as the Model denotes, and multiplied into the reſpective Co-efficients, will then produce theſe Numbers,

Viz. $1,9563638967 + 164111168e + 6883341606628 9ee = 2$

Then $164111168e + 6883341606628 9ee = ,043636 1033$

And $,0000002384e + ee = ,0000000000000063393 = D$

$$\textit{Conſequently} \left\{ \frac{D}{,0000002384 + e} = e \right.$$

. . . . . .

,0000000000000063393 = D

,0000002384 480 (,00000000263 = e

$+ e =$,0000000026 — 15393

Diviſor ,000000240 — 14400

Diviſor ,0000002410 — 9330

7230

0 0 0

Laſt

$$\begin{array}{rl} \text{Last } r = & 1{,}00000008 \\ + e = & {,}00000000263 \\ \hline \text{New } r = & 1{,}00000008263 \end{array}$$

I take only $1{,}0000000268 = r$; the which being involved, and ordered as before, will produce these following Numbers, *viz.*

$1{,}999503684867 + 16773028e + 70351267454084\,ee = 2$

Then $16773028\,e + 70351267454084\,ee = {,}000496315133$

${,}0000002384186e + ee = 0000000000000000000705481443 = D$

$$\textit{Consequently} \left\{ \frac{D}{{,}0000002384186 + e} = e \right.$$

. . . . . .

${,}0000000000000000000705481443 = D$

$$\begin{array}{rl} & {,}0000002384186 \\ e = & {,}0000000000295 \\ \hline \text{Divisor} & {,}00000023843 \\ \hline \text{Divisor} & {,}000000238447 \\ \hline \text{Divisor} & {,}0000002384481^{*} \end{array}$$

$$\begin{array}{r} 47686 \quad ({,}0000000000295 = e \\ \hline 2286214 \\ 2146023 \\ \hline 14019143 \\ 11922405 \\ \hline {}^{*}20967380 \quad (879 = e \\ 19075848 \\ \hline 891532 \\ 1669136 \\ \hline 222396 \\ 214596 \\ \hline \end{array}$$

Here I desist forming a new Divisor and make use of the Abridgement.

$$\begin{array}{rl} \text{Last } r = & 1{,}0000000826 \\ + e = & {,}0000000000295879 \\ \hline a = & 1{,}0000000826295879 \end{array}$$

This Value of $a = 1{,}0000000826295879$ is the Geometrical Mean betwixt 2 and 1, as was required; (agreeable to that before found, by twenty-three several Extractions.) And by this Method of proceeding, may be found the Mean betwixt 10 and 1, *viz.* 1,00000006862238, or betwixt any other of the (before-mentioned) Prime Numbers and Unity, as might easily be shewed. But for Brevity Sake, I shall omit giving more Examples thereof, this one being sufficient (especially to the Ingenious) if well considered, and but once understood, to shew the Nature of, and Manner how to proceed on the like Occasion, of finding any proposed Mean.

Mean. The next Thing will be to find the Logarithm of the Number from whence ſuch Mean was produced, which may be thus performed:

Firſt, find its correſponding Arithmetical Mean, or Logarithm, by Proportion (as in *Page* 462.) Then multiply that correſponding Mean, ſo found, into the Index Number of ſuch Power as the Geometrical Mean was produced from; that Product will be the Logarithm of the given Number (without a continued Doubling and Redoubling, as before.) For the clearing of this, let it be required to complete the Logarithm of 2.

Having firſt found 1,00000006862238, the proper Geometrical Mean betwixt 10 and 1; alſo its correſponding Logarithm ,00000002980232 (as before directed) with them and the Mean betwixt 2 and 1, laſt found, *viz.* 1,00000008262958 79; make uſe of the above-mentioned Proportion (as in *Page* 463.) *viz.*

6862238 : 2980232 : : 826295879 : 358855729

To which prefix ſeven Cyphers to complete it (as before.) Then it will become ,00000003588557 29. This Number being multiplied into the Power of *a* (what that is, ſee *Page* 465.) will produce the Logarithm of 2.

viz. 00000003588557 29 × 8388608 = 0,30103000391352

But according to the firſt Deſign, it is required to have but ſeven Places, *viz.* 0301300; which is the true Logarithm of 2 without any Defect.

Thus I have preſented you with a new and expeditious Method of making Logarithms; which if they were required to fourteen or fifteen Places (I can modeſtly ſay) they might then be made with one twentieth Part of the Time and Trouble required by the firſt Method.

METHOD III.

A New Table of Logarithms, *Compos'd by Mr.* Long. *Finding the* Logarithm *by* Division *only, and the Natural Number belonging to a* Logarithm, *by* Multiplication *only.*

Log.	*Nat. Num.*	*Log.*	*Nat. Num.*
0,9	7.943282347	0,00009	1.000207254
0,8	6.309573445	0,00008	1.000184224
0,7	5.011872336	0,00007	1.000161194
0,6	3.981071706	0,00006	1.000138165
0,5	3.162277660	0,00005	1.000115136
0,4	2.511886432	0,00004	1.000092106
0,3	1.995262315	0,00003	1.000069080
0,2	1.584893193	0,00002	1.000046053
0,1	1.258925412	0,00001	1.000023026
0,09	1.230268771	0,000009	1.000020724
0,08	1.202264435	0,000008	1.000018421
0,07	1.174897555	0,000007	1.000016118
0,06	1.148153621	0,000006	1.000013816
0,05	1.122018454	0,000005	1.000011513
0,04	1.096478196	0,000004	1.000009210
0,03	1.071519305	0,000003	1.000006908
0,02	1.047128548	0,000002	1.000004605
0,01	1.023292992	0,000001	1.000002302
0,009	1.020939484	0,0000009	1.000002072
0,008	1.018591388	0,0000008	1.000001842
0,007	1.016248694	0,0000007	1.000001611
0,006	1.013911380	0,0000006	1.000001381
0,005	1.011579454	0,0000005	1.000001151
0,004	1.009252886	0,0000004	1.000000921
0,003	1.006931669	0,0000003	1.000000690
0,002	1.004615794	0,0000002	1.000000460
0,001	1.002305238	0,0000001	1.000000230
0,0009	1.002074475	0,00000009	1.000000207
0,0008	1.001843766	0,00000008	1.000000184
0,0007	1.001613109	0,00000007	1.000000161
0,0006	1.001382506	0,00000006	1.000000138
0,0005	1.001151956	0,00000005	1.000000115
0,0004	1.000921459	0,00000004	1.000000092
0,0003	1.000691015	0,00000003	1.000000069
0,0002	1.000460623	0,00000002	1.000000046
0,0001	1.000230285	0,00000001	1.000000023

This

This Table I ſometimes make uſe of for finding the Logarithm of any Number propos'd, and *vice verſa*. Suppoſe I had Occaſion to find the Logarithm of 2000. I look in the firſt Claſs of my Table (the whole Table conſiſts of 8 Claſſes) for the next leſs to 2, which is 1.995262315, and againſt it is 3, which conſequently is the firſt Figure of the Logarithm ſought. Again, dividing the Number propos'd 2, by 1.995262315 the Number found in the Table, the Quotient is 1.002374467; which being look'd for in the ſecond Claſs of the Table, and finding neither its Equal, nor a Leſſer, I add 0 to the Part of the Logarithm before found, and look for the ſaid Quotient, 1.002374467 in the third Claſs, where the next leſs is 1.002305238, and againſt it is 1, to be added to the Part of the Logarithm already found; and dividing the Quotient 1.002374467, by 1.002305238, laſt found in the Table, the Quotient is 1.000069070; which being ſought in the fourth Claſs gives 0, but being ſought in the fifth Claſs gives 2, to be added to the Part of the Logarithm already found; and dividing the laſt Quotient by the Number laſt found in the Table, *viz.* 1.000046053, the Quotient is 1.000023015, which, being ſought in the ſixth Claſs, gives 9 to the Part of the Logarithm already found: And dividing the laſt Quotient by the new Diviſor, *viz.* 1.000002072, the Quotient is 1.000000219, which being greater than 1.000000115 ſhews that the Logarithm already found, *viz.* 3.3010299, is leſs than the Truth by more than half an Unit; wherefore adding 1, you have *Briggs*'s Logarithm of 2000, *viz.* 3.3010300.

If any Logarithm be given, ſuppoſe 3,3010300, throw away the Characteriſtic, then overagainſt theſe Figures 3...0..1..0..0, you have in their reſpective Claſſes 1,995262315.....0..... 1,002305238.....0.....1,000069080....0...0 which multiply'd continually into one another, the Product is 2.000000019966, which, by reaſon the Characteriſtic is 3, becomes 2,000,000019966, *&c.* that is, 2000, the Natural Number deſired. I ſhall not mention the Method by which this Table is fram'd, becauſe you will eaſily ſee that from the Uſe of it.

It is obvious to the intelligent Reader, that theſe Claſſes of Numbers are no other than ſo many Scales of mean Proportionals: in the firſt Claſs, between 1 and 10; ſo that the laſt Number thereof, *viz.* 1,258925412, is the tenth Root of 10, and the reſt in order aſcending are the Powers thereof. So in the ſecond Claſs, the laſt Number 1,023292992 is the hundreth Root of 10, and the reſt in the ſame Manner are Powers thereof. So 1.002305238, in the third Claſs, is the tenth Root of the laſt of the ſecond, and the

the reſt its Powers, *&c.* Or, which is all one, each Number, in the preceding Claſs, is the tenth Power of the correſponding Number in the next following Claſs: Whence 'tis plain, that to conſtruct theſe Tables requires no more than one Extraction of the fifth or ſurſolid Root for each Claſs, the reſt of the Work being done by the common Rules of Arithmetick.

METHOD IV.

Their Conſtruction, according to the common Rules, given by many Extractions of Roots, is tedious; the beſt Way yet known is this which follows.

To make a Table of Logarithms.

Firſt, Put for the Logarithm of 1 a Cypher for the Index, and a competent Number of Cyphers for the Logarithm, according to the Number of Places you would have your Logarithms conſiſt of; for 10 an Unit, with the ſame Number of Cyphers; for 100, 2, with as many Cyphers; for 1000, 3, with as many Cyphers, *&c.*

Secondly, Find the Difference between ſome two Logarithms above 1000, or rather above 10000, that differ by Unity; thus multiply the two Numbers together, and that Product you muſt multiply again by 43429448190325183896 * which laſt Product divided by the Arithmetical Mean between both Numbers, the Quotient is the Difference ſought.

Suppoſe we would find the Difference between the Log. 10000, and 10001, the Product of theſe two Numbers is 1.00010000, which multiplied by 4343 produced 43434343; this divided by 10000.5, quotes 4343. Now if to the Logarithm of 10000, which is 4.0000000, you add the Difference before found, to wit, 434, the Sum 4.0000434 is the true Logarithm of 10001 to 7 Places.

Thirdly, Having thus found the Difference of any two Logarithms differing by Unity, and conſequently ſome of the Logarithms by dividing the Difference found by the Arithmetical Mean, between any two Numbers differing by Unity, you ſhall have the Difference of the Logarithm of thoſe two Numbers.

Thus to find the Difference betwixt the Logarithm 274, and 275; divide 4343, the Difference of the Logarithm of 10000, and 10001 by 2745 the Quotient 15821, is the Difference ſought.

Fourthly, Having by this Means found a few of the prime Logarithms, the reſt are made by Addition and Subtraction, and having

* *Which is the Subtangent of the Curve expreſſing* Briggs's *Logarithms. See* Keil's *Trig.* Pag. 135, 140, *&c.*

ing made the Canon upwards, above 1000 to 10000, by Conſequence it is made for all inferior Numbers.

The prime Numbers to which Logarithms muſt be found, in the firſt Place are theſe, 2 . 3 . 7 . 11 . 13 . 17 . 19 . 23 . 29 . 31 . 37 . 41 . 43 . 47 . 53 . 59 . 61 . 67 . 71 . 73 . 79 . 89 . 97, *&c.* or the ſame Numbers with Cyphers.

But ſince it was very tedious and laborious, to find the Logarithms of the prime Numbers, and not eaſy to compute Logarithms by Interpolation, by firſt, ſecond, and third, *&c.* Differences, therefore the great Men, Sir *Iſaac Newton*, *Mercator*, *Gregory*, *Wallis*, and laſtly, Dr. *Halley*, have publiſhed infinite converging Series, by which the Logarithms of Numbers to any Number of Places may be had more expeditiouſly and truer: Concerning which Series, Dr. *Halley* has written a learned Tract, in the *Philoſophical Tranſactions*, wherein he has demonſtrated thoſe Series after a new Way, and ſhews how to compute the Logarithms by them. But I think it may be more proper here to add a new Series, by Means of which may be found, eaſily and expeditiouſly, the Logarithms of large Numbers.

Let z be an odd Number, whoſe Logarithm is ſought; then ſhall the Number $z - 1$ and $z + 1$ be even, and accordingly their Logarithms, and the Difference of the Logarithms will be had, which let be called y: Therefore, alſo the Logarithm of a Number which is a Geometrical Mean between $z - 1$ and $z + 1$ will be given, *viz.* equal to the half Sum of the Logarithms. Now the Series $y \times \frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5} + \frac{181}{151200z^7} + \frac{13}{25200z^9}$ *&c.* ſhall be equal to the Logarithm of the Ratio, which the Geometrical Mean between the Numbers $z - 1$ and $z + 1$, has to the Arithmetical Mean, *viz.* to the Number z.

If the Number exceeds 1000, the firſt Term of the Series $\frac{y}{4z}$ is ſufficient for producing the Logarithm to 13 or 14 Places of Figures, and the ſecond Term will give the Logarithm to 20 Places of Figures. But if z be greater than 10000, the firſt Term will exhibit the Logarithm to 18 Places of Figures; and ſo this Series is of great Uſe in filling up the Logarithms of the Chiliads omitted by *Briggs*. For Example: It is required to find the Logarithm of 20001. The Logarithm of 20000 is the ſame as the Logarithm of 2, with the Index 4 prefix'd to it; and the Difference of the Logarithms of 20000 and 20002, is the ſame as the Difference of the Logarithms of the Numbers 10000 and 10001, *viz.* 0.0000434272 7687. And if this Difference be divided by $4z$, or 80004, the

Quotient $\frac{y}{4z}$ ſhall be ——————— 0. 00000 0000542814

And if the Logarithm or the Geometricall Mean be added to the Quotient, the Sum will be the Logarithm of 20001. 4. 30105 1709302416

4. 30105 1709845230

Wherefore it is manifeſt, that to have the Logarithm to 14 Places of Figures, there is no Neceſſity of continuing out the Quotient beyond 6 Places of Figures. But if you have a Mind to have the Logarithm to 10 Places of Figures only, as they are in *Vlacq*'s Tables, the two firſt Figures of the Quotient are enough. And if the Logarithms of the Numbers above 20000 are to be found by this Way, the Labour of doing them will moſtly conſiſt in ſetting down the Numbers. *Note*, This Series is eaſily deduced from that found out by Dr. *Halley*; and thoſe who have a Mind to be inform'd more in this Matter, let them conſult his abovenamed Treatiſe.

Mr. *WARD*'s *Eaſy* Method of making the Canon of Sines, Tangents, &c.

FIRST, let me premiſe two Things, that the Periphery of a Circle, whoſe Radius is Unity or 1, is 6.283185, &c. and that the natural Sine of one Minute doth ſo inſenſibly differ from the Length of the Arch of one Minute, that it may be taken for the ſame.

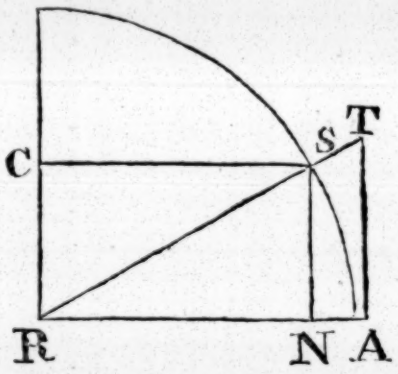
Conſequently, { *As the Periphery in Minutes* : *is to the Periphery in equal Parts of the Radius* : : *ſo is one Minute* : *to the Parts agreeing to that Minute.*

That is, 21600′ : 6,283185 : : 1′ : 0,000290888 = the Natural Sine of one Minute; which agrees with the largeſt Table of Sines I ever ſaw.

Having thus got the Sine of one Minute, its Co-ſine may be thus found:

Suppoſe

Suppoſe $RA = RS$ the Radius of any Circle, $SN =$ the Sine of the Arch SA. Then $RN = CS$ is the Co-Sine of that Arch. But $\square RS - \square SN = \square RN$, conſequently $\sqrt{\square RS - \square SN} = RN$.



That is, From the Square of the Radius, ſubtract the Square of the Sine of 1′, the ſquare Root of the Remainder will be the Co-Sine of 1′, per *Chap.* 9. *Prop.* 1. In Numbers, the Sine of 1′ is 000290885, its Square is 0,00000000846 12; and 1 — 0,00000000846 12 = 0,99999991 5388, the Square Root thereof is ,99999995 = the Co-Sine required.

The Sine and Co-Sine of one Minute being thus obtain'd, all the reſt of the Sines in the Quadrant may be gradually calculated by Mr. *Michael Dary*'s Sinical Proportions; which I ſhall here inſert, to the ſame Effect as they are in his Miſcellanies; and then explain and demonſtrate the Truth of thoſe Proportions.

If a Rank of Arches be equi-different;

Then { *As the Sine of any Arch in that Rank: is to the Sum of the Sines of any two Arches equally remote from it on each Side*: : *ſo is the Sine of any other Arch in the ſaid Rank: to the Sum of the Sines of two Arches next it on each Side; having the ſame common Diſtance.*

Immediately after theſe Proportions, he lays down the following Æquations:

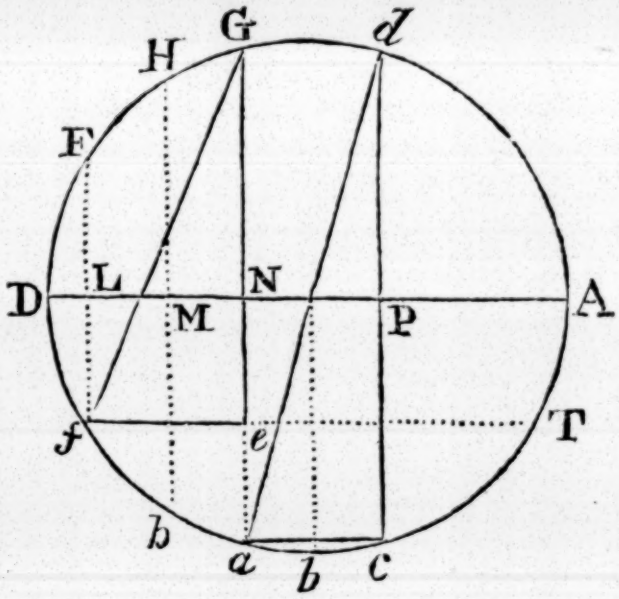
Three Arches equi-different being propoſed; if (ſaith he) you put $Z =$ the Sine of the great Extreme, $y =$ the Sine of the leſſer Extreme; $M =$ the Sine of the Mean; $m =$ the Co-Sine thereof; D the Sine of the common Difference; $d =$ the Co-Sine thereof; and $R =$ the Radius.

1. Then $Z + y = \frac{2Md}{R}$. 2. Then $Z - y = \frac{2mD.}{R}$

3. Then $Zy = MM - DD$. 4. Then $\frac{Z}{y} = \frac{Md + mD.}{Md - Md}$

From the foregoing it is evident (ſaith he) that if two Thirds, *viz.* either the former or latter 60 Degrees, or the former 30 Degrees, and the latter 30 Degrees of the Quadrant be completed with Sines; the remaining Part of the Quadrant may be completed by Addition, or Subtraction only.

Thus far is from the ingenious Mr. *Dary*, concerning theſe excellent Proportions; the Truth whereof I ſhall thus demonſtrate.



In the annexed Circle $DA = da$ are Diameters, $fh = ha = ab = bc$, are equal Arches.

Draw fT parallel to DA; then will $Ne = Lf$. And the $\triangle dac$, like the $\triangle Gfe$, being both right-angled at c and e, and $\angle d = \angle G$ becauſe ſubtended by the equal Arches $ac = fa$.

Therefore $da : dc :: Gf : Ge$.

Conſequently $\frac{1}{2} da : dc :: \frac{1}{2} Gf : Ge$. But $Hb = Gf$, whence $HM = \frac{1}{2} Gf$, and $\frac{1}{2} da =$ the Radius, $dp = \frac{1}{2} dc$. Therefore it will be, Radius : $2dp :: HM = \frac{1}{2} Gf : GN + Ne = GN + Lf$. That is, as the Radius : is to twice the Sine dp : : ſo is the Sine HM : to the Sum of the two Sines GN and $FL = fL$. Q. E. D.

I ſhall now explain theſe Proportions, and ſhew how they may be applied in Practice: Having the Sine of one Minute, and its Co-Sine as before; let the Radius be made the mean or middle Term between thoſe two Extremes; then the Proportions will run

Thus { *As the Radius : is to the double Co-ſine of one Minute : : ſo is the Sine of one Minute : to the Sine of two Minutes, and of* 00′ : *and ſo is the Sine of* 2′ : : *to the Sum of the Sines of* 3′ *and* 1′ : : *and ſo is the Sine of* 3′ : *to the Sum of the Sine of* 4′ *and* 2′.

And ſo on in a ſucceſſive Order, from Minute to Minute.

And then, if from the Sum of the Sines of 3′ and 1′ be taken the Sine of 1′, the Remainder will be the Sine of 3′: And the like, if, from the Sum of the Sines of 4′ and 2′, be taken the Sine of 2′ there will remain the Sine of 4′, *&c.*

Proceeding on by this Method, all the Natural Sines in the Quadrant may be eaſily calculated by Addition, and Subtraction only. For the Radius, or Firſt Term in the Proportion, being 1,0000000

or

or Unity, Diviſion is wholly avoided: And becauſe the ſecond Term in the Proportion varies not, if a Tariffa, or ſmall Table be made thereof, to all the nine Digits, then Multiplication is alſo avoided. For, by the Help of that Tariffa, the whole Work may be perform'd by Addition and Subtraction, until all the Sines are gradually made.

Thus you have an eaſy Way of making the Canon of Sines; which being once done, the Tangents and Secants may be found by the following

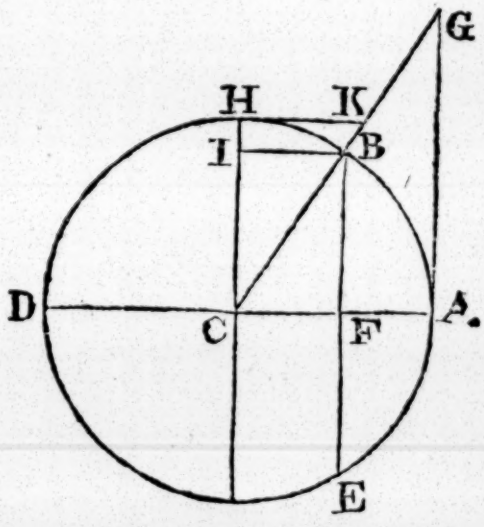
Proportions { *As the Co-ſine of any Arch: is to the Sine of that Arch:: ſo is the Radius: to the Tangent of the ſame Arch.*

That is, by *the firſt Scheme of this Problem*, $RN : SN :: RA : TA$. And $RN : RS :: RA : RT =$ the Secant of that Arch.

Plane Trigonometry.

Definitions.

1. A Circle is ſuppos'd to be divided into 360 equal Parts, called Degrees; and each Degree into 60 equal Parts, called Minutes; and each Minute into 60 equal Parts, called Seconds, *&c.* Any Portion of whoſe Circumference is called an Arch, and is meaſured by the Number of Degrees it contains.



2. A Chord or Subtenſe is a ſtraight Line, connecting the Extremities of an Arch; as BE is the Chord of the Arches BAE, BDE.

3. A Sine (or Right-ſine) is a ſtraight Line drawn from one End of an Arch perpendicular to that Diameter paſſing thro' the other End; or it is half the Chord of twice the Arch; ſo BF is the Sine of the Arches BA, BD. And here it is evident, that the Sine of 90 Degrees (which is equal to the Radius or Semi-Diameter of the Circle) is the greateſt of all Sines, the Sine of an Arch greater than a Quadrant being leſs than the Radius.

4. The Difference of an Arch from a Quadrant, whether it be greater or leſs, is call'd its Complement; ſo HB is the Complement of the Arches BA, BD; BI is the Sine of that Complement,

and therefore it is called the Co-ſine, or Sine-Complement of the Arches BA, BD.

5. The Secant of an Arch is a ſtraight Line drawn from the Center thro' one End of the Arch till it meet with the Tangent, which is a ſtraight Line touching the Circle at the Extremity of that Diameter which cuts the other End of the Arch; ſo CG is the Secant, and AG the Tangent of the Arches BA, BD: And CK is the Coſecant, and HK the Co-tangent of the ſaid Arches.

6. A Verſed Sine is the Segment of the Diameter intercepted between the Arch and its Sine: Thus FA is the Verſed Sine of the Arch BA, and FD of the Arch BD.

7. Whatever Number of Degrees an Arch wants of a Semi-Circle is called its Supplement.

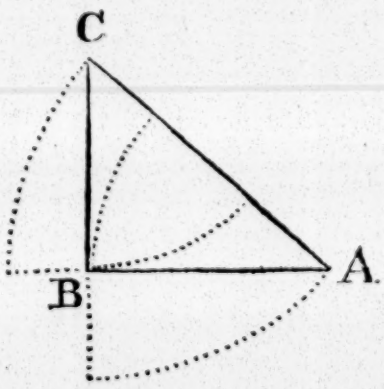
8. That Part of the Radius which is betwixt the Center and Sine is equal to the Co-ſine; thus CF is = IB.

9. If an Arch be greater or leſs than a Quadrant, the Sum or Difference of the Radius or Co-ſine is equal to the Verſed Sine.

In a Triangle are ſix Parts, *viz.* three Sides and three Angles: Any three of which being given, except the three Angles of a Plain Triangle, the other three may be found either Mechanically, by the Help of a Scale of equal Parts and Line of Chords, or by an Arithmetical Calculation, if, ſuppoſing the Radius divided into any Number of equal Parts, we know how many of thoſe equal Parts are in the Sine, Tangent, or Secant of any Arch propos'd: The Art of inferring which is called *Trigonometry*, and it is either Plane or Spherical.

Plane Trigonometry is ſolv'd by the Help of four fundamental Propoſitions, call'd *Axioms*.

Axiom I.



In a Right-angled Triangle ABC, if one Leg of the Right-angle, as AB or CB, be made the Radius of a Circle, then ſhall the other Leg CB or AB be the Tangent of the Angle oppoſite to it, and the Hypothenuſe AC (or Side oppoſite to the Right-angle) its Secant (by Definition 5.)

But if the Hypothenuſe AC be made the Radius of a Circle, then will the Legs (or Sides including the Right-angle) to wit, CB and AB be the Sines of the Angles oppoſite (by Definition 3.)

Upon

Upon this *Axiom* depends the Solution of the ſeven Caſes of Right-angled Plane Triangles.

Note, That the three Angles of a Plane Triangle make two Right-Angles, or 180 Degrees, by 32. 1 *Eucl.*

For the more eaſy making the Proportions for the Solution of Right-angled Triangles, obſerve, that as different Sides are made Radius, ſo the other Sides acquire different Names, which Names are either Sines, Tangents, or Secants, and are to be taken out of your Table.

To find a Side, any Side may be made Radius: Then ſay, as the Name of the Side given is to the Name of the Side required; ſo is the Side given to the Side required.

But to find an Angle, one of the given Sides muſt be made Radius; then, as the Side made Radius, is to the other Side; ſo is the Name of the firſt Side (which is Radius) to the Name of the ſecond Side; which fourth Proportional muſt be found among the Sines or Tangents, *&c.* to be determined by the Side made Radius, and againſt it is the Angle required.

The Proportions for the Solution of ſeven Caſes of Plane Right-angled Triangles. [*See the next foregoing* Fig.]

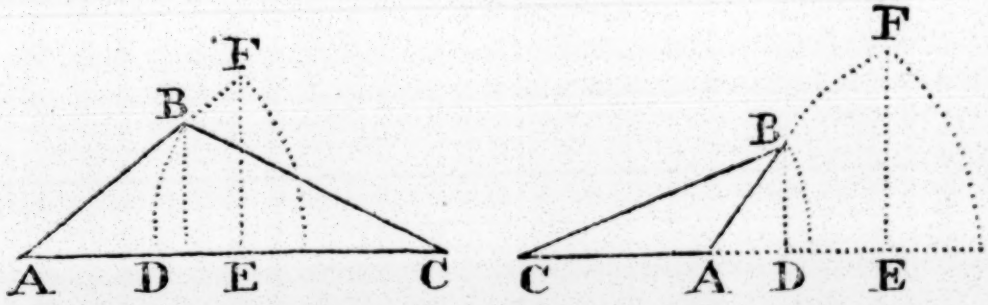
Given.	*Reqd.*	*Proportions.*	*Rad.*	*Caſe.*
AB A and C	BC	Coſi. A : Si. A :: AB : BC. R : Tan. A :: AB : BC. Cot. A : R :: AB : BC.	AC AB BC	1
AB A and C	AC	Coſi. A : R :: AB : AC. R : Sec. A :: AB : AC. Tan. A : Coſe. A :: AB : AC.	AC AB BC	2
AB BC	A and C	AB : BC :: R : Tan. A. Complement is C. BC : AB :: R : Tan. C. Complement is A.	AB BC	3
AB BC	AC	AB : BC :: R : Tan. A; then Coſi. A : R :: AB : AC, or $\surd$: $\square$ AB + $\square$ BC : = AC (*per* 47. 1. *Eucl.*	AB AC	4
AB AC	A and C	AC : BC :: R : Coſi. A. AB : AC :: R : Secant A.	AC AB	5
AB AC	BC	AC : AB :: R : Coſi. A; then R : Tan. A :: AB : BC, or $\surd$: $\square$ AC — $\square$ AB : = BC.	AC AB	6
AC A and C	AB	R : Coſi. A :: AC : AB. Sec. A : R :: AC : AB. Coſ. A : Cot. A :: AC : AB.	AC AB BC	7

Axiom

Axiom II.

In any Triangle the Sides are proportional to the Sines of the oppoſite Angles.

Demonſtration.



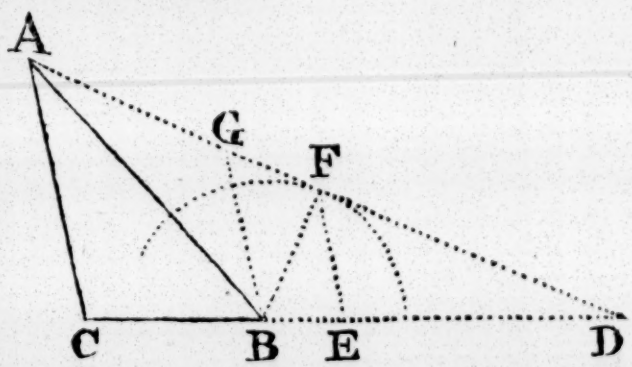
Produce the leſſer Side of the Triangle ABC, to wit AB to F, making AF = BC: Let fall the Perpendiculars BD, FE, upon the Side CA produc'd if Need be; then will FE be the Sine of the Angle A, and BD the Sine of the Angle C, to the Radius BC = AF.

Now the Triangles ABD and AFE, having the ∠ A common to them both, and the ∠ D = ∠ E = to a Right-Angle, are ſimilar; wherefore (by 4. 6 *Eucl. Elem.*) AF (BC): AB :: FE : BD; *viz.* : : Si. A : Si. C. *Q. E. D.* Otherwiſe thus; by *Ax.* I. AB : R : : BD : Si. A, and BC : R : : BD : Si. C; therefore AB × Si. A (= R × BD) = BC × Si. C; wherefore AB : BC : : Si. C : Si. A. *Q. E. D.*

Axiom III.

The Sum of the Legs of any Angle of a Plane Triangle is to their Difference, as the Tangent of half the Sum of the Angles oppoſite to thoſe Legs is to the Tangent of half their Difference.

Demonſtration.



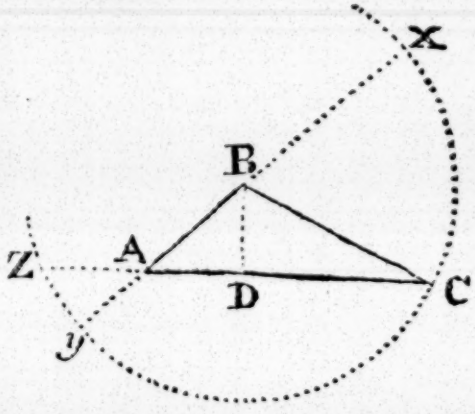
In the Triangle ABC produce CB, the leſſer Leg of the Angle B, till BD becomes = BA, the greater Leg, and then biſect CD in E; join A D and biſect it alſo in F; draw BF, which (by 8. 1 *Eucl. El.*) will be perpen. to AD; and draw EF, which (by 2. 6 *Eucl. Elem.*) will be parallel to AC. Then will the Angle ABF = FBD = ½ ABD, which external Angle ABD is (by 32. 1 *Eucl. Elem.*) = BAC + C, that is to the Sum of the oppoſite Angles required.

Draw then BG parallel to CA, ſo will the Angle GBA be (by 29. 1 *Eucl. Elem.*) equal to its Alternate one BAC; and if from half the Sum

Sum of the oppoſite Angles you take the leſſer Angle, *i. e.* If from ∠ ABF you take the ∠ GBA, there will remain ∠ GBF = half the Difference of the oppoſite Angles: And ſo alſo, if from CE half the Sum of the Legs, you take CB the leſſer Leg, there will remain BE equal to half the Difference of the Legs. And then ſince the △ ABF is Right-angled, if BF be made Radius, AF will be the Tangent of ∠ ABF (*i. e.* the Tangent of half the Sum of the oppoſite Angles); and in the little △ GBF, FG will be the Tangent of the ∠GBF (*i. e.* the Tangent of half the Difference of the oppoſite Angles): But the Segments of the Legs of any Triangle cut by Lines parallel to the Baſe, being (by *Schol.* to 2. 6 *Eucl. El.*) proportional; EC : EB : : FA : FG; that is in Words, half the Sum of the Legs is to half their Difference, as the Tangent of half the Sum of the oppoſite Angles is to the Tangent of half their Difference: But Wholes are as their Halves; wherefore the Sum of the Legs is to their Difference, as the Tangent of half the Sum of the Angles oppoſite is to the Tangent of half their Difference. *Q. E. D.*

Axiom IV.

The Baſe or greateſt Side of any Plane Triangle is to the Sum of the Legs, as the Difference of the Legs is to the Difference of the Segments of the Baſe made by a Perpendicular let fall from the Angle oppoſite to the Baſe.



Demonſtration.

From the ∠ B, on the Baſe AC, of the △ ABC, let fall the Perpendicular BD; on B, as a Center, with the greater Leg BC, as a Radius, deſcribe the Circle B*x*C*y* Z; and produce AB to *x* and *y*, and CA to Z. Then by the 35. 3 *Eucl. Elem.* A*y* × A*x* is = AC × AZ; *viz.* : BC—BA : × : BC + BA : = AC × : DC — DA : therefore AC : BC + BA : : BC — BA : DC — DA. *Q. E. D.* Otherwiſe, let the Difference of the Squares of the Sides BC and AB be taken and divided by the Baſe AC, the Quotient ſhall be the Difference of the Segments of the Baſe aforeſaid: Or, ſquare all the 3 Sides, and deduct the Square of one of the leſs Sides out of the Sum of the other two Squares, divide half the Remainder by the longeſt Side, the Quotient is the Alternate Segment of the Baſe.

The Proportions for the Solution of the ſix Caſes of Plane oblique Triangles. [*See the laſt* Fig.]

Given.

Given	*Reqd.*	*Proportions.*	*Ax.*	*Caſe.*
AB BC and C	A	AB : BC :: Si. C : Si. A.	2	1

N. B. 1ſt, If the given Angle be Obtuſe, the other two Angles then are each Acute.

2dly, If the Side oppoſite to the given Angle is longer than the Side oppoſite to the Angle ſought, then is the Angle ſought Acute; but if ſhorter, then is the ſaid Angle doubtful, and may be either Acute or Obtuſe, becauſe both the Sine and its Complement to two Right-Angles are the ſame: Wherefore to be certain of what Quality the Angle oppoſite to the greateſt Side is: Take the Sum and Difference of the greateſt Side and Middle (or leaſt) and their Logarithms, if the Half of them be equal to the Logarithm of the third Side, the Angle oppoſite to the greateſt Side is a Right-Angle; but if the Logarithm of the third Side be greater than the Half, it is Acute, if leſs, it is Obtuſe: Or, without Logarithms, multiply the ſaid Sum by the Difference aboveſaid, and extract the Square Root,

which if { Equal to / Greater than / Leſs than } the third Side, then is the greateſt Angle { Right / Obtuſe / Acute }

AB BC and C	AC	AB : BC :: Si. C : Si. A. Hence, by Subtraction, the ∠ B will be known. Si. A : Si. B :: BC : AC.	2	2
A, C and BC	AB	Si. A : Si. C :: BC : AB.	2	3
B AB BC	A and C	BC + AB : BC — AB :: Tan. ½ Sum of the ∠s oppoſite : Tan. ½ Difference of the ∠s oppoſite. Then ½ Sum + ½ Difference = greater ∠ A; and ½ Sum — ½ Difference = leſſer ∠ C.	3	4
B AB BC	AC	Firſt, find the Angles by the laſt; then Si. C : Si. B :: AB : AC.	3 2	5
AB BC AC	A B C	AC : BC + BA :: BC — BA : DC — DA: Then ½ AC + ½ DC — ½ DA = DC. And ½ AC — : ½ DC — ½ DA : = DA. Then AB : AD :: R : Coſi. A. And CB : DC :: R : Coſi. C. And 180° — ∠ A — ∠ C = ∠ B.	4 1 1	6

Or more readily at one Operation.

From half the Sum of the Sides ſubduct each particular Side, and let the Sum of the Logarithm of the half Sum and Difference of the Side ſubtending the enquired Angle be ſubducted from the Sum of the Log. of the other Difference and the double Radius, half the Remainder ſhall be the Log. of the Tangent of half the enquired Angle.

Agreeable to this Axiom *in* Gellibrand's Trig. Britann. p. 46.

As the Rectangle of half the Sum of the Sides and the Difference between that half Sum and the Side oppoſite to the Angle required, is to the Rectangle of the other two Remainders; ſo is the Square of Radius to the Square of the Tangent of half the Angle ſought.

Ex Angulis latera, vel ex lateribus Angulos & mixtim in Triangulis tam planis quam Sphæricis aſſequi, Summa Gloria Mathematici eſt: Sic enim Cœlum & Terras & Maria felici & admirando calculo menſurat. Fran. Vieta.

FINIS.

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FINIS.

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PREFACE.

THE French Tongue being confeſſedly an eſſential Part of a liberal Education, every attempt to render the Study leſs painful, and the attainment of that faſhionable Language more expeditious, cannot but meet with Encouragement from the judicious Part of the Public.

Dialogues have, at ſeveral Times, been publiſhed, deſignedly for the Uſe of Schools; but, it muſt be acknowledged, that they are, for the moſt Part, injudiciouſly collected, containing, at the very Beginning, idiomatical Expreſſions, which, far from forwarding the Student in his Purſuits, ſerve only to perplex his Underſtanding: In the following Sheets Care has been taken to lead him gradually, and the Vocabularies ſuitable to and containing almoſt all the Words of each ſucceeding Dialogue, muſt of Courſe greatly help his Memory.

The Dialogues, as well as the Plan of the whole Work, are entirely new, and I doubt not but they may be improved, and, if rightly purſued, will render this

Under

Undertaking, not only the moſt uſeful, but likewiſe the firſt Book to be put in the Hands of all thoſe who learn the French Language.

Conciſeneſs, Eaſe, and Perſpicuity ought to be aimed at in all elementary Performances; how far I have ſucceeded in theſe Particulars, is left to the Determination of candid and impartial Judges.

N. B. The Words of the Vocabularies, to make Senſe in ſome Places, are generally to be aſked from the Left to the right, which a Maſter will eaſily perceive.

ERRATA.

Page 17. Line 9. Peignez votre tête, *read* Peignez-vous.
—— 19. —— 42. Ecalez les pois, *ſhell the peaſe, read* Ecalez les noix, *ſhell the walnuts.*
—— 26. —— 4. Nous avons écalé les pois, *we have ſhell'd the peaſe, read* Nous avons écalé les noix, *We have ſhelled the walnuts.*

THE ELEMENTS OF FRENCH CONVERSATION.

THE VOCABULARY.

DU papier, *paper.*
Des plumes, *pens.*
Un canif, *a penknife.*
Des pommes, *apples.*
Des prunes, *plumbs.*
Du chagrin, *ſorrow.*
De l'or, *gold.*
Des amandes, *almonds.*
Des framboiſes, *raſberries.*
Des raiſins, *grapes.*
Des ennemis, *enemies.*
Des égaux, *equals.*

AVOIR,

J'ai du papier.
Il a de l'encre.
Nous avons des plumes.
Vous avez un canif.
Ils ont des oranges.

J'avois des poires.
Il avoit des pommes.
Nous avions des ceriſes.
Vous aviez des prunes.
Ils avoient des citrons.

De l'encre, *ink.*
Des oranges, *oranges.*
Des poires, *pears.*
Des ceriſes, *cherries.*
Des citrons, *lemons.*
Du change, *change.*
De l'argent, *money.*
Des groſeilles, *gooſeberries.*
Des amis, *friends.*
Des ſupérieurs, *ſuperiors.*
Des inférieurs, *inferiors.*

TO HAVE.

I Have paper.
He has ink.
We have pens.
You have a penknife.
They have oranges.

I had pears.
He had apples.
We had cherries.
You had plumbs.
They had lemons.

J'eus de la joie.	*I had joy.*
Il eut du chagrin.	*He had ſorrow.*
Nous eûmes du change.	*We had change.*
Vous eûtes de l'or.	*You had gold.*
Ils eurent de l'argent.	*They had money.*
J'aurai des amandes.	*I ſhall have almonds.*
Il aura des groſeilles.	*He ſhall have gooſeberries.*
Nous aurons des framboiſes.	*We ſhall have raſberries.*
Vous aurez des fraiſes.	*You will have ſtrawberries.*
Ils auront des raiſins.	*They will have grapes.*
J'aurois des amis.	*I ſhould have friends.*
Il auroit des ennemis.	*He ſhould have enemies.*
Nous aurions des ſupérieurs.	*We would have ſuperiors.*
Vous auriez des inférieurs.	*You would have inferiors.*
Ils auroient des égaux.	*They would have equals.*

THE VOCABULARY.

UNE épée, *a ſword.*	De la modeſtie, *modeſty.*
Une épingle, *a pin.*	Du dégoût, *diſguſt.*
Un congé, *a holiday.*	Des avelines, *filberts.*
Des pêches, *peaches.*	Des cocombres, *cucumbers.*
Des figues, *figs.*	Des tulippes, *tulips.*
Des chataignes, *cheſnuts.*	Des fleurs, *flowers.*
Des noix, *nuts.*	Du profit, *profit.*
Des noiſettes, *ſmall nuts.*	Des roſes, *roſes.*
De la Sincérité, *ſincerity.*	Des marchandiſes, *wares.*
De la politeſſe, *politeneſs.*	De l'appétit, *appetite.*
INTERROGATIVEMENT.	INTERROGATIVELY.
A-t-il-une épée ?	*Has he a ſword ?*
Avons-nous congé ?	*Have we a holiday ?*
Avez-vous une épingle ?	*Have you a pin ?*
Ont-ils des pêches ?	*Have they peaches.*
Avoit-il des figues ?	*Had he figs ?*
Avions-nous des chataignes ?	*Had we cheſnuts ?*
Aviez-vous des noix ?	*Had you nuts ?*
Avoient-ils des noiſettes ?	*Had they ſmall nuts ?*

Eut-il de la fincérité ?	*Had he ſincerity ?*
Eûmes-nous du dégoût ?	*Had we diſguſt ?*
Eûtes-vous de la politeſſe ?	*Had you politeneſs ?*
Eurent-ils de la modeſtie ?	*Had they modeſty ?*
Aura-t-il-des avelines ?	*Will he have filberts ?*
Aurons-nous des cocombres ?	*Shall we have cucumbers ?*
Aurez-vous des tulippes ?	*Shall you have tulips ?*
Auront-ils des fleurs ?	*Shall they have flowers ?*
Auroit-il du profit ?	*Would he have a profit ?*
Aurions-nous des roſes ?	*Would we have roſes ?*
Auriez-vous des marchandiſes ?	*Would you have wares ?*
Auroient-ils de l'appétit ?	*Would they have appetite ?*

THE VOCABULARY.

UN mouchoir, *a handkerchief.*	Des ſerviettes, *napkins.*
Des jarretières, *garters.*	Un maître, *a maſter.*
	De la reconnoiſſance, *gratitude.*
Des ſouliers, *ſhoes.*	Un habit, *a ſuit of cloaths.*
Des boucles, *buckles.*	Des bijoux, *jewels.*
Un chapeau, *a hat.*	Des cartes, *carts.*
Des gants, *gloves.*	Des rivaux, *rivals.*
Un peigne, *a comb.*	Du fruit, *fruit.*
Des bottes, *boots.*	Des domeſtiques, *ſervants.*
Une montre, *a watch.*	Des richeſſes, *riches.*
Une tabatière, *a ſnuff-box.*	Du thé, *tea.*
Du plaiſir, *pleaſure.*	Du caffé, *coffee.*
De l'amitié, *friendſhip.*	Du ſucre, *ſugar.*

NEGATIVEMENT.	NEGATIVELY.
JE n'ai pas de mouchoir.	*I Have no handkerchief.*
Il n'a pas de jarretières.	*He has no garters.*
Nous n'avons pas de ſouliers.	*We have no ſhoes.*
Vous n'avez pas de boucles.	*You have no buckles.*
Ils n'ont pas de chapeaux.	*They have no hats.*
Je n'avois pas de gants.	*I had no gloves.*
Il n'avoit pas de peigne.	*He had no comb.*
Nous n'avions pas de bottes.	*We had no boots.*
Vous n'aviez pas de montre.	*You had no watch.*
Ils n'avoient pas de tabatière.	*They had no ſnuff box.*

Je n'eus pas de plaiſir.	*I had no pleaſure.*
Il n'eut pas d'amitié.	*He had no friendſhip.*
Nous n'eûmes pas de ſerviettes.	*We had no napkins.*
Vous n'eûtes pas de maître.	*You had no maſter.*
Ils n'eurent pas de reconnoiſſance.	*They had no gratitude.*
Je n'aurai pas d'habit.	*I ſhall have no ſuit of cloaths.*
Il n'aura pas de bijoux.	*He will have no jewels.*
Nous n'aurons pas de cartes.	*We ſhall have no carts.*
Vous n'aurez pas de rivaux.	*You will have no rivals.*
Ils n'auront pas de fruit.	*They will have no fruit.*
Je n'aurois pas de domeſtiques.	*I ſhould have no ſervants.*
Il n'auroit pas de richeſſes.	*He would have no riches.*
Nous n'aurions pas de thé.	*We ſhould have no tea.*
Vous n'auriez pas de caffé.	*You ſhould have no coffee.*
Ils n'auroient pas de ſucre.	*They would have no ſugar.*

THE VOCABULARY.

DU velour, *velvet.*	Une place, *a place.*
Des couteaux, *knives.*	Une penſion, *a penſion.*
Une cuiller, *a ſpoon.*	Un coffre, *a trunk.*
De la poudre, *powder.*	Des rideaux, *curtains.*
Des ciſeaux, *ſciſſars.*	Des bougies, *wax candles.*
Un miroir, *a looking-glaſs.*	Des chandeliers, *candleſticks.*
Des rubans, *ribbands.*	Du pâté, *pye.*
Des chaiſes, *chairs.*	Des jardins, *gardens.*
Une récompenſe, *a reward.*	Des maiſons, *houſes.*
Un caroſſe, *a coach.*	Les ſuffrages, *the ſuffrages.*

NEGATIVEMENT ET INTERROGATIVEMENT.	NEGATIVELY AND INTERROGATIVELY.
N'A-t-il pas du velour ?	*HAS he no velvet ?*
N'avons-nous pas de couteaux ?	*Have we no knives ?*
N'avez-vous pas de cuillers ?	*Have you no ſpoons ?*
N'ont-ils pas de poudre ?	*Have they no powder ?*
N'avoit-il pas de ciſeaux ?	*Had he no ſciſſars ?*
N'avions-nous pas de miroir ?	*Had we no looking-glaſs ?*
N'aviez-	*Had*

N' aviez-vous pas de rubans ? *Had you no ribbands ?*
N'avoient-ils pas de chaiſes ? *Had they no chairs ?*

N'eut-il pas de récompenſe ? *Had he no reward ?*
N'eûmes-nous pas de caroſſe ? *Had we no coach ?*
N'eûtes-vous pas de place ? *Had you no place ?*
N'eurent-ils pas de penſion ? *Had they no penſion ?*

N'aura-t-il pas de coffre ? *Will he have no trunk ?*
N'aurons-nous pas de rideaux ? *Shall we not have curtains ?*
N'aurez-vous pas de bougies ? *Will you not have wax candles ?*
N'auront-ils pas de chandeliers ? *Will they not have candleſticks ?*

N'auroit-il pas de pâté ? *Would he not have pye ?*
N'aurions-nous pas de jardins ? *Should we not have gardens ?*
N'auriez-vous pas de maiſons ? *Would you not have houſes ?*
N'auroient ils pas les ſuffrages ? *Would they not have the ſuffrages ?*

THE VOCABULARY.

BIEN aiſe, *very glad.*
Sincère, *ſincere.*
Occupé, *buſy.*
Agréable, *agreeable.*
Timide, *fearful.*
Surpris, *ſurpriſed.*
Aimable, *lovely.*
Tranquille, *quiet.*
Conſcientieux, *conſcientious.*
Conſtant, *conſtant.*
Curieux, *curious.*
Deſpotique, *deſpotical.*
Diligent, *diligent.*
Pareſſeux, *idle.*
Dangereux, *dangerous.*
Diſcret, *diſcreet.*
Docte, *learned.*
Econome, *ſaving.*
Envieux, *envious.*
Expert, *expert.*
Malade, *ſick.*
Suſpect, *ſuſpected.*
Trompé, *deceived.*
Méthodique, *methodical.*
Admirable, *admirable.*

ETRE.	TO BE.
JE ſuis bien aiſe.	*I Am very glad.*
Il eſt ſincère.	*He is ſincere.*
Nous ſommes occupés.	*We are buſy.*
Vous êtes agréables.	*You are agreeable.*
Ils ſont timides.	*They are fearful.*
J'étois ſurpris.	*I was ſurpriſed.*
Il étoit aimable.	*He was lovely.*
Nous étions tranquilles.	*We were quiet.*
Vous étiez conſcientieux.	*You were conſcientious.*
Ils étoient conſtans.	*They were conſtant.*
Je fus curieux.	*I was curious.*
Il fut deſpotique.	*He was deſpotical.*
Nous fûmes diligens.	*We were diligent.*
Vous fûtes pareſſeux.	*You were idle.*
Ils furent dangereux.	*They were dangerous.*
Je ſerai diſcret.	*I ſhall be diſcreet.*
Il ſera docte.	*He will be learned.*
Nous ſerons économes.	*We ſhall be ſaving.*
Vous ſerez envieux.	*You will be envious.*
Ils ſeront experts.	*They will be expert.*
Je ſerois malade.	*I ſhould be ſick.*
Il ſeroit ſuſpect.	*He would be ſuſpected.*
Nous ſerions trompés.	*We ſhould be deceived.*
Vous ſeriez méthodiques.	*You would be methodical.*
Ils ſeroient admirables.	*They would be admirable.*

THE VOCABULARY.

FACILE, *eaſy.*
Fantaſtique, *fantaſtical.*
Zélé, *zealous.*
Favorable, *favourable.*
Ferme, *firm.*
Fier, *proud.*
Flatteur, *flattering.*
Foible, *weak.*
Frugal, *frugal.*
Généreux, *generous.*
Triſte, *ſorrowful.*
Grave, *grave.*
Habile, *ſkillful.*
Illuſtre, *illuſtrious.*
Immodeſte, *immodeſt.*
Imparfait, *imperfect.*
Mélancolique, *melancholical.*
Juſte, *juſt.*
Judicieux, *judicious.*
Laborieux, *laborious.*

INTER-

INTERROGATIVEMENT.	INTERROGATIVELY.
EST-il facile ?	*IS he eaſy ?*
Sommes-nous fantaſtiques ?	*Are we fantaſtical ?*
Etes-vous zélés ?	*Are you zealous ?*
Sont-ils favorables ?	*Are they favourable ?*
Etoit-il ferme ?	*Was he firm ?*
Etions-nous fiers ?	*Were we proud ?*
Etiez-vous flatteurs ?	*Were you flatterers ?*
Etoient ils foibles ?	*Were they weak ?*
Fut-il frugal ?	*Was he frugal ?*
Fûmes-nous généreux ?	*Were we generous ?*
Fûtes-vous triſtes ?	*Were you ſorrowful ?*
Furent-ils graves ?	*Were they grave ?*
Sera-t-il habile ?	*Will he be ſkillful ?*
Serons-nous illuſtres ?	*Shall we be illuſtrious ?*
Serez-vous immodeſtes ?	*Will you be immodeſt ?*
Seront-ils imparfaits ?	*Will they be imperfect ?*
Seroit-il mélancolique ?	*Would he be melancholical ?*
Serions-nous juſtes ?	*Should we be juſt ?*
Seriez-vous judicieux ?	*Would you be judicious ?*
Seroient-ils laborieux ?	*Would they be laborious ?*

THE VOCABULARY.

INconſtant, *inconſtant.*
Imprudent, *imprudent.*
Impudent, *impudent.*
Incurable, *incurable.*
Incorrigible, *incorrigible.*
Indigent, *indigent.*
Inépuiſable, *inexhauſtible.*
Inhumain, *inhuman.*
Inquiet, *uneaſy.*
Invincible, *invincible.*
Inutile, *uſeleſs.*
Joyeux, *joyful.*
Laborieux, *laborious.*
Lent, *ſlow.*
Equitable, *equitable.*
Jeune, *young.*
Laid, *ugly.*
Savant, *learned.*
Riche, *rich.*
Malicieux, *malicious.*
Litigieux, *litigious.*
Obſtiné, *obſtinate.*
Imbécille, *ſilly.*
Excuſable, *excuſable.*
Ingénieux, *ingenious.*

NEGATIVEMENT.	NEGATIVELY.
JE ne ſuis pas inconſtant.	*I Am not inconſtant.*
Il n'eſt pas imprudent.	*He is not imprudent.*
Nous ne ſommes pas impudens.	*We are not impudent.*
Vous n'êtes pas incurables.	*You are not incurable.*
Ils ne ſont pas incorrigibles.	*They are not incorrigible.*
Je n'étois pas indigent.	*I was not indigent.*
Il n'étoit pas inépuiſable.	*He was not inexhauſtible.*
Nous n'étions pas inhumains.	*We were not inhuman.*
Vous n'étiez pas inquiets.	*You were not uneaſy.*
Ils n'étoient pas invincibles.	*They were not invincible.*
Je ne fus pas inutile.	*I was not uſeleſs.*
Il ne fut pas joyeux.	*He was not joyful.*
Nous ne fûmes pas laborieux.	*We were not laborious.*
Vous ne fûtes pas lents.	*You were not ſlow.*
Ils ne furent pas équitables.	*They were not equitable.*
Je ne ſerai pas jeune.	*I ſhall not be young.*
Il ne ſera pas laid.	*He will not be ugly.*
Nous ne ſerons pas riches.	*We ſhall not be rich.*
Vous ne ſerez pas ſavans.	*You will not be learned.*
Ils ne ſeront pas malicieux.	*They will not be malicious.*
Je ne ſerois pas litigieux.	*I ſhould not be litigious.*
Il ne ſeroit pas obſtiné.	*He would not be obſtinate.*
Nous ne ſerions pas imbécilles.	*We ſhould not be ſilly.*
Vous ne ſeriez pas excuſables.	*You would not be excuſable.*
Ils ne ſeroient pas ingénieux.	*They would not be ingenious.*

THE VOCABULARY.

MODESTE, *modeſt.*
Joli, *pretty.*
Mortel, *mortal.*
Nonchalant, *careleſs.*
Langoureux, *languiſhing.*
Pâle, *pale.*
Morne, *ſullen.*
Opulent, *opulent.*
Peremptoire, *peremptory.*
Orgueilleux, *proud.*
Ponctuel, *punctual.*
Pointilleux, *cavelling.*
Poli, *polite.*
Raiſonnable, *reaſonable.*
Quinteux, *whimſical.*
Régulier, *regular.*
Utile, *uſeful.*
Sage, *wiſe.*
Querelleux, *quarrelſome.*
Téméraire, *raſh.*
Vigoureux,